Mathematics of complete fluid systems

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Everything should be made as simple as possible, but not one bit simpler.

A. Einstein

Contents

1 Introduction

This lecture series is a brief introduction to the mathematics of complete fluid systems. By *complete* we mean the systems that are energetically closed; the total energy of the fluid is a constant of motion. We stay on the platform of *classical continuum mechanics* - a fluid in motion will be described by means of phenomenological quantities - *fields* - that are continous in space and time and obey the basic principles of *classical physics*.

We introduce a mathematical theory of complete fluid systems in the framework of *weak solutions*. The main advantage of this approach is the fact that the associated mathematical problems are solvable without any artificial (smallness and smoothness) restrictions imposed on the data and on arbitrary lapses of time, including infinite time intervals.

1.1 Balance laws

The time evolution of systems in continuum mechanics is determined by the physical principles expressed mathematically as *balance laws*. A balance law includes three basic quantities:

- *a density* d of a balanced quantity;
- *a flux vector field* $\mathbf{F}$;
- *a source* s;

where all quantities depend on the time t and the spatial variable $x \in \Omega \subset R^3$. Here, the symbol Ω denotes the *physical space* - a domain in the Euclidean space occupied by the fluid.

A general form of a balance law reads

Balance law for d

$$[\text{total amount of } d \text{ in } B \text{ at the time } t_2] - [\text{total amount of } d \text{ in } B \text{ at the time } t_1] = \tag{1.1}$$

$$\int_B d(t_2, x)\ \mathrm{d}x - \int_B d(t_1, x)\ \mathrm{d}x = -\int_{t_1}^{t_2} \int_{\partial B} \mathbf{F}(t, x) \cdot \mathbf{n}\ \mathrm{dS}_x\ \mathrm{d}t + \int_{t_1}^{t_2} \int_B s(t, x) \mathrm{d}x\ \mathrm{d}t$$

$$= [\text{total flux through the boundary } \partial B \text{ during the time interval } [t_1, t_2]] + [\text{source}]$$

1.1.1 Balance law - differential form

The following argument is classical. Assuming that all quantities in (1.1) are smooth (differentiable) we may apply the Gauss-Green theorem to deduce that

$$\int_B d(t_2, x)\ \mathrm{d}x - \int_B d(t_1, x)\ \mathrm{d}x = -\int_{t_1}^{t_2} \int_B \mathrm{div}_x \mathbf{F}(t, x)\ \mathrm{d}x\ \mathrm{d}t + \int_{t_1}^{t_2} \int_B s(t, x) \mathrm{d}x\ \mathrm{d}t,$$

furthermore, letting $t_2 \to t_1 \equiv t$,

$$\int_B \partial_t d(t, x)\ \mathrm{d}x = -\int_B \mathrm{div}_x \mathbf{F}(t, x)\ \mathrm{d}x + \int_B s(t, x) \mathrm{d}x,$$

yielding, finally,

Balance law for d - differential form

$$\partial_t d(t, x) + \mathrm{div}_x \mathbf{F}(t, x) = s(t, x) \tag{1.2}$$

1.1.2 Balance law - weak form

There are two ways how to view the so-called *weak formulation* of the balance law (1.1). First, we multiply equation (1.2) by a smooth (differentiable) *test function* $\varphi = \varphi(t, x)$, with a compact support. Integrating by parts, we obtain

Balance law for d - weak form

$$\int_0^T \int_\Omega \Big[d(t,x)\partial_t \varphi(t,x) + \mathbf{F}(t,x)\cdot\nabla_x \varphi(t,x) + s(t,x)\varphi(t,x) \Big] \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{1.3}$$

for any $\varphi \in C_c^\infty((0,T)\times\Omega)$.

It seems a bit strange that the "weak" form was obtained from the "strong" differentiable form. Another possibility how to see (1.3) is to go back to (1.1) that can be viewed as

$$-\lim_{\varepsilon\to 0}\int_{t_1}^{t_2}\int_B d(t,x)\partial_t \varphi_\varepsilon(t,x) \ \mathrm{d}x \ \mathrm{d}t$$

$$\int_B d(t_2,x) \ \mathrm{d}x - \int_B d(t_1,x) \ \mathrm{d}x = -\int_{t_1}^{t_2}\int_{\partial B} \mathbf{F}(t,x)\cdot\mathbf{n} \ \mathrm{dS}_x \ \mathrm{d}t + \int_{t_1}^{t_2}\int_B s(t,x)\mathrm{d}x \ \mathrm{d}t$$

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\varphi_\varepsilon(t,x) \ \mathrm{d}x \ \mathrm{d}t + \lim_{\varepsilon\to 0}\int_0^T\int_\Omega s(t,x)\varphi_\varepsilon(t,x) \ \mathrm{d}x \ \mathrm{d}t,$$

where $\varphi_\varepsilon \in C_c^\infty((t_1,t_2)\times B)$ is a suitable family of smooth functions,

$$0 \le \varphi_\varepsilon \le 1, \ \varphi_\varepsilon \nearrow 1 \text{ as } \varepsilon \to 0.$$

1.2 Systems of balance laws - constitutive relations

Mathematical formulation of problems in continuum fluid mechanics consists of a system of balance laws for quantities $d_1,\dots,d_N$, fluxes $\mathbf{F}_1,\dots,\mathbf{F}_N$, and sources $s_1,\dots,s_N$. Clearly, as we would always have more unknowns than equations, the system must be *closed* by determining the fluxes and sources in terms of the unknowns $d_1,\dots,d_N$. These relations characterize *material properties* of a specific fluid and are termed constitutive relations.

2 Field equations of continuum fluid mechanics

The basic equations of continuum fluid mechanics correspond to the physical principles of balance of *mass*, *momentum*, and *energy*.

2.1 Fluid description - thermostatics

The problem of time in physics and chemistry is closely related to the formulation of the second law of thermodynamics.

I. Prigogine, Nobel Lecture, 8 December 1977

Consider, for a moment, a fluid at rest. We suppose that the *state* of the fluid is completely determined by two fields: the *mass density* ϱ and the temperature ϑ. The (internal) *energy* density $e = e(\varrho, \vartheta)$ and the *pressure* $p = p(\varrho, \vartheta)$ are uniquely determined by ϱ and ϑ. In accordance with *Second law of thrmodynamics*, there is another thermodynamic varible $s = s(\varrho, \vartheta)$ called (specific) *entropy*. The internal energy e, the pressure p, and the entropy s are interrelated through

GIBBS' EQUATION

$$\vartheta D s(\varrho, \vartheta) = D e(\vartheta, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right). \tag{2.1}$$

Relation (2.1) should be viewed as the stipulation that

$$\frac{1}{\vartheta} \left[D e(\vartheta, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right) \right] \text{ is a perfect differential.}$$

Note that the entropy s is determined modulo an additive constant. The latter can be fixed by means of Third law of thermodynamic that requires the entropy to approach zero for $\vartheta \to 0$.

2.2 Transport - bulk velocity

The observable (macroscopic) motion of the fluid is described by means of the *velocity field* $\mathbf{u} = \mathbf{u}(t, x)$. The streamlines are obtained by solving a system of ordinary differential equations:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{u}(t, \mathbf{X}(t, x)), \ \mathbf{X}(0) = \mathbf{x}_0.$$

The velocity describes *mass transport*, the quantity $\varrho\mathbf{u}$ is the mass flux, and the physical principle of mass conservation reads

Equation of continuity:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.2}$$

or, in the weak form,

$$\int_0^T \int_\Omega \Big[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big] \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{2.3}$$

for any test function $\varphi \in C_c^\infty((0,T) \times \Omega)$.

2.3 Balance of linear momentum

Similarly to the preceding part, we derive the balance of linear momentum in the (differential) form

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho \mathbf{f},$$

where $\mathbb{T}$ is the so-called *Cauchy stress*, and $\mathbf{f}$ is a (given) external force acting on the fluid.

Fluids are characterized among other materials by

Stokes' law:

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}, \tag{2.4}$$

where $\mathbb{S}$ is the *viscous stress* and p the pressure.

Accordingly, the balance of momentum in an isotropic fluid can be written as

Balance of momentum - Newton's second law:

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x\mathbb{S} + \varrho\mathbf{f}, \tag{2.5}$$

or, in the weak form

$$\int_0^T\int_\Omega \Big[\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho(\mathbf{u}\otimes\mathbf{u}):\nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}\Big]\ \mathrm{d}x\ \mathrm{d}t = \int_0^T\int_\Omega \Big[\mathbb{S}:\nabla_x\mathbf{u} - \varrho\mathbf{f}\cdot\varphi\Big]\ \mathrm{d}x\ \mathrm{d}t \tag{2.6}$$

for any test function $\varphi\in C_c^\infty((0,T)\times\Omega;R^3)$.

We have noticed that, unlike in the mass conservation, the momentum flux consists of two components: the *convective* one representes by the tensor $\varrho\mathbf{u}\otimes\mathbf{u}$ that corresponds, similarly to (2.2) to the mass transport, and the diffusive one $\mathbb{S}-p\mathbb{I}$. As we shall se below, the energy is transported in a similar way.

2.4 Balance of energy

Determining a suitable form of the balance of energy is one of the cornerstones of the theory. Our approach leans on a formulation based on *Second law of thermodynamics*, specifically, on the entropy production equation formulated below.

2.4.1 Kinetic energy balance

Taking the scalar product of the momentum balance equation (2.5) with $\mathbf{u}$ we obtain

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \mathrm{div}_x\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\right) + \mathrm{div}_x\Big(p(\varrho,\vartheta)\mathbf{u}\Big) - \mathrm{div}_x\Big(\mathbb{S}\mathbf{u}\Big) = -\mathbb{S}:\nabla_x\mathbf{u} + p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u}. \tag{2.7}$$

Note that, in accordance with the equation of continuity (2.2), we have

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = \Big(\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u})\Big)\mathbf{u} + \varrho\partial_t\mathbf{u} + \varrho\mathbf{u}\cdot\nabla_x\mathbf{u}.$$

The quantity

$$\frac{1}{2}\varrho|\mathbf{u}|^2 \text{ is called } \textit{kinetic energy},$$

whereas equation (2.7) represent the kinetic energy balance. Obviously, equation (2.7) contains a source term

$$-\mathbb{S}:\nabla_x\mathbf{u} + p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u}.$$

In many cases, the external force $\mathbf{f}$ is a gradient of a scalar potential, specifically,

$$\mathbf{f} = \nabla_x F,\ \ F = F(x).$$

Accordingly, making use of the continuity equation (2.2), we get

$$\varrho \mathbf{f} \cdot \mathbf{u} = \varrho \nabla_x F \cdot \mathbf{u} = \mathrm{div}_x \Big(\varrho F \mathbf{u} \Big) - F \mathrm{div}_x (\varrho \mathbf{u}) == \partial_t \Big(\varrho F \Big) + \mathrm{div}_x \Big(\varrho F \mathbf{u} \Big);$$

whence (2.7) reads

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 - \varrho F \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 - \varrho F \right) \mathbf{u} \right) + \mathrm{div}_x \Big(p(\varrho, \vartheta) \mathbf{u} \Big) - \mathrm{div}_x \Big(\mathbb{S} \mathbf{u} \Big) \tag{2.8}$$

$$= -\mathbb{S} : \nabla_x \mathbf{u} + p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}.$$

The presence of the source term on the right-hand side of (2.8) indicates that the kinetic energy is not conserved, in particular, the *total energy balance* should also take into account the changes of the internal energy. Note, however, that if the fluid is inviscid (idealization), we would have

$$\mathbb{S} \equiv 0.$$

If, in addition, the fluid is barotropic, meaning the pressure $p = p(\varrho)$ depends solely on the density, we have, making once more use of the continuity equation,

$$p(\varrho) \mathrm{div}_x \mathbf{u} = -\partial_t \Big(\varrho P(\varrho) \Big) - \mathrm{div}_x \Big(\varrho P(\varrho) \Big),$$

where

$$P(\varrho) = \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z.$$

In such a case, equation (2.8) would reduce to a *conservation law*

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho P(\varrho) - \varrho F \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho P(\varrho) - \varrho F \right) \mathbf{u} \right) + \mathrm{div}_x \Big(p(\varrho, \vartheta) \mathbf{u} \Big) = 0 \tag{2.9}$$

for the energy

$$\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho P(\varrho) - \varrho F$$

of an inviscid barotropic fluid.

2.4.2 Internal energy balance

In accordance with *First law of thermodynamics*, the *total energy* of the system is conserved. Thus the internal energy balance must contain the source term appearing in (2.8) as a "sink" on the right-hand side. Specifically, the internal energy balance equation reads

$$\partial_t \left(\varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left(\varrho e(\varrho, \vartheta) \mathbf{u} \right) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}. \tag{2.10}$$

In addition to the convective transport term $\varrho e \mathbf{u}$, the internal energy can be transported without mass transfer by diffusion, represented by the extra internal energy flux $\mathbf{q}$.

Summing up (2.8), (2.10) we obtain the *total energy balance* equation

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \mathbf{u} \right) \tag{2.11}$$

$$+ \mathrm{div}_x \Big(p(\varrho, \vartheta) \mathbf{u} \Big) - \mathrm{div}_x \Big(\mathbb{S} \mathbf{u} \Big) + \mathrm{div}_x \mathbf{q} = 0.$$

2.4.3 Entropy production

Using again the continuity equation (2.2), we can rewrite the internal energy equation (2.11) in the form

$$\varrho De(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \mathbf{u} \cdot De(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix} + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}. \tag{2.12}$$

Since e, p are related to the entropy s by Gibbs'equation (2.1), we get

$$\varrho De(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \mathbf{u} \cdot De(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix}$$

$$= \varrho \vartheta Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \vartheta \mathbf{u} \cdot Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix} + \frac{p(\varrho, \vartheta)}{\varrho} \left(\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho \right)$$

$$= \varrho \vartheta Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \vartheta \mathbf{u} \cdot Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u};$$

whence (2.12) reads as

$$\varrho \vartheta Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \partial_t \varrho \\ \partial_t \vartheta \end{bmatrix} + \varrho \vartheta \mathbf{u} \cdot Ds(\varrho, \vartheta) \cdot \begin{bmatrix} \nabla_x \varrho \\ \nabla_x \vartheta \end{bmatrix} + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u}.$$

In other words, we have deduced

Entropy balance equation:

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma \tag{2.13}$$

or, in the weak form,

$$\int_0^T \int_\Omega \left[\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\varphi\right] \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_\Omega \sigma\varphi \,\mathrm{d}x\,\mathrm{d}t \tag{2.14}$$

for any $\varphi \in C_c^\infty((0,T)\times\Omega)$, with the *entropy production rate*

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right). \tag{2.15}$$

Although the internal energy balance (2.10), the total energy conservation (2.11), and the entropy production equation (2.13) are, in fact, *equivalent* in the framework of regular solutions, it may not be the case for the weak solutions. As we shall see, it is the entropy that provides the most relevant piece of information in the weak formulation.

2.5 Constitutive equations for the viscous stress and the internal energy flux

In accordance with *Second law of thermodynamics*, the entropy production rate σ in (2.15) must be a non-negative quantity for any admissible physical process. This fact imposes certain restrictions on the specific form of $\mathbb{S}$ and $\mathbf{q}$. Here, we suppose that $\mathbb{S}$ is a linear function of the velocity gradient $\nabla_x\mathbf{u}$ while || is a linear function of $\nabla_x\vartheta$. Specifically, we impose

Newton's rheological law:

$$\mathbb{S} = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{2.16}$$

together with

Fourier's law:

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{2.17}$$

Note that the entropy production rate σ takes the form

$$\sigma = \frac{1}{\vartheta} \left(\frac{\mu}{2} \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right|^2 + \eta |\mathrm{div}_x \mathbf{u}|^2 + \frac{\kappa(\vartheta) |\nabla_x \vartheta|^2}{\vartheta} \right); \tag{2.18}$$

whence, in particular, the *shear viscosity coefficient* μ, the *bulk viscosity coefficient* η as well as the *heat conductivity coefficient* κ must be non-negative. In this lecture, we always assume *strict* positivity of μ and κ.

2.6 Boundary conditions

In this series of lectures, we always assume that the physical space containing the fluid is a *bounded* domain $\Omega \subset R^3$. Accordingly, some kind of boundary behavior of certain fields must be specified. Our principal assumption is that the total energy of the fluid is conserved. A short inspection of equation (2.11) reveals that a suitable condition could be *impermeability of the boundary*:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{2.19}$$

together with

Thermal insulation:

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{2.20}$$

In the presence of *viscosity*, the impermeability hypothesis (2.19) is not sufficient to determine uniquely the behavior of the fluid. Alternatively, we may take the most common

No-slip boundary condition:

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{2.21}$$

or

Complete slip boundary condition:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0. \tag{2.22}$$

3 Weak formulation

The system of equations (2.2), (2.5), (2.13), with the constitutive relations for $\mathbb{S}$ and $\mathbf{q}$ given by (2.16), (2.17), will be called *Navier-Stokes-Fourier system*. Our goal in the present section is to introduce a suitable *weak* formulation of the problem that takes into account the satisfaction of the boundary conditions (2.20 - 2.22). In order to have, at least formally, a well-posed problem, we introduce the *initial conditions* that characterize the original state of the fluid system at the time $t = 0$:

$$\varrho(0, \cdot) = \varrho_0, \ \varrho\mathbf{u}(0, \cdot) = \varrho_0\mathbf{u}_0, \ \varrho s(\varrho, \vartheta) = \varrho_0 s(\varrho_0, \vartheta_0). \tag{3.1}$$

In order to comply with the basic physical principles, we suppose that

$$\varrho_0 > 0, \ \vartheta_0 > 0 \text{ in } \Omega. \tag{3.2}$$

3.1 Weak continuity

We solutions are usually quantities that are only (locally) integrable with respect to the space and time variables, satisfying some differential equations in the sense of generalized derivatives. Consider, for instance, the weak formulation of a general conservation law (1.3). Here, it is enough that d, $\mathbf{F}$, and s be integrable functions in the space-time cylinder $(0, T) \times \Omega$,

$$d \in L^1((0, T) \times \Omega), \ \mathbf{F} \in L^1((0, T) \times \Omega; R^3), \ s \in L^1((0, T) \times \Omega).$$

Take the quantity $\varphi(t,x) = \psi(t)\phi(x)$ as a test function in (1.3) to obtain

$$\int_0^T \left(\int_\Omega d(t,\cdot)\phi \ \mathrm{d}x \right) \partial_t \psi \ \mathrm{d}t = - \int_0^T \left(\int_\Omega \left(\mathbf{F} \cdot \nabla_x \phi - s\phi \right) \ \mathrm{d}x \right) \psi \ \mathrm{d}t. \tag{3.3}$$

As relation (3.3) holds for any smooth $\psi = \psi(t)$, we deduce that the function of time:

$$t \mapsto \int_\Omega d(t,\cdot)\phi \ \mathrm{d}x \text{ is absolutely continuous}$$

for any $\phi \in C_c^\infty(\Omega)$. In other words, we are able to identify the *instantaneous values* of the averages $\int_\Omega d\phi \ \mathrm{d}x$ provided ϕ is differentiable and compactly supported.

In addition, suppose that d enjoys some extra integrability, say

$$\operatorname{ess} \sup_{t \in (0,T)} \|d(t,\cdot)\|_{L^p(\Omega)} \leq c \text{ for a certain } p > 1,$$

in other words,

$$d \in L^\infty(0,T; L^p(\Omega)).$$

Since the smooth functions $\phi \in C_c^\infty(\Omega)$ are dense in the dual space $L^{p'}(\Omega)$, $1/p + 1/p' = 1$, we get

$$t \mapsto \int_\Omega d(t,\cdot)\phi \ \mathrm{d}x \text{ is absolutely continuous for any } \phi \in L^{p'}(\Omega) \approx [L^p(\Omega)]^*,$$

in other words, the density d, viewed as a function of the time t with values in the Banach space $L^p(\Omega)$, is continuous with respect to the weak topology in $L^p(\Omega)$:

$$d \in C_{\text{weak}}([0,T]; L^p(\Omega)).$$

3.2 Equation of continuity - weak formulation

Suppose that $\varrho \geq u$ and that

$$\varrho \in L^\infty(0,T; L^\gamma(\Omega)), \ \ \varrho\mathbf{u} \in L^\infty(0,T; L^\beta(\Omega; R^3)) \text{ for certain } \beta, \gamma > 1.$$

We say that ϱ, $\varrho\mathbf{u}$ is a weak solution of equation (2.2), supplemented with the boundary condition (2.19) and the initial condition (3.1), if the integral identity

$$\int_\Omega \left[\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0\varphi(0,\cdot) \right] \ \mathrm{d}x = \int_0^\tau \int_\Omega \left(\varrho\partial_t\varphi + \varrho\mathbf{u} \cdot \nabla_x \varphi \right) \mathrm{d}x \ \mathrm{d}t \tag{3.4}$$

is satisfied for any $\tau \in [0,T]$ and any test function $\varrho \in C_c^\infty([0,T] \times \overline{\Omega})$.

It is worth-noting that (3.4) holds in the whole physical space $(0,T) \times R^3$ provided ϱ was extended to be zero outside Ω.

3.3 Momentum equation - weak formulation

In order to formulate a weak analogue of the momentum equation, we need the stress tensor to be al least integrable. As the latter is given by Newton's law (2.16), we need the spatial gradient of $\mathbf{u}$ to be at least integrable. Accordingly, we suppose that

$$\mathbf{u} \in L^q(0,T;W^{1,q}(\Omega;R^3)) \text{ for a certain } q>1.$$

As the Sobolev functions in $W^{1,q}$ possess well-defined *traces*, the no-slip boundary condition (2.21) may be incorporated in the function space, requiring

$$\mathbf{u} \in L^q(0,T;W_0^{1,q}(\Omega;R^3)) \text{ for a certain } q>1. \tag{3.5}$$

Similarly to the preceding section, we say that ϱ, $\mathbf{u}$, ϑ is a weak solution of the momentum equation (2.5), (2.16), with the boundary conditions (2.21) and the initial condition (3.1) if the velocity field belongs to the class specified in (3.5), $\mathbb{S}$ is given by (2.16), and if the integral identity

$$\int_\Omega \Big[\varrho\mathbf{u}(\tau,\cdot)\cdot\varphi(\tau,\cdot) - \varrho_0\mathbf{u}_0\varphi(0,\cdot)\Big]\, \mathrm{d}x = \int_0^\tau \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\varphi \Big)\, \mathrm{d}x\, \mathrm{d}t \tag{3.6}$$

$$-\int_0^\tau \int_\Omega \Big(\mathbb{S} : \nabla_x\varphi - \varrho\nabla_x F\cdot\varphi \Big)\, \mathrm{d}x\, \mathrm{d}t$$

holds for any $\tau \in [0,T]$ and any test function $\varphi \in C_c^\infty([0,T]\times\Omega;R^3)$.

In (3.6), we tacitly assume that all quantities are at least integrable in the set $(0,T)\times\Omega$. If the no-slip condition (2.21) is replaced by the complete slip (2.22), we have to replace (3.5) by

$$\mathbf{u} \in L^q(0,T;W^{1,2}(\Omega;R^3)),\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{3.7}$$

and to extend the class of admissible test functions to

$$\varphi \in C_c^\infty([0,T]\times\overline{\Omega};R^3),\ \varphi\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

3.4 Entropy production

We have reached the crucial moment of the mathematical theory. In the *weak formulation* of the entropy production equation (2.14), we replace *equality* (2.15) by *inequality*

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta} \right). \tag{3.8}$$

To this end, we must have $\vartheta > 0$ a.a. in $(0,T) \times \Omega$.

Accordingly, the weak formulation of the entropy balance reads

$$\int_0^\tau \int_\Omega \Big[\varrho s(\varrho,\vartheta)(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0 s(\varrho_0,\vartheta_0)\varphi(0,\cdot)\Big] \ \mathrm{d}x \ \mathrm{d}t - \int_0^\tau \int_\Omega \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$\geq \int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\varphi\right) \ \mathrm{d}x \ \mathrm{d}t \tag{3.9}$$

for a.a. $\tau \in (0,T)$ and any $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$, $\varphi \geq 0$.

Of course, replacing (2.15) by (3.8) means a crude simplifaction and considerable extension of the class of possible solutions. In order to compensate for this lack of information, the resulting system must be supplemented by an extra condition formulated in the following section.

3.5 Total energy balance

Thanks to our choice of the boundary conditions, we may integrate the total energy balance (2.11) to obtain

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right](\tau,\cdot) \ \mathrm{d}x = \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) - \varrho_0 F\right] \ \mathrm{d}x \tag{3.10}$$

for a.a. $\tau \in (0,T)$.

To conclude, we recall that the *weak formulation* of the *Navier-Stokes-Fourier system* consists of

- MASS CONSERVATION. Equation of continuity in the sense of generalized derivatives.
- NEWTON'S SECOND LAW. Momentum equation + boundary conditions satisfied in the weak sense.
- ENTROPY PRODUCTION. Entropy *inequality* with a non-negative production rate.
- TOTAL ENERGY BALANCE. Conservation of the total energy in time.

4 Weak vs. strong solutions

However beautiful the strategy, you should occasionally look at the results.

Sir Winston Churchill, 1874-1965

Our concept of *weak solutions* to the Navier-Stokes-Fourier system apparently extends that of the strong solutions, in the sense that, evidently, strong solutions satisfying all equations in the classical sense, including the entropy *equation* (2.13), (2.15) are also weak solutions in the sense specified in Section 3.

4.1 Vacuum problem

One of the major open problems of the theory is the possibility for ϱ to become zero in a finite time. Although weak solutions can be constructed with $\varrho \geq 0$, we are not able to show that, in fact,

$$\varrho(t,\cdot) > 0 \text{ as long as } \varrho_0 > 0.$$

Suppose that solutions are smooth. We can rewrite the equation of continuity (2.2) in the form

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u},$$

or, equivalently,

$$\partial_t \log(\varrho) + \mathbf{u} \cdot \nabla_x \log(\varrho) = -\mathrm{div}_x \mathbf{u}.$$

Thus, defining the characteristic field

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{u}(t, \mathbf{X}(t)),$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \log(\varrho(t, \mathbf{X}(t)) = -\mathrm{div}_x \mathbf{u}(t, \mathbf{X}(t));$$

whence

$$\exp(-Lt) \inf_{x\in\Omega} \varrho_0(x) \leq \varrho(t,x) \leq \exp(Lt) \sup_{x\in\Omega} \varrho_0(x) \text{ for all } t > 0, \tag{4.1}$$

where

$$L = \sup_{(t,x)\in(0,T)\times\Omega} |\mathrm{div}_x \mathbf{u}(t,x)|.$$

Unfortunately, *uniform* bounds on $\mathrm{div}_x \mathbf{u}$ are beyond reach of the present theory so (4.1) remains valid only in the class of strong solutions.

4.2 Weak-strong compatibility

A more delicate task is to show that a weak solution that is sufficiently regular, represents a strong solution. In particular, the entropy inequality (3.9) can be replaced by (2.13), (2.15). In order to see this, we first observe that the kinetic energy balance (2.8) holds as soon as the weak solution is smooth, in particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\left(\frac{1}{2}\varrho|\mathbf{u}|^2-\varrho F\right)\ \mathrm{d}x=\int_{\Omega}\Big(-\mathbb{S}:\nabla_x\mathbf{u}+p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}\Big)\ \mathrm{d}x. \tag{4.2}$$

The next step is to take $\vartheta\varphi$, $\varphi\geq 0$, $\varphi\in C_c^\infty((0,T)\times\overline{\Omega})$, as a test function in (3.9) to obtain:

$$\int_0^\tau\int_{\Omega}\mathbb{S}:\nabla_x\mathbf{u}\varphi\ \mathrm{d}x\ \mathrm{d}t\leq\int_0^\tau\int_{\Omega}\Big(\vartheta\Big[\partial_t(\varrho s(\varrho,\vartheta)+\mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u})\Big]\varphi-\mathbf{q}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t. \tag{4.3}$$

On the other hand, Gibbs' equation (2.1) yields

$$\begin{aligned}\vartheta\Big[\partial_t(\varrho s(\varrho,\vartheta))+\mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u})\Big]&=\varrho\Big[\vartheta\partial_t s(\varrho,\vartheta)+\vartheta\mathbf{u}\cdot\nabla_x s(\varrho,\vartheta)\Big]\\&=\varrho\Big[\partial_t e(\varrho,\vartheta)+\mathbf{u}\nabla_x e(\varrho,\vartheta)\Big]+p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u},\end{aligned}$$

where we have also used the equation of continuity (2.2), together with the fact that the (regular) density remains bounded below away from zero provided we have chosen $\varrho_0>0$, see (4.1).

We infer that (4.3) gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\varrho e(\varrho,\vartheta)\ \mathrm{d}x\geq\int_{\Omega}\Big(\mathbb{S}:\nabla_x\mathbf{u}-p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}\Big)\ \mathrm{d}x. \tag{4.4}$$

Thus the total energy balance imposed through (3.10) is compatible with (4.2), (4.4) only if the sign $\geq$ is replaced by $=$ in (4.4), meaning relation (3.9) holds with equality sign.

Weak solutions of the Navier-Stokes-Fourier system introduced in Section 4 coincide with the strong solutions as soon as they are regular, specifically, if $\varrho>0$, $\vartheta>0$, and ϱ, ϑ, $\mathbf{u}$ are continuously differentiable in t and twice continuously differentiable in x.

4.3 Exercises

4.3.1 Maxwell's equation

Suppose that $p=p(\varrho,\vartheta)$, $e=e(\varrho,\vartheta)$, and $s=s(\varrho,\vartheta)$ satisfy Gibbs' relation (2.1). Show Maxwell's equation

$$\frac{\partial e(\varrho,\vartheta)}{\partial\varrho}=\frac{1}{\varrho^2}\left(p(\varrho,\vartheta)-\vartheta\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}\right). \tag{4.5}$$

4.3.2 Entropy vs. pressure

Using Gibbs' equation (2.1) and Maxwell's equation (4.5) show that

$$\frac{\partial s(\varrho,\vartheta)}{\partial \varrho} = -\frac{1}{\varrho^2}\frac{\partial p(\varrho,\vartheta)}{\partial \vartheta}.$$

4.3.3 Monoatomic gas state equation

A general *monoatomic gas* is characterized by the relation

$$p(\varrho,\vartheta) = \frac{2}{3}\varrho e(\varrho,\vartheta). \tag{4.6}$$

Suppose, in addition to (4.6), that p and e satisfy Gibbs' equation (2.1). Show that p takes the form

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) \text{ for a certain function } P.$$

5 A priori bounds

A priori bounds are constraints imposed on the solution set by the data and the differential identities satisfied. As is well known in the standard theiry of *elliptic* equations, *a priori* bounds determine the function spaces framework in which the solutions are looked for. Evolutionary equations arising in continuum fluid mechanics are known to possess only very poor *a rpriori* bounds, which makes the rigorous analysis rather delicate. To begin, we point out that *a priori* bounds are purely formal, derived under the hypothesis of smoothness of all quantities in question.

5.1 Total mass conservation

The simplest bound follows directly by integrating the equation of continuity (2.2) over the spatial domain Ω. Supposing that the velocity field $\mathbf{u}$ obeys the impermeability condition

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

we immediately obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = 0,$$

in other words

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho_0\ \mathrm{d}x, \text{ for any } t\in[0,T]. \tag{5.1}$$

The physical meaning of (5.1) is obvious - the *total mass* of the fluid contained in the physical domain Ω is a constant of motion. Note that the same conclusion holds even in the class of the weak solutions satisfying (3.4), where it is enough to take $\varphi \equiv 1$.

Since the density is *non-negative* we get

$$\sup_{t \in [0,T]} \|\varrho(t,\cdot)\|_{L^1(\Omega)} = \|\varrho_0\|_{L^1(\Omega)} \text{ for any } t \in [0,T]. \tag{5.2}$$

5.2 Energy estimates

The total energy balance (3.10) yields immediately

$$\sup_{t \in [0,T]} \|\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^2(\Omega;R^3)} \leq c(\text{initial data}), \tag{5.3}$$

and

$$\sup_{t \in [0,T]} \|\varrho e(\varrho,\vartheta)(t,\cdot)\|_{L^1(\Omega)} \leq c(\text{initial data}) \tag{5.4}$$

provided, say,

$$F \in L^\infty(\Omega).$$

Of course, we have tacitly assumed that $e \geq 0$. Similarly to the preceding part, the energy estimates remain valid even in the class of weak solutions in the sense specified in Section 4.

5.3 Dissipation and entropy estimates

Our next step is to exploit the entropy balance equation (2.13). Integrating by parts and making use of the boundary conditions we deduce that

$$\int_\Omega \varrho s(\varrho,\vartheta)(\tau,\cdot)\ \mathrm{d}x = (\geq) \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\ \mathrm{d}x + \int_0^\tau \int_\Omega \frac{1}{\vartheta}\left[\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}.\cdot \nabla_x \vartheta}{\vartheta}\right]\ \mathrm{d}x\ \mathrm{d}t. \tag{5.5}$$

5.3.1 Thermodynamic stability

In accordance with *Second law of thermodynamics*, we can control the total entropy production rate provided we have an upper bound on the total entropy of the system. Since we already know that the total internal energy is bounded, we may attempt to obtain upper bound on $\varrho s(\varrho, \vartheta)$ in terms of $\varrho e(\varrho, \vartheta)$. To this end, we introduce

HYPOTHESIS OF THERMODYNAMIC STABILITY

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \tag{5.6}$$

$$\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0. \tag{5.7}$$

Condition (5.6) asserts that *compressibility* of the fluid is always positive while $\partial_\vartheta e$ is the *specific heat at constant volume*.

5.3.2 Ballistic free energy

Let us introduce a thermodynamic potential

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta), \text{ with } \Theta > 0 \text{ a constant}, \tag{5.8}$$

termed *ballistic free energy*.

It follows from the hypotheses (5.6), (5.7) that

- $$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ is strictly convex;} \tag{5.9}$$

- $$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ is decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta. \tag{5.10}$$

5.3.3 Total dissipation balance

Combining the total energy conservation (3.10) with (5.5) we deduce

Total dissipation balance:

$$\int_\Omega \left[H_\Theta(\varrho,\vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_\Theta(\overline{\varrho},\Theta)}{\partial \varrho} - H_\Theta(\overline{\varrho},\Theta) - \varrho F \right] (\tau,\cdot)\ \mathrm{d}x \tag{5.11}$$
$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta} \right)\ \mathrm{d}x\ \mathrm{d}t$$
$$= (\leq) \int_\Omega \left[H_\Theta(\varrho_0,\vartheta_0) - (\varrho_0 - \overline{\varrho}) \frac{\partial H_\Theta(\overline{\varrho},\Theta)}{\partial \varrho} - H_\Theta(\overline{\varrho},\Theta) - \varrho_0 F \right]\ \mathrm{d}x,$$

where the constant $\overline{\varrho} > 0$ has been chosen so that

$$\int_\Omega (\varrho(t,\cdot) - \overline{\varrho})\ \mathrm{d}x = 0.$$

Consequently, making use of the coercivity properties of the ballistic free energy H_Θ established in Section 5.3.2, we deduce from (5.11) the following *a priori* bounds:

$$-c(\text{data}) \leq \int_\Omega \varrho s(\varrho,\vartheta)(\tau,\cdot)\ \mathrm{d}x \leq c(\text{data}) \text{ for all } \tau \in [0,T], \tag{5.12}$$

and

$$\int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta} \right)\ \mathrm{d}x\ \mathrm{d}t \leq c(\text{data}). \tag{5.13}$$

5.3.4 Korn inequality

Our next goal is to use (5.13) in order to derive bounds on $\nabla_x \mathbf{u}$ and $\nabla_x \vartheta$, respectively. We start with $\nabla_x \vartheta$ seeing that, in accordance with Fourier's law (2.17),

$$\int_0^T \int_\Omega \frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta^2}\ \mathrm{d}x\ \mathrm{d}t \leq c(\text{data}).$$

Consequently, assuming

$$\underline{\kappa}(1+\vartheta^2) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^2) \tag{5.14}$$

for certain positive constants $\underline{\kappa}$, $\overline{\kappa}$ we deduce that

$$\int_0^T \int_\Omega |\nabla_x \vartheta|^2 \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega |\nabla_x \log(\vartheta)|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c(\mathrm{data}) \tag{5.15}$$

Similar estimates on the velocity gradient are more delicate. By virtue of Newton's law (2.16), we get

$$\mathbb{S} : \nabla_x \mathbf{u} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \mathbf{u}$$

$$= \frac{\mu(\vartheta)}{2} \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right|^2;$$

whence, assuming, say,

$$\underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \tag{5.16}$$

we deduce from (5.13) that

$$\int_0^T \int_\Omega \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c \tag{5.17}$$

Suppose, for a moment, that $\mathbf{u}$ vanishes on $\partial\Omega$, meaning $\mathbf{u}$ satisfies the no-slip boundary condition (2.21). By simple computation, we get

$$\left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right|^2 = |\nabla_x \mathbf{u}|^2 + |\nabla_x^t \mathbf{u}|^2 + 2 \nabla_x \mathbf{u} \cdot \nabla_x^t \mathbf{u} - \frac{4}{3} |\mathrm{div}_x \mathbf{u}|^2,$$

where, by means of a simple by parts integration,

$$\int_\Omega \nabla_x \mathbf{u} \cdot \nabla_x^t \mathbf{u} \ \mathrm{d}x = \int_\Omega |\mathrm{div}_x \mathbf{u}|^2 \ \mathrm{d}x.$$

Thus the bound (5.13) yields

$$\int_0^T \int_\Omega |\nabla_x \mathbf{u}|^2 \ \mathrm{d}x \leq c(\mathrm{data}). \tag{5.18}$$

Finally, since we have assumed that $\mathbf{u}|_{\partial\Omega} = 0$, we may combine (5.18) with Poincaré's inequality to conclude that

$$\int_0^T \int_\Omega \left(|\mathbf{u}|^2 + |\nabla_x \mathbf{u}|^2 \right) \, \mathrm{d}x \leq c(\text{data}). \tag{5.19}$$

We note that
$$\int_\Omega \left(|\mathbf{u}|^2 + |\nabla_x \mathbf{u}|^2 \right) \, \mathrm{d}x \equiv \|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)},$$
where $W^{1,2}$ stands for the standard *Sobolev space* of functions having first (spatial) partial derivatives square integrable.

5.4 More hypotheses imposed on constitutive equations

The bounds derived in the preceding part are the best (known) *a priori* bounds available under the general conditions imposed on the initial data and the existence time $T > 0$. Still, they are not strong enough, in general, to render *all* terms appearing in the weak formulation at least *equi-integrable* in order to ensure their stability. Besides the general hypothesis of thermodynamic stability (5.6), (5.7), more restrictions will be imposed on the constitutive relations in order to obtain better estimates to make the problem mathematically tractable.

5.4.1 General pressure law

A general pressure law of a *monoatomic gas* reads

$$p(\varrho, \vartheta) = \frac{2}{3} \varrho e(\varrho, \vartheta), \tag{5.20}$$

see Eliezer, Ghatak, Hora [3].

It is straightforward to check that (5.20) is compatible with the general Gibbs' relation (2.1) if the pressure takes the form
$$p(\varrho, \vartheta) = \vartheta^{5/2} P \left(\frac{\varrho}{\vartheta^{3/2}} \right) \tag{5.21}$$
for a certain function $P : [0, \infty) \to R$. Now, the hypothesis (5.6) can be interpreted as

$$P = C^1[0, \infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for any } Z \geq 0. \tag{5.22}$$

Going back to (5.20), the specific internal energy reads

$$e(\varrho, \vartheta) = \frac{3}{2} \varrho \left(\frac{\vartheta^{3/2}}{\varrho} \right) P \left(\frac{\varrho}{\vartheta^{3/2}} \right). \tag{5.23}$$

Now, the second condition of thermodynamic stability (5.7) translates to

$$\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} > 0 \text{ for any } Z > 0, \tag{5.24}$$

in particular, the function

$$Z \mapsto \frac{P(Z)}{Z^{5/3}} \text{ is non-increasing,} \tag{5.25}$$

and we set

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{5.26}$$

Under the present circumstances, the entropy is determined, up to an additive constant, by the Gibbs' equation (2.1), specifically,

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{5.27}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0. \tag{5.28}$$

In accordance with *Third law of thermodynamic*, we may fix S by postulating

$$\lim_{Z\to\infty} S(Z) = 0. \tag{5.29}$$

5.4.2 Effect of thermal radiation

The pressure p can be augmented by a component $p_R = p_R(\vartheta)$ resulting from *thermal radiation*, namely,

$$p_R = \frac{a}{3}\vartheta^4, \ a > 0,$$

with a the Stefan-Boltzmann constant. Consequently, we end up with the following specific form of the constitutive equations:

Pressure:

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \tag{5.30}$$

Internal energy:

$$e(\varrho,\vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4, \tag{5.31}$$

Entropy:

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3\varrho}\vartheta^3. \tag{5.32}$$

5.5 Exercises

5.5.1 Thermodynamic stability for monaotomic gases

Show the the hypothesis of thermodynamic stability (5.7) implies relations (5.24), (5.25).

5.6 Energy estimates revisited

From now on, we shall assume that the pressure p, the internal energy e, and the entropy s are given through (5.30 - 5.32). Thus we get, as a consequence of (5.26), the following bounds

$$\sup_{t\in[0,T]} \|\varrho(t,\cdot)\|_{L^{5/3}(\Omega)} \leq c(\text{data}), \tag{5.33}$$

and, because of the "radiation" component of the internal energy,

$$\sup_{t\in[0,T]} \|\vartheta(t,\cdot)\|_{L^4(\Omega)} \leq c(\text{data}). \tag{5.34}$$

5.7 Pressure estimates

The *a priori* bounds should be at least so strong for all the expressions appearing in the weak formulation to make sense (to be at least integrable in the space-time cylinder $(0,T)\times\Omega$. As a matter of fact, slightly more is needed, namely the *equi-integrability* property in order to perform the limits with respect to the weak topology of the Lebesgue space L^1. Note that the estimates (5.33), (5.34) guarantee only that

$$p(\varrho,\vartheta),\, \varrho e(\varrho,\vartheta)\in L^\infty(0,T;L^1(\Omega)).$$

Better estimates can be obtained by "computing" the pressure by means of the continuity equation (2.5):

$$\Delta p(\varrho,\vartheta)=\mathrm{div}_x\mathrm{div}_x\mathbb{S}-\mathrm{div}_x\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})-\mathrm{div}_x\partial_t(\varrho\mathbf{u}),$$

where, for the sake of simplicity, we have take $\mathbf{f}\equiv 0$. Then, very formally indeed,

$$p(\varrho,\vartheta)=\mathrm{div}_x\Delta^{-1}\mathrm{div}_x\mathbb{S}-\mathrm{div}_x\Delta^{-1}\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})-\Delta^{-1}\mathrm{div}_x\partial_t(\varrho\mathbf{u}).$$

Let us remark that both $\varrho\mathbf{u}\otimes\mathbf{u}$ and $\mathbb{S}$ are already known to be bounded in the Lebesgue space $L^q((0,T)\times\Omega$ for a certain $q>1$. Indeed, by virtue of the Sobolev imbedding relation

$$W^{1,2}(\Omega)\hookrightarrow L^6(\Omega), \tag{5.35}$$

combined with the uniform bounds established in (5.3), (5.19), and (5.32), we have

$$\|\varrho\mathbf{u}\otimes\mathbf{u}\|_{L^q(\Omega;R^{3\times3})}=\|\sqrt{\varrho}\sqrt{\varrho}\mathbf{u}\otimes\mathbf{u}\|_{L^q(\Omega;R^{3\times3})}\le\|\varrho\|_{L^{5/3}(\Omega)}\|\sqrt{\varrho}\mathbf{u}\|_{L^2(\Omega;R^3)}\|\mathbf{u}\|_{W^{1,2}(\Omega;R^3)},$$

with

$$\frac{1}{\gamma}=\frac{3}{10}+\frac{1}{2}+\frac{1}{6},\ \text{meaning } \gamma=\frac{30}{29}>1;$$

whence

$$\varrho\mathbf{u}\otimes\mathbf{u}\in L^2(0,T;L^{30/29}(\Omega;R^{3\times3})). \tag{5.36}$$

Similarly, by virtue of (5.19) , (5.34), and hypothesis (5.16)

$$\|\mathbb{S}\|_{L^q(\Omega,R^{3\times3})}\le c\|\vartheta\|_{L^4(\Omega)}\|\nabla_x\mathbf{u}\|_{L^2(\Omega;R^{3\times3})},\ q=\frac{4}{3};$$

whence

$$\mathbb{S}\in L^2(0,T;L^{4/3}(\Omega;R^{3\times3})). \tag{5.37}$$

Seeing that the operator $\mathrm{div}_x\Delta^{-1}\mathrm{div}_x$ is of zero-th order, the estimates (5.35), (5.36) would guarantee boundedness of the pressure provided we could handle the time derivative $\Delta^{-1}\mathrm{div}_x\partial_t(\varrho\mathbf{u})$.

In order to make these arguments rigorous, we proceed in a slightly different way. Specifically, we multiply the momentum equation by the expression

$$\varphi \nabla_x \Delta^{-1}[1_\Omega b(\varrho)],$$

where 1_Ω denotes the characterictic function of thw domain Ω and $\varphi \in C_c^\infty(\Omega)$. The obvious advantage of such a choice is the fact that the function vanishes on $\partial\Omega$ and one can perform by parts integration by means of the Gauss-Green theorem. The operator $\nabla_x \Delta^{-1}$ may be viewed as an inverse div_x^{-1} and can be rigorously defined as a Fourier multiplier

$$\partial_{x_j} \Delta^{-1}[v] = \mathcal{F}^{-1}_{\xi \to x} \left[\frac{\mathrm{i}\xi_j}{|\xi|^2} \mathcal{F}_{x\xi}[v] \right], \ j = 1, 2, 3,$$

where $\mathcal{F}$ denotes the Fourier transform in the spatial variable x.

The following properties of $\nabla_x \mathrm{div}_x^{-1}$ are standard (see, for instance, [4, Chapter 5]):

$$\|\nabla_x \Delta^{-1}[v]\|_{L^\infty + L^2(R^3;R^3)} \leq c\|v\|_{L^2 \cap L^1(R^3)}, \tag{5.38}$$

$$\|\nabla_x \nabla_x \Delta^{-1}[v]\|_{L^p(R^3;R^{3\times 3})} \leq c\|v\|_{L^p(R^3)} \text{ for any } 1 < p < \infty. \tag{5.39}$$

Taking the scalar product of the momentum equation with $\varphi \nabla_x \Delta^{-1}[b(\varrho)]$, where $b(0) = 0$ and ϱ was extended to be zero outside Ω; or equivalently, using this quantity as a test function in (3.6), and performing obvious by-parts integration, we obtain

$$\int_0^\tau \int_\Omega \varphi p(\varrho, \vartheta) b(\varrho) \ \mathrm{d}x \ \mathrm{d}t \tag{5.40}$$

$$= - \int_0^\tau \int_\Omega p(\varrho, \vartheta) \nabla_x \varphi \cdot \nabla_x \Delta^{-1}[b(\varrho)] \ \mathrm{d}x \ \mathrm{d}t - \int_0^\tau \int_\Omega \varphi \varrho \mathbf{u} \cdot \partial_t \nabla_x \Delta^{-1}[b(\varrho)] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \Big[\Big(\varrho \mathbf{u} \otimes \mathbf{u} + \mathbb{S} \Big) : \nabla_x \Big(\varphi \nabla_x \Delta^{-1}[b(\varrho)] \Big) - \varphi \varrho \nabla_x F \cdot \nabla_x \Delta^{-1}[b(\varrho)] \Big] \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_\Omega \Big[\varphi \varrho \mathbf{u} \nabla_x \Delta^{-1}[b(\varrho)](\tau, \cdot) - \varphi \varrho_0 \mathbf{u}_0 \nabla_x \Delta^{-1}[b(\varrho_0)] \Big] \ \mathrm{d}x.$$

Our goal is to take

$$b(\varrho) = \min\{\varrho, \varrho^\alpha\} \text{ with } \alpha > 0,$$

where $\alpha > 0$ is chosen so small that all terms on the right-hand side of (5.40) may be bounded in terms of the previously established energy estimates. To this end, however, we have to compute $\partial_t \nabla_x \Delta^{-1}[b(\varrho)]$.

5.7.1 Renormalized equation of continuity

Multiplying equation (2.2) on $b'(\varrho)$ we obtain

RENORMALIZED EQUATION OF CONTINUITY:

$$\partial_t b(\varrho) + \mathrm{div}_x\Big(b(\varrho)\mathrm{div}_x\mathbf{u}\Big) + \Big(b'(\varrho)\varrho - b(\varrho)\Big)\mathrm{div}_x\mathbf{u} = 0. \tag{5.41}$$

The renormalized equation was introduced by DiPerna and Lions [2]. It can be derived from the equation of continuity as long as all quantities are sufficiently smooth. In the framework of weak solutions, equation (5.41) can be taken as a kind of supplementary condition to be satisfied by the weak solutions. We point out that, since $\mathbf{u}$ vanishes on $\partial\Omega$, equation (5.41) holds (a.a.) in the whole physical space provided ϱ, $\mathbf{u}$ were extended to be zero outside Ω. In particular, we may compute

$$\partial_t \nabla_x \Delta^{-1}[b(\varrho)] = -\nabla_x\Delta^{-1}[\mathrm{div}_x(b(\varrho)\mathbf{u})] + \nabla_x\Delta^{-1}\left[\Big(b(\varrho) - b'(\varrho)\varrho\Big)\mathrm{div}_x\mathbf{u}\right]. \tag{5.42}$$

Going back to (5.40) we observe that all terms on the right-hand side will be bounded in terms of the energy estimates (5.19), (5.32), (5.34) as soon as we take $b(\varrho) = \min\{\varrho, \varrho^\alpha\}$, with $\alpha > 0$ sufficiently small. Indeed, given $q > 1$, we can find $\alpha > 0$ such that, in view of (5.38), (5.39),

$$\nabla_x\Delta^{-1}[b(\varrho)] \in L^\infty(0,T; W^{1,q}(\Omega; R^3)), \text{ with } W^{1,q}(\Omega) \subset C(\overline{\Omega}) \text{ for } q > 3. \tag{5.43}$$

Similarly, for $\alpha = \alpha(q) > 0$ small enough, we have

$$\nabla_x\Delta^{-1}[\mathrm{div}_x(b(\varrho)\mathbf{u})] \in L^2(0,T; L^q(\Omega; R^3)) \text{ for any } q < 6, \tag{5.44}$$

where we have used the embedding relation

$$W^{1,2}(\Omega) \hookrightarrow L^6(\Omega).$$

Finally, by the same token

$$\nabla_x\Delta^{-1}\left[\Big(b'(\varrho)\varrho - b(\varrho)\Big)\mathrm{div}_x\mathbf{u}\right] \in L^2(0,T; L^q(\Omega; R^3)) \text{ for any } q < 6. \tag{5.45}$$

Thus we conclude that all integrals on the right-hand side of (5.40) can be bounded in terms of the available estimates; whence we conclude that

$$\int_0^T \int_K p(\varrho,\vartheta)\varrho^\alpha \ \mathrm{d}x \ \mathrm{d}t \leq c(K) \text{ for any compact set } K \subset \Omega \text{ and for a certain } \alpha > 0. \tag{5.46}$$

In accordance with the hypotheses (5.22), (5.25), we have

$$p(\varrho,\vartheta) \geq c\vartheta^{5/3}, \ c > 0;$$

whence

$$\int_0^T \int_K \varrho^{\alpha+\frac{5}{3}} \ \mathrm{d}x \ \mathrm{d}t \leq c(K) \text{ for any compact } K \subset \Omega. \tag{5.47}$$

Finally, by the same token

$$p(\varrho,\vartheta) \leq c(1 + \vartheta^4 + \varrho^{5/3}),$$

where

$$\vartheta^4 = \vartheta^3\vartheta, \text{ with } \vartheta^3 \text{ bounded in } L^\infty(0,T;L^{4/3}(\Omega)), \ \vartheta \text{ bounded in } L^2(0,T;L^6(\Omega)).$$

Therefore we conclude that

$$\int_0^T \int_\Omega \vartheta^{4+\beta} \ \mathrm{d}x \ \mathrm{d}t \leq c(\text{data}) \text{ for a certain } \beta > 0. \tag{5.48}$$

5.8 Boundedness of entropy

We stengthen (5.24) to

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{5.49}$$

meaning that the specific heat at constant volume is positive and uniformly bounded. Assuming the entropy complies with Third law of thermodynamics (5.29) we obtain

$$0 \leq \varrho s(\varrho,\vartheta) \leq \varrho s(\varrho,1) \leq \varrho|\log(\varrho)| \text{ for } 0 < \vartheta \leq 1,$$

and

$$\varrho s(\varrho,\vartheta) \leq \varrho(\log(\varrho) + [\log(\vartheta)]^+) \text{ for any } \varrho,\vartheta > 0. \tag{5.50}$$

Consequently, the uniform bounds established in the previous section are strong enough in order to ensure integrability of all terms appearing in the entropy balance (3.9). Indeed the entropy diffusive flux reads

$$\frac{\kappa(\vartheta)}{\vartheta}\nabla_x\vartheta;$$

where $\kappa(\vartheta)$ satisfies (5.14) and the desired bounds follow from (5.15).

6 Weak sequential stability

The problem of *weak sequential stability* may be stated as follows:

WEAK SEQUENTIAL STABILITY:

Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of (strong or weak) solutions of the Navier-Stokes-Fourier system, emanating from the initial data

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon},\ (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_{0,\varepsilon},\ \vartheta_\varepsilon(0,\cdot) = \vartheta_{0,\varepsilon},$$

and bounded by the *a priori* bounds derived in the previous section.

The goal is to show that

$$\varrho_\varepsilon \to \varrho,\ \mathbf{u}_\varepsilon \to \mathbf{u},\ \vartheta_\varepsilon \to \vartheta \text{ as } \varepsilon \to 0$$

in a certain sense, where ϱ, $\mathbf{u}$ ϑ is a weak solution of the same system.

Although showing *weak sequential stability* does not provide an explicit proof of *existence* of the weak solutions, its verification represents one of the prominent steps towards a rigorous *existence theory* for a given system of equations.

6.1 Preliminary observations

In view of the uniform bounds established in the previous section, specifically (5.15), (5.19), (5.32), and (5.34), we may assume that

$$\varrho_\varepsilon \to \varrho \text{ weakly-}(*) \text{ in } L^\infty(0,T;L^{5/3}(\Omega)), \tag{6.1}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;R^3)) \tag{6.2}$$

$$\vartheta_\varepsilon \to \vartheta \text{ weakly-}(*) \text{ in } L^\infty(0,T;L^4(\Omega)) \text{ and weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{6.3}$$

passing to suitable subsequences as the case may be.

Moreover, since ϱ_ε satisfies the equation of continuity (2.3), convergence in (6.1) may be strengthened to

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T]; L^{5/3}(\Omega)). \tag{6.4}$$

Let us recall that (6.4) simply means

$$\left\{ t \mapsto \int_\Omega \varrho_\varepsilon(t,\cdot)\varphi \ {\rm d}x \right\} \to \left\{ t \mapsto \int_\Omega \varrho(t,\cdot)\varphi \ {\rm d}x \right\} \text{ in } C[0,T]$$

for any $\varphi \in L^{5/2}(\Omega)$.

Finally, combining (5.3), (5.33) we deduce that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u}} \text{ weakly-(*) in } L^\infty(0,T; L^{5/4}(\Omega; R^3)). \tag{6.5}$$

Here and hereafter, we use the symbol $\overline{b(v)}$ to denote a limit of a composition of a (nonlinear) function b with a weakly (in L^1) converging sequence $v_\varepsilon \to v$. As is well know, in general,

$$\overline{b(v)} \neq b(v)$$

unless an extra piece of information on oscillations of v_ε is available.

6.2 Weak continuity of the convective terms

Our next goal is to establish convergence of the convective terms, specifically, to show that

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}, \ \overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u}, \text{ and } \overline{\varrho s(\varrho, \vartheta) \mathbf{u}} = \overline{\varrho s(\varrho, \vartheta)} \mathbf{u}.$$

This can be done by means of several rather similar arguments.

Suppose we want to show that

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}.$$

This can be observed in several ways. Seeing that

$$W^{1,2}_0(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \text{ compactly for } 1 \leq q < 6,$$

we deduce that

$$L^p(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega) \text{ compactly whenever } p > \frac{6}{5}. \tag{6.6}$$

In particular, relation (6.4) yields

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u}} \text{ in } C([0,T]; W^{-1,2}(\Omega))),$$

which, combined with (6.2), gives rise to the desired conclusion

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}.$$

Relation

$$\overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u}$$

can be shown in a similar way.

6.2.1 Compactness via Div-Curl lemma

Div-Curl lemma, developed by Murat and Tartar [6], [8], represents an efficient tool for handling compactness in non-linear problems, where the classical Rellich-Kondraschev argument is not applicable.

Div-Curl lemma:

Lemma 6.1 *Let $B \subset R^M$ be an open set. Suppose that*

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^p(B; R^3),$$

$$\mathbf{w}_n \to \mathbf{w} \text{ weakly in } L^q(B; R^3)$$

as $n \to \infty$, where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Let, moreover,

$$\{\mathrm{div}[\mathbf{v}]\}_{n=1}^{\infty} \text{ be precompact in } W^{-1,s}(B),$$

$$\{\mathrm{curl}[\mathbf{w}]\}_{n=1}^{\infty} \text{ be precompact in } W^{-1,s}(B, R^{M \times M})$$

for a certain $s > 1$.

Then

$$\mathbf{v}_n \cdot \mathbf{w}_n \to \mathbf{v} \cdot \mathbf{w} \text{ weakly in } L^r(B).$$

We apply Div-Curl lemma in order to show compactness of all convective terms. Let us start with $\varrho_\varepsilon \mathbf{u}_\varepsilon$. In order to comply with the hypotheses, we introduce cut-off functions:

$$T_k(z) \in C^\infty(R),\ T_k(z) = -T_k(-z),\ T_k(z) \text{ concave in } (0,\infty),\ T_k(z) = z \text{ whenever } |z| \leq k,$$
$$T_k \text{ strictly increasing in } R,\ \lim_{z\to\infty} T_k(z) = k+1. \tag{6.7}$$

We use Div-Curl lemma in the (four-dimensional) space-time cylinder $(0,T)\times\Omega$ setting

$$\mathbf{v}_\varepsilon = [\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon],\ \mathbf{w}_\varepsilon = [T_k[u^j_\varepsilon], 0, 0, 0],\ j = 1,2,3.$$

Consequently,

$$\varrho_\varepsilon T_k[u^j_\varepsilon] \to \varrho \overline{T_k[u^j]} \text{ weakly in } L^{5/4}((0,T)\times\Omega).$$

On the other hand, we have

$$\|u^j_\varepsilon - T_k[u^j_\varepsilon]\|^q_{L^q(\Omega)} \leq 2\int_{\{|u^j_\varepsilon|\geq k\}} |u^j_\varepsilon|^q \, \mathrm{d}x$$

$$= 2\int_{\{|u^j_\varepsilon|\geq k\}} |u^j_\varepsilon|^{6-q}|u^j_\varepsilon|^6 \, \mathrm{d}x \leq \frac{2}{k^{q-6}}\|u^j_\varepsilon\|^q_{L^6(\Omega)} \text{ whenever } 1 \leq q < 6.$$

Consequently,

$$\|u^j_\varepsilon - T_k[u^j_\varepsilon]\|^q_{L^2(0,T;L^q(\Omega))} \leq \left(\frac{1}{k^{6-q}}\right)^{1/q} \|u^j_\varepsilon\|_{L^2(0,T;L^6(\Omega))},\ 1 \leq q < 6,$$

and we may write

$$\varrho_\varepsilon u^j_\varepsilon = \varrho_\varepsilon T_k[u^j_\varepsilon] + \varrho_\varepsilon \left(u^j_\varepsilon - T_k[u^j_\varepsilon]\right) \to \varrho\overline{T_k[u^j]} + \chi_k = \overline{\varrho \mathbf{u}} \text{ weakly in } L^r((0,T)\times\Omega) \text{ for a certain } r > 1,$$

where

$$\|\chi_k\|_{L^r((0,T)\times\Omega)} \to 0 \text{ as } k \to \infty,$$

and, by the same token

$$\|\varrho\overline{T_k[u^j]} - \varrho u^j\|_{L^r((0,T)\times\Omega)} \to 0 \text{ as } k \to \infty.$$

Thus we may infer that

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}.$$

6.2.2 Compactness of the remaining convective terms

The Div-Curl lemma argument can be successively applied to

$$\mathbf{v}_\varepsilon = [\varrho_\varepsilon u^j_\varepsilon, \varrho \mathbf{u}_\varepsilon u^j_\varepsilon - S_{\cdot,j} + p\delta_{\cdot,j}],\ j = 1,2,3,\ \mathbf{w}_\varepsilon = [T_k[u^i_\varepsilon], 0, 0, 0],$$

and

$$\mathbf{v}_\varepsilon = [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon), \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon - \frac{\kappa(\vartheta_\varepsilon)\nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}],\quad \mathbf{w}_\varepsilon = [T_k[\vartheta_\varepsilon], 0, 0, 0],$$

to conclude that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^r((0,T)\times\Omega; R^{3\times 3}), \tag{6.8}$$

$$\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon \to \overline{\varrho s(\varrho, \vartheta)}\mathbf{u} \text{ weakly in } L^r((0,T)\times\Omega; R^3) \tag{6.9}$$

$$\text{for a certain } r > 1.$$

The convergence stated in (6.9) deserves some comments. Here, we have

$$\mathrm{DIV}_{t,x}\left[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon), \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon - \frac{\kappa(\vartheta_\varepsilon)\nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\right] = \sigma_\varepsilon,$$

with, in accordance with the total dissipation balance (5.11),

$$\{\sigma_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^1((0,T)\times\Omega).$$

However, since

$$W^{1,p}_0((0,T)\times\Omega) \hookrightarrow\hookrightarrow \text{ compactly } C([0,T]\times\overline{\Omega}) \text{ for } p > 4,$$

we have

$$L^1((0,T)\times\Omega) \hookrightarrow\hookrightarrow W^{-1,s}((0,T)\times\Omega),\ s < \frac{4}{3}.$$

6.3 Poinwise (a.a.) convergence of the temperature field

Our goal is to show that

$$\vartheta_\varepsilon \to \vartheta \text{ a.a. in } (0,T)\times\Omega.$$

The pointwise convergence of the temperature is necessary in order to pass to the the limit in the non-linear terms. As a matter of fact, we show a stronger statement, specifically,

$$\vartheta_\varepsilon \to \vartheta \text{ in } L^4((0,T)\times\Omega). \tag{6.10}$$

In order to show (6.10), we use monotonicity of the entropy $s(\varrho,\vartheta)$ with respect to ϑ, together with compactness of $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ in the space variable. To begin, we claim that it is enough to show that

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta)\Big)\Big(\vartheta_\varepsilon - \vartheta\Big) \ \mathrm{d}x \ \mathrm{d}t \to 0 \tag{6.11}$$

as $\varepsilon \to 0$. Indeed, since the entropy is given by (5.32), we have

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta)\Big)\Big(\vartheta_\varepsilon - \vartheta\Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_0^T \int_\Omega \left(\varrho_\varepsilon S\left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon^{3/2}}\right) - \varrho_\varepsilon S\left(\frac{\varrho_\varepsilon}{\vartheta^{3/2}}\right)\right)\Big(\vartheta_\varepsilon - \vartheta_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t + \frac{4a}{3}\int_0^T \int_\Omega \Big(\vartheta_\varepsilon^3 - \vartheta^3\Big)\Big(\vartheta_\varepsilon - \vartheta\Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$\geq \frac{4a}{3}\int_0^T \int_\Omega \left(\vartheta_\varepsilon - \vartheta\right)^4 \ \mathrm{d}x \ \mathrm{d}t.$$

We proceed by several steps.

Step 1:

The argument based on Div-Curl Lemma can be applied in a similar way as in the preceding section to obtain:

$$\overline{\varrho s(\varrho,\vartheta)\vartheta} = \overline{\varrho s(\varrho,\vartheta)}\vartheta,$$

in other words,

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\Big(\vartheta_\varepsilon - \vartheta\Big) \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0. \tag{6.12}$$

Thus it remains to show that

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta)\Big(\vartheta_\varepsilon - \vartheta\Big) \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0. \tag{6.13}$$

Step 2:

Since $b(\varrho_\varepsilon)$ satisfy the renormalized equation (5.41), we can apply once more the Div-Curl lemma argument in order to show that

$$\overline{b(\varrho)\vartheta} = \overline{b(\varrho)}\vartheta \text{ for any bounded and smooth (Lipschitz) } \ b,$$

in particular, (6.13) would follow should s depend only on ϱ. As a matter of fact, the same argument yield a slightly better result, namely,

$$\overline{b(\varrho)g(\vartheta)} = \overline{b(\varrho)}\ \overline{g(\vartheta)} \text{ for any } b, g \text{ bounded and Lipschitz.} \tag{6.14}$$

Step 3:

In view of (6.14), the desired conclusion (6.13 follows from the following result - fundamental theorem of the theory of Young measures, see Ball [1], Pedregal [7].

FUNDAMENTAL THEOREM ON YOUNG MEASURES

Theorem 6.1 *Let $\mathbf{v}_\varepsilon : Q \subset R^N \to R^M$ be a sequence of vector fields bounded in $L^1(Q; R^M)$.*

Then there exists a subsequence (not relabeled) and a family of probability measures $\{\nu_y\}_{y\in Q}$ on R^M such that:

For any Carathéodory function $\Phi = \Phi(y, Z)$, $y \in Q$, $Z \in R^M$ such that

$$\Phi(\cdot, \mathbf{v}_\varepsilon) \to \overline{\Phi} \text{ weakly in } L^1(Q)$$

we have

$$\overline{\Phi}(y) = \int_{R^M} \Phi(y, Z) \, \mathrm{d}\nu_y(Z) \text{ for a.a. } y \in Q.$$

Taking

$$\varrho \mapsto \varrho s(\varrho, \vartheta(t, x)) \text{ as a Caratheodory function of the argument } \varrho$$

we deduce from (6.14) the desired conclusion

$$\overline{\varrho s(\varrho, \vartheta(t, x))\vartheta} = \overline{\varrho s(\varrho, \vartheta(t, x))}\vartheta$$

yielding (6.13). Thus we have shown (6.11).

6.4 Pointwise (a.a.) convergence of the densities

Our goal is to show that

$$\varrho_\varepsilon \to \varrho \text{ a.a. in } (0, T) \times \Omega. \tag{6.15}$$

We start by introducing a linear (zero-th) order operator

$$\mathcal{R}_{i,j} = \partial_{x_i} \Delta^{-1} \partial_{x_j},$$

or, in terms of Fourier mutlipliers,

$$\mathcal{R}_{i,j}[v] = \mathcal{F}_{\xi \to x} \left[\frac{\xi_i \xi_j}{|\xi|^2} \S \to \xi[v] \right].$$

6.4.1 Weak continuity of the effective viscous pressure

We start with a celebrated result of Lions [5] on the "weak continuity" of the quantity

$$p(\varrho, \vartheta) - \left(\frac{4}{3} \mu(\vartheta) + \eta(\vartheta) \right) \mathrm{div}_x \mathbf{u}$$

termed *effective viscous pressure*, see Lions [5].

We introduce

$$\varphi_\varepsilon = \phi \nabla_x \Delta^{-1} [T_k(\varrho_\varepsilon)], \ \ \varrho_\varepsilon \equiv 0 \text{ outside } \Omega, \ \ \phi \in C_c^\infty(\Omega),$$

where T_k are the cut-off functions defined through (6.7), to be used as test functions in the momentum equation (3.6). Similarly to (5.40), we obtain

$$\int_{\tau_1}^{\tau_2} \int_\Omega \phi p(\varrho_\varepsilon, \vartheta_\varepsilon) T_k(\varrho_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t - \int_{\tau_1}^{\tau_2} \int_\Omega \phi \mathbb{S}_\varepsilon : \mathcal{R}[T_k(\varrho_\varepsilon)] \ \mathrm{d}x \ \mathrm{d}t \tag{6.16}$$

$$= - \int_{\tau_1}^{\tau_2} \int_\Omega p(\varrho_\varepsilon, \vartheta_\varepsilon) \nabla_x \phi \cdot \nabla_x \Delta^{-1} [T_k(\varrho_\varepsilon)] \ \mathrm{d}x \ \mathrm{d}t + \int_{\tau_1}^{\tau_2} \int_\Omega \phi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \nabla_x \Delta^{-1} [T_k(\varrho_\varepsilon)] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_{\tau_1}^{\tau_2} \int_\Omega \phi (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \mathcal{R}[T_k(\varrho_\varepsilon)] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_{\tau_1}^{\tau_2} \int_\Omega \Big[\Big(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + \mathbb{S}_\varepsilon \Big) \cdot \nabla_x \phi \cdot \nabla_x \Delta^{-1} [T_k(\varrho_\varepsilon)] \Big) - \phi \varrho_\varepsilon \nabla_x F \cdot \nabla_x \Delta^{-1} [T_k(\varrho_\varepsilon)] \Big] \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_\Omega \Big[\phi \varrho_\varepsilon \mathbf{u}_\varepsilon \nabla_x \Delta^{-1} [T_k(\varrho_\varepsilon)](\tau_2, \cdot) - \phi \varrho_\varepsilon \mathbf{u}_\varepsilon \nabla_x \Delta^{-1} [T_k(\varrho_\varepsilon)](\tau_1, \cdot) \Big] \ \mathrm{d}x,$$

where we have set

$$\mathbb{S}_\varepsilon = \mu(\vartheta_\varepsilon) \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right) + \eta(\vartheta_\varepsilon) \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I}.$$

We remark that, in accordance with (6.10),

$$\mathbb{S}_\varepsilon \to \mathbb{S} \equiv \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I} \text{ weakly in } L^r((0,T) \times \Omega; R^3)$$

for a certain $r > 1$.

Now, we can let $\varepsilon \to 0$ in (3.6) and take

$$\varphi = \phi \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}]$$

as a test function in the resulting integral identity to obtain

$$\int_{\tau_1}^{\tau_2} \int_\Omega \phi \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\ {\rm d}x\ {\rm d}t - \int_{\tau_1}^{\tau_2} \int_\Omega \phi \mathbb{S} : \mathcal{R}[\overline{T_k(\varrho)}]\ {\rm d}x\ {\rm d}t \tag{6.17}$$

$$= - \int_{\tau_1}^{\tau_2} \int_\Omega \overline{p(\varrho,\vartheta)} \nabla_x \phi \cdot \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}]\ {\rm d}x\ {\rm d}t + \int_{\tau_1}^{\tau_2} \int_\Omega \phi \varrho \mathbf{u} \cdot \partial_t \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}]\ {\rm d}x\ {\rm d}t$$

$$+ \int_{\tau_1}^{\tau_2} \int_\Omega \phi (\varrho \mathbf{u} \otimes \mathbf{u}) : \mathcal{R}[\overline{T_k(\varrho)}]\ {\rm d}x\ {\rm d}t$$

$$+ \int_{\tau_1}^{\tau_2} \int_\Omega \Big[\Big(\varrho \mathbf{u} \otimes \mathbf{u} + \mathbb{S} \Big) \cdot \nabla_x \phi \cdot \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}] \Big) - \phi \varrho \nabla_x F \cdot \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}] \Big]\ {\rm d}x\ {\rm d}t$$

$$- \int_\Omega \Big[\phi \varrho \mathbf{u} \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}](\tau_2, \cdot) - \phi \varrho \mathbf{u} \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}](\tau_1, \cdot) \Big]\ {\rm d}x.$$

At this stage, we use the renormalized equation (5.41) to compute

$$\partial_t \nabla_x \Delta^{-1}[T_k(\varrho_\varepsilon)] = - \nabla_x \Delta^{-1}[{\rm div}_x (T_k(\varrho_\varepsilon) \mathbf{u}_\varepsilon)] + \nabla_x \Delta^{-1} \left[(T_k(\varrho_\varepsilon) - T_k'(\varrho_\varepsilon) \varrho_\varepsilon)\, {\rm div}_x \mathbf{u}_\varepsilon \right]$$

and, similarly,

$$\partial_t \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}] = - \nabla_x \Delta^{-1}[{\rm div}_x (\overline{T_k(\varrho)} \mathbf{u}_\varepsilon)] + \nabla_x \Delta^{-1} \overline{[(T_k(\varrho) - T_k'(\varrho) \varrho)\, {\rm div}_x \mathbf{u}}].$$

Seeing that, by virtue of (5.38), (5.39),

$$\nabla_x \Delta^{-1}[T_k(\varrho_\varepsilon)] \to \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}] \text{ in } C([0,T] \times \overline{\Omega}),$$

we can let $\varepsilon \to 0$ in (6.16) and compare the resulting expression with (6.17) to conclude

$$\lim_{\varepsilon \to 0} \int_{\tau_1}^{\tau_2} \int_\Omega \phi p(\varrho_\varepsilon, \vartheta_\varepsilon) T_k(\varrho_\varepsilon)\ {\rm d}x\ {\rm d}t - \lim_{\varepsilon \to 0} \int_{\tau_1}^{\tau_2} \int_\Omega \phi \mathbb{S}_\varepsilon : \mathcal{R}[T_k(\varrho_\varepsilon)]\ {\rm d}x\ {\rm d}t \tag{6.18}$$

$$= \int_{\tau_1}^{\tau_2} \int_\Omega \phi \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\ {\rm d}x\ {\rm d}t - \int_{\tau_1}^{\tau_2} \int_\Omega \phi \mathbb{S} : \mathcal{R}[\overline{T_k(\varrho)}]\ {\rm d}x\ {\rm d}t$$

$$+ \lim_{\varepsilon \to 0} \int_{\tau_1}^{\tau_2} \int_\Omega \phi (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \mathcal{R}[T_k(\varrho_\varepsilon)]\ {\rm d}x\ {\rm d}t - \lim_{\varepsilon \to 0} \int_{\tau_1}^{\tau_2} \int_\Omega \phi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1}[{\rm div}_x (T_k(\varrho_\varepsilon) \mathbf{u}_\varepsilon)]\ {\rm d}x$$

$$- \int_{\tau_1}^{\tau_2} \int_\Omega \phi (\varrho \mathbf{u} \otimes \mathbf{u}) : \mathcal{R}[\overline{T_k(\varrho)}]\ {\rm d}x\ {\rm d}t + \int_{\tau_1}^{\tau_2} \int_\Omega \phi \varrho \mathbf{u} \cdot \nabla_x \Delta^{-1}[{\rm div}_x (\overline{T_k(\varrho)} \mathbf{u})]\ {\rm d}x.$$

Rewriting

$$\int_\Omega \Big(\phi(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \mathcal{R}[T_k(\varrho_\varepsilon)] - \phi\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1}[\mathrm{div}_x(T_k(\varrho_\varepsilon)\mathbf{u}_\varepsilon)]\Big)\ \mathrm{d}x$$

$$= \int_\Omega \sum_{i,j=1}^{3} \Big(\phi\varrho_\varepsilon u^i_\varepsilon u^j_\varepsilon : \partial_{x_i}\Delta^{-1}\partial_{x_j}[T_k(\varrho_\varepsilon)] - \phi\varrho_\varepsilon u^i_\varepsilon \partial_{x_i}\Delta^{-1}\partial_{x_j}(T_k(\varrho_\varepsilon)u^j_\varepsilon)\Big)\ \mathrm{d}x$$

$$= \int_\Omega \sum_{i,j=1}^{3} \Big(\phi\varrho_\varepsilon u^i_\varepsilon u^j_\varepsilon : \partial_{x_i}\Delta^{-1}\partial_{x_j}[T_k(\varrho_\varepsilon)] - \partial_{x_j}\Delta^{-1}\partial_{x_i}[\phi\varrho_\varepsilon u^i_\varepsilon]T_k(\varrho_\varepsilon)u^j_\varepsilon),\Big)\ \mathrm{d}x$$

we examine the bilinear form

$$[\mathbf{v},\mathbf{w}] = \sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{i,j}[w^j] - w^i \mathcal{R}_{i,j}[v^j]\Big),$$

where we may write

$$\sum_{i,j=1}^{3} \Big(v^i \mathcal{R}_{i,j}[w^j] - w^i \mathcal{R}_{i,j}[v^j]\Big)$$

$$= \sum_{i,j=1}^{3} \Big((v^i - \mathcal{R}_{i,j}[v^j])\mathcal{R}_{i,j}[w^j] - (w^i - \mathcal{R}_{i,j}[w^j])\mathcal{R}_{i,j}[v^j]\Big)$$

$$= \mathbf{U}\cdot\mathbf{V} - \mathbf{W}\cdot\mathbf{Z},$$

where

$$U^i = \sum_{j=1}(v^i - \mathcal{R}_{i,j}[v^j]),\ W^i = \sum_{j=1}(w^i - \mathcal{R}_{i,j}[w^j]),\ \mathrm{div}_x\mathbf{U} = \mathrm{div}_x\mathbf{W} = 0,$$

and

$$V^i = \partial_{x_i}\left(\sum_{j=1}^{3}\Delta^{-1}\partial_{x^j}w^j\right),\ Z^i = \partial_{x_i}\left(\sum_{j=1}^{3}\Delta^{-1}\partial_{x^j}v^j\right),\ i = 1,2,3.$$

Thus a direct application of Div-Curl lemma (Lemma 6.1) yields

$$[\mathbf{v}_\varepsilon,\mathbf{w}_\varepsilon] \to [\mathbf{v},\mathbf{w}] \text{ weakly in } L^s(R^3)$$

$$\text{whenever } \mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } L^p(R^3;r^3),\ \mathbf{w}_\varepsilon \to \mathbf{w} \text{ weakly in } L^q(R^3;R^3),$$

and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Seeing that

$$T_k(\varrho_\varepsilon) \to \overline{T_k(\varrho)} \text{ in } C_{\rm weak}([0,T]; L^q(\Omega)) \text{ for any } 1 \leq q < \infty,\ \varrho_\varepsilon \mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } C_{\rm weak}([0,T]; L^{5/4}(\Omega))$$

we conclude that

$$T_k(\varrho_\varepsilon(t,\cdot))\nabla_x\Delta^{-1}[{\rm div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon)(t,\cdot)] - (\varrho_\varepsilon\mathbf{u}_\varepsilon)(t,\cdot)\cdot\nabla_x\Delta^{-1}\nabla_x[T_k(\varrho_\varepsilon(t,\cdot))] \tag{6.19}$$

$$\to$$

$$\overline{T_k(\varrho)}(t,\cdot)\nabla_x\Delta^{-1}[{\rm div}_x(\varrho\mathbf{u})(t,\cdot)] - (\varrho\mathbf{u})(t,\cdot)\cdot\nabla_x\Delta^{-1}\nabla_x[\overline{T_k(\varrho)}(t,\cdot)]$$
$$\text{weakly in } L^s(\Omega;R^3) \text{ for all } t\in[0,T],\ 1\leq s<5/4.$$

Thus we conclude that the convergence in (6.19) takes place in the space

$$L^q(0,T;W^{-1,2}(\Omega;R^3)) \text{ for any } 1\leq q<\infty;$$

whence, going back to (6.18) we may infer that

Effective viscous flux weak continuity:

$$\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{\mathbb{S}:\mathcal{R}[T_k(\varrho)]} = \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} - \mathbb{S}:\mathcal{R}[\overline{T_k(\varrho)}] \text{ for any } k\geq 1. \tag{6.20}$$

6.4.2 Commutator estimates

Our aim is to show the commutator estimates for the vector field

$$Z_i = \sum_{j=1}^3 \partial_{x_i}\Delta^{-1}\partial_{x_j}[\nu Y_j] - \nu\partial_{x_i}\Delta^{-1}\sum_{j=1}^3\partial_{x_j}Y_j,\ i=1,2,3.$$

where $Y=[Y_1,Y_2,Y_3]$ is a vector field in $C_c^\infty(R^3;R^3)$ and $\nu\in C_c^\infty(R^3)$. Computing

$${\rm div}_x\mathbf{Z} = \nabla_x\nu\cdot\Big[\mathbf{Y} - \nabla_x\Delta^{-1}[{\rm div}_x\mathbf{Y}]\Big],$$

$$\mathbf{curl}\ \mathbf{Z} = \Big\{\partial_{x_i}\nu\partial_{x_j}\Delta^{-1}[{\rm div}_x\mathbf{Y}]\Big\}_{i,j=1}^3,$$

we may use the bound

$$\|\nabla_x\mathbf{Z}\|_{L^q(R^3;R^{3\times 3})} \leq c\Big(\|{\rm div}_x\mathbf{Z}\|_{L^q(R^3)} + \|\mathbf{curl}\ \mathbf{Z}\|_{L^q(R^3;R^{3\times3})}\Big),\ 1<q<\infty$$

to deduce that

$$\|\nabla_x \mathbf{Z}\|_{L^q(R^3;R^{3\times 3})} \leq c\|\nabla_x \nu\|_{L^p(R^3;R^3)}\|\mathbf{Y}\|_{L^{r_1}(R^3;R^3)}, \ \frac{1}{p}+\frac{1}{r_1}=\frac{1}{q}\in(0,1). \tag{6.21}$$

Moreover, by means of the standard interpolation argument,

$$\|\mathbf{Z}\|_{L^q(R^3;R^3)} \leq c\|\nabla_x \nu\|_{L^p(R^3;R^3)}\|\mathbf{Y}\|_{L^{r_2}(R^3;R^3)}, \ \frac{3-p}{3p}+\frac{1}{r_2}=\frac{1}{q}\in(0,1), \ 1\leq p<3. \tag{6.22}$$

Finally, interpolating (6.21), (6.22) we obtain the following result:

Commutator lemma:

Lemma 6.2 *Let $\nu \in C_c^\infty(R^3)$, $\mathbf{Y}\in C_c^\infty(R^3;R^3)$ be given fields, and let*

$$1<p<3, \ \frac{1}{q}-\frac{1}{p}<\frac{1}{r}<\frac{1}{q}-\frac{3-p}{3p}, \ q\in(0,1),$$

and

$$\mathbf{Z}=\nabla_x\Delta^{-1}\mathrm{div}_x[\nu\mathbf{Y}]-\nu\nabla_x\Delta^{-1}\mathrm{div}_x[\mathbf{Y}].$$

Then

$$\|\mathbf{Z}\|_{D^{\alpha,q}(R^3)}\leq c\|\nu\|_{D^{1,p}(R^3)}\|\mathbf{U}\|_{L^r(R^3;R^3)} \text{ for a certain } \alpha>0.$$

Now, if $\{Z_\varepsilon\}_{\varepsilon>0}$ is a sequence of functions bounded in $L^1(0,T;D^{\alpha,q}(R^3))\cap L^r((0,T)\times\Omega)$ for certain $\alpha>0$, $q,r>1$, we may use compactness of the embedding

$$D^{\alpha,q}(\Omega)\hookrightarrow\hookrightarrow L^q(\Omega),$$

and the convergence

$$T_k(\varrho_\varepsilon)\to\overline{T_k(\varrho)} \text{ in } C_{\rm weak}([0,T;L^p(\Omega)) \text{ for any } 1\leq p<\infty$$

to deduce that

$$\overline{ZT_k(\varrho)}=\overline{Z}\ \overline{T_k(\varrho)}.$$

Applying Lemma 6.2 to the fields

$$\nu=\mu(\vartheta_\varepsilon),\eta(\vartheta_\varepsilon), \ \mathbf{Y}=\nabla_x u^i, \ i=1,2,3, \ p=r=2,$$

and noting that

$$\sum_{i,j=1}^{3} \partial_{x_i} \Delta^{-1} \partial_{x_j} \frac{\partial u^i}{\partial x_j} = \mathrm{div}_x \mathbf{u},$$

we conclude that relation (6.20) gives rise to

EFFECTIVE VISCOUS FLUX WEAK CONTINUITY REVISITED:

$$\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} = \left(\mu(\vartheta) + \frac{4}{3}\eta(\vartheta)\right)\left(\overline{\mathrm{div}_x \mathbf{u}[T_k(\varrho)]} - \mathrm{div}_x \mathbf{u}\overline{T_k(\varrho)}\right) \text{ for any } k \geq 1. \tag{6.23}$$

6.4.3 Density oscillations

A suitable tool for describing the density oscillations is the renormalized equation (5.41). In particular, we get

$$\int_\Omega \varrho_\varepsilon L_k(\varrho_\varepsilon)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega T_k(\varrho_\varepsilon)\mathrm{div}_x \mathbf{u}_\varepsilon\ \mathrm{d}x\ \mathrm{d}t = \int_\Omega \varrho_{0,\varepsilon} L_k(\varrho_{0,\varepsilon})\ \mathrm{d}x \text{ for any } \tau \geq 0 \text{ and for any } k \geq 1, \tag{6.24}$$

where

$$L_k(\varrho) = \int_1^\varrho \frac{T_k(z)}{z^2}\ \mathrm{d}z.$$

Now, seeing that

$$\varrho L_k(\varrho) \to \overline{\varrho L_k(\varrho)} \text{ in } C_{\mathrm{weak}}([0,T]; L^{5/3}(\Omega)),$$

we may let $\varepsilon \to 0$ in (6.24) to deduce that

$$\int_\Omega \overline{\varrho L_k(\varrho)}(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}}\ \mathrm{d}x\ \mathrm{d}t = \int_\Omega \varrho_0 L_k(\varrho_0)\ \mathrm{d}x \text{ for any } k \geq 1. \tag{6.25}$$

since we have assumed strong (pointwise a.a.) convergence of the initial data.

Suppose, for a while, that the limit ϱ, $\mathbf{u}$ also satisfies the renormalized equation - a statement far from obvious. Then we would get, by the same argument,

$$\int_\Omega \varrho L_k(\varrho)(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega T_k(\varrho)\mathrm{div}_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t = \int_\Omega \varrho_0 L_k(\varrho_0)\ \mathrm{d}x \text{ for any } k \geq 1;$$

whence

$$\int_\Omega \left[\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\right](\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \tag{6.26}$$

$$= \int_0^\tau \int_\Omega \Big(T_k(\varrho) - \overline{T_k(\varrho)} \Big) \operatorname{div}_x \mathbf{u} \ {\rm d}x \ {\rm d}t \text{ for any } k \geq 1.$$

Now, we claim that

$$\overline{T_k(\varrho)\operatorname{div}_x \mathbf{u}} - \overline{T_k(\varrho)}\operatorname{div}_x \mathbf{u} \geq 0$$

as a direct consequence of (6.23). Indeed, as the pressure p is a non-decreasing function of the density, we may write

$$\Big(p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(T_k^{-1}(\overline{T_k(\varrho)}); \vartheta_\varepsilon) \Big) \Big(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \Big)$$

$$= \Big(p(T_k^{-1}(T_k(\varrho_\varepsilon))\varrho_\varepsilon, \vartheta_\varepsilon) - p(T_k^{-1}(\overline{T_k(\varrho)}); \vartheta_\varepsilon) \Big) \Big(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \Big),$$

where, by means of convexity of T_k^{-1},

$$T_k^{-1}(\overline{T_k(\varrho)}) \leq \varrho.$$

Letting $\varepsilon \to 0$ and using strong convergence of $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ we deduce the desired conclusion

$$\overline{p(\varrho, \vartheta) T_k(\varrho)} - \overline{p(\varrho, \vartheta)} \ \overline{T_k(\varrho)}.$$

Thus relation (6.26) gives rise to

$$\int_\Omega \Big[\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho) \Big] (\tau, \cdot) \ {\rm d}x \leq \int_0^\tau \int_\Omega \Big(T_k(\varrho) - \overline{T_k(\varrho)} \Big) \operatorname{div}_x \mathbf{u} \ {\rm d}x \ {\rm d}t \text{ for any } k \geq 1. \tag{6.27}$$

Finally, assume that we can show

$$\int_0^\tau \int_\Omega \Big(T_k(\varrho) - \overline{T_k(\varrho)} \Big) \operatorname{div}_x \mathbf{u} \ {\rm d}x \ {\rm d}t \to 0 \text{ as } k \to \infty. \tag{6.28}$$

Consequently, letting $k \to \infty$ in (6.27) we would obtain that

$$\overline{\varrho \log(\varrho)}(\tau, \cdot) = \varrho \log(\varrho)(\tau, \cdot) \text{ for any } \tau \geq 0. \tag{6.29}$$

Relation (6.29) implies the desired strong (a.a. pointwise) convergence of the sequence $\{\varrho_\varepsilon\}_{\varepsilon>0}$. In order to see that, we need the following result.

6.4.4 Exercise - convexity and strong convergence

Lemma 6.3 *Let $Q \subset R^M$ be a bounded domain and $\{\varrho_\varepsilon\}_{\varepsilon>0}$ a family of non-negative functions such that*

$$\varrho_\varepsilon \to \varrho \text{ weakly in } L^p(Q),$$

and

$$\varrho_\varepsilon \log(\varrho_\varepsilon) \to \varrho \log(\varrho) \text{ weakly in } L^p(Q)$$

for some $p > 1$.

Then

$$\varrho_\varepsilon \to \varrho \text{ (strongly) in } L^1(Q).$$

In this section, we have "almost" proved the strong convergence of the densities, however, two fundamental questions were left open:

- validity of the renormalized equation of continuity for the limit functions ϱ, $\mathbf{u}$;
- relation (6.28).

These issues will be addressed in the following section.

6.4.5 Oscillation defect measure

Let $\varrho_\varepsilon \to \varrho$ be a weakly converging sequence in $L^1(Q)$. We define

Oscillation defect measure:

$$\mathbf{osc}_q[\varrho_\varepsilon \to \varrho](Q) = \sup_{k\geq 1} \left(\limsup_{\varepsilon\to 0} \int_Q |T_k(\varrho_\varepsilon) - T_k(\varrho)|^q \ \mathrm{d}x \ \mathrm{d}t \right). \tag{6.30}$$

Suppose that we can show

$$\mathbf{osc}_q[\varrho_\varepsilon \to \varrho](Q) = \sup_{k\geq 1} \left(\limsup_{\varepsilon\to 0} \int_Q |T_k(\varrho_\varepsilon) - T_k(\varrho)|^q \ \mathrm{d}x \ \mathrm{d}t \right) < \infty \text{ for } Q = (0,T)\times\Omega \text{ and some } q > 2. \tag{6.31}$$

We claim that (6.31) yields the relation (6.28) in the preceding section. Indeed we may write

$$\left| \int_0^\tau \int_\Omega \left(\overline{T_k(\varrho)} - T_k(\varrho) \right) \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \right|$$

$$\leq \|\mathrm{div}_x \mathbf{u}\|_{L^2((0,T)\times\Omega)} \left\| \overline{T_k(\varrho)} - T_k(\varrho) \right\|^{\lambda}_{L^1((0,T)\times\Omega)} + \left\| \overline{T_k(\varrho)} - T_k(\varrho) \right\|^{1-\lambda}_{L^q((0,T)\times\Omega)} \ \text{ for a certain } \lambda > 0;$$

whence, since

$$\left\| \overline{T_k(\varrho)} - T_k(\varrho) \right\|_{L^1((0,T)\times\Omega)} \to 0 \text{ as } k \to \infty,$$

and

$$\left\| \overline{T_k(\varrho)} - T_k(\varrho) \right\|_{L^q((0,T)\times\Omega)} \leq \liminf_{\varepsilon \to 0} \|T_k(\varrho_\varepsilon) - T_k(\varrho)\|_{L^q((0,T)\times\Omega)},$$

relation (6.31) yields the desired conclusion.

Furthermore, we have the following result:

Proposition 6.1 *Suppose that ϱ_ε, $\mathbf{u}_\varepsilon$ are weak renormalized solutions to the equation of continuity in a space-time cylinder $(0,T) \times \Omega$ such that*

$$\varrho_\varepsilon \to \varrho \ \textit{weakly-(*) in} \ L^\infty(0,T; L^\gamma(\Omega)), \ \gamma > \frac{6}{5},$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \ \textit{weakly in} \ L^2(0,T; W_0^{1,2}(\Omega; R^3)),$$

and

$$\mathbf{osc}_q[\varrho_\varepsilon \to \varrho]((0,T)\times\Omega) < \infty \ \textit{for some} \ q > 2. \tag{6.32}$$

Then the limit functions ϱ, $\mathbf{u}$ represent a renormalized solution of the equation of continuity.

Proof:

Extending ϱ_ε, $\mathbf{u}_\varepsilon$ to be zero outside Ω we may replace $\Omega \approx R^3$. Since ϱ_ε, $\mathbf{u}_\varepsilon$ are renormalized solutions we get

$$\partial_t T_k(\varrho_\varepsilon) + \mathrm{div}_x(T_k(\varrho_\varepsilon)\mathbf{u}_\varepsilon) + \Big(T_k'(\varrho_\varepsilon)\varrho_\varepsilon - T_k(\varrho_\varepsilon) \Big) \mathrm{div}_x \mathbf{u}_\varepsilon = 0$$

in the weak sense. The limit for $\varepsilon \to 0$ therefore reads

$$\partial_t \overline{T_k(\varrho)} + \mathrm{div}_x(\overline{T_k(\varrho)}\mathbf{u}) + \overline{\Big(T_k'(\varrho)\varrho - T_k(\varrho) \Big) \mathrm{div}_x \mathbf{u}} = 0 \tag{6.33}$$

Now, consider a convolution of equation (6.33) with a family $\{\omega_\delta\}_{\delta>0}$ of regularizing kernels in the x-variable. Consequently, denoting $[v]_\delta = \omega_\delta * v$ we obtain

$$\partial_t \left[\overline{T_k(\varrho)}\right]_\delta + \mathrm{div}_x(\left[\overline{T_k(\varrho)}\right]_\delta \mathbf{u}) + \left[\overline{\left(T_k'(\varrho)\varrho - T_k(\varrho)\right)\mathrm{div}_x \mathbf{u}}\right]_\delta = \mathrm{div}_x(\left[\overline{T_k(\varrho)}\right]_\delta \mathbf{u}) - \left[\mathrm{div}_x(\overline{T_k(\varrho)}\mathbf{u})\right]_\delta , \tag{6.34}$$

where, in contrast with (6.33), the equation is satisfied a.a. pointwise. Since

$$\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^3)) \text{ and } \overline{T_k(\varrho)} \in L^\infty((0,T)\times\Omega),$$

it is possible to show that

$$\mathrm{div}_x(\left[\overline{T_k(\varrho)}\right]_\delta \mathbf{u}) - \left[\mathrm{div}_x(\overline{T_k(\varrho)}\mathbf{u})\right]_\delta \to 0 \text{ in } L^2((0,T)\times\Omega) \text{ as } \delta \to 0.$$

Consequently, multiplying (6.34) by $b'(\overline{T_k(\varrho)})$ and letting $\delta \to 0$ we may infer that

$$\partial_t b\left(\overline{T_k(\varrho)}\right) + \mathrm{div}_x\left(b\left(\overline{T_k(\varrho)}\right)\mathbf{u}\right) + \left(b'\left(\overline{T_k(\varrho)}\right)\overline{T_k(\varrho)} - b\left(\overline{T_k(\varrho)}\right)\right)\mathrm{div}_x \mathbf{u} \tag{6.35}$$

$$= b'\left(\overline{T_k(\varrho)}\right)\left[\overline{\left(T_k(\varrho) - T_k'(\varrho)\varrho\right)\mathrm{div}_x \mathbf{u}}\right]$$

in the weak sense, where b can be taken such that $b'(z) \equiv 0$ as soon as $z \geq M$.

Thus our ultimate goal will be to show that the expression on the right-hand side of (6.35) vanishes as $k \to \infty$, which yields the desired renormalized equation. To this end, we denote

$$Q_{k,M} = \{(t,x) \in (0,T)\times\Omega \mid \overline{T_k(\varrho)} \leq M\}.$$

We have

$$\left\| b'\left(\overline{T_k(\varrho)}\right)\left[\overline{\left(T_k(\varrho) - T_k'(\varrho)\varrho\right)\mathrm{div}_x \mathbf{u}}\right]\right\|_{L^1((0,T)\times\Omega)}$$

$$\leq \left(\sup_{z\in[0,M]} |b'(z)|\right)\left(\sup_{\varepsilon>0} \|\mathrm{div}_x \mathbf{u}_\varepsilon\|_{L^2((0,T)\times\Omega)}\right) \liminf_{\varepsilon\to 0} \|T_k(\varrho_\varepsilon) - T_k'(\varrho_\varepsilon)\varrho_\varepsilon\|_{L^2(Q_{k,M})} ,$$

where, by interpolation,

$$\|T_k(\varrho_\varepsilon) - T_k'(\varrho_\varepsilon)\varrho_\varepsilon\|_{L^2(Q_{k,M})}$$

$$\leq \|T_k(\varrho_\varepsilon) - T_k'(\varrho_\varepsilon)\varrho_\varepsilon\|^\beta_{L^1(Q_{k,M})} \|T_k(\varrho_\varepsilon) - T_k'(\varrho_\varepsilon)\varrho_\varepsilon\|^{1-\beta}_{L^q(Q_{k,M})} ,\ \frac{1}{2} = \beta + \frac{(1-\beta)}{q}.$$

Since the family of densities $\{\varrho_\varepsilon\}_{\varepsilon>0}$ is equi-integrable, we deduce that

$$\|T_k(\varrho_\varepsilon) - T_k'(\varrho_\varepsilon)\varrho_\varepsilon\|_{L^1((0,T)\times\Omega)} \to 0 \text{ as } k \to \infty \text{ uniformly in } \varepsilon.$$

Thus the proof reduces to showing that the expression

$$\|T_k(\varrho_\varepsilon) - T_k'(\varrho_\varepsilon)\varrho_\varepsilon\|_{L^q(Q_{k,M})}$$

is bounded uniformly with respect to $k \to \infty$ and $\varepsilon > 0$. Seeing that

$$|T_k'(\varrho_\varepsilon)\varrho| \leq T_k(\varrho_\varepsilon)$$

we get

$$\|T_k(\varrho_\varepsilon) - T_k'(\varrho_\varepsilon)\varrho_\varepsilon\|_{L^q(Q_{k,M})} \leq 2\, \|T_k(\varrho_\varepsilon)\|_{L^q(Q_{k,M})}$$

$$\leq 2\left(\|T_k(\varrho_\varepsilon) - T_k(\varrho)\|_{L^q((0,T)\times\Omega)} + \left\|T_k(\varrho) - \overline{T_k(\varrho)}\right\|_{L^q((0,T)\times\Omega)} + \left\|\overline{T_k(\varrho)}\right\|_{L^q(Q_{k,M})}\right)$$

$$\leq 4\mathbf{osc}_q[\varrho_\varepsilon \to \varrho]((0,T)\times\Omega) + 2Mc(|(0,T)\times\Omega|).$$

Q.E.D.

Consequently, the strong (a.a. pointwise) convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$ will follow from the arguments presented in this part as soon as we establish (6.30). This goal will be accomplished in the following section.

6.4.6 Boundedness of the oscillations defect measure

Our goal is to establish (6.30). The basic idea is to use the effective viscous flux identity (6.23). To begin, observe that the pressure p can be written in the form

$$p(\varrho,\vartheta) = b\varrho^{5/3} + p_m(\varrho,\vartheta),\ \ b > 0,$$

where the mapping $\varrho \mapsto p_m(\varrho,\vartheta)$ is non-decreasing for any fixed ϑ. Similarly to Section 6.4.3, we therefore obtain that

$$\overline{\varrho^{5/3}T_k(\varrho)} - \overline{\varrho^{5/3}}\ \overline{T_k(\varrho)} \leq \frac{1}{b}\left(\mu(\vartheta) + \frac{4}{3}\eta(\vartheta)\right)\left(\overline{\mathrm{div}_x\mathbf{u}[T_k(\varrho)]} - \mathrm{div}_x\mathbf{u}\overline{T_k(\varrho)}\right) \text{ for any } k \geq 1. \tag{6.36}$$

Next, as $\varrho \mapsto \varrho^{5/3}$ is convex and $\varrho \mapsto T_k(\varrho)$ concave,

$$\limsup_{\varepsilon\to 0} \int_0^T \int_\Omega \varphi |T_k(\varrho_\varepsilon) - T_k(\varrho)|^{8/3}\ \mathrm{d}x\ \mathrm{d}t \tag{6.37}$$

$$\limsup_{\varepsilon\to 0} \int_0^T \int_\Omega \varphi\left(\varrho_\varepsilon^{5/3} - \varrho^{5/3}\right)\left(T_k(\varrho_\varepsilon) - T_k(\varrho)\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$\leq \int_0^T \int_\Omega \varphi \left(\overline{\varrho^{5/3} T_k(\varrho)} - \overline{\varrho^{5/3}}\ \overline{T_k(\varrho)} \right) \ \mathrm{d}x\ \mathrm{d}t \text{ for any } \varphi \in C_c^\infty((0,T) \times \Omega).$$

Comparing (6.36), (6.37) we get

$$\int_0^T \int_\Omega \frac{1}{1+\vartheta} \overline{G_k(t,x,\varrho)} \ \mathrm{d}x\ \mathrm{d}t \leq c \left(\sup_{\varepsilon > 0} \|\mathrm{div}_x \mathbf{u}_\varepsilon\|_{L^2((0,T)\times\Omega} \limsup_{\varepsilon \to 0} \|T_k(\varrho_\varepsilon) - T_k(\varrho)\|_{L^2((0,T)\times\Omega)} \right), \tag{6.38}$$

where we have set

$$G_k(t,x,z) = |T_k(z) - T_k(\varrho(t,x))|^{8/3}.$$

Finally, for

$$2 < q < \frac{8}{3},$$

we get

$$\int_0^T \int_\Omega |T_k(\varrho_\varepsilon) - T_k(\varrho)|^q \ \mathrm{d}x\ \mathrm{d}t \tag{6.39}$$

$$= \int_0^T \int_\Omega (1+\vartheta)^\beta (1+\vartheta)^{-\beta} |T_k(\varrho_\varepsilon) - T_k(\varrho)|^q \ \mathrm{d}x\ \mathrm{d}t$$

$$\leq \int_0^T \int_\Omega (1+\vartheta)^{-1} |T_k(\varrho_\varepsilon) - T_k(\varrho)|^{8/3} \ \mathrm{d}x\ \mathrm{d}t + \int_0^T \int_\Omega (1+\vartheta)^{3q/(8-3q)} \ \mathrm{d}x\ \mathrm{d}t,\ \beta = \frac{3q}{8}.$$

Making use of the uniform bound (5.34), we can take

$$q = \frac{32}{15} > 2$$

and combine (6.38), (6.39) to obtain (6.30). We conclude with

$$\varrho_\varepsilon \to \varrho \text{ a.a. in } (0,T) \times \Omega. \tag{6.40}$$

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