Mathematical theory of complete fluid systems

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

based on joint work with
Antonín Novotný, Hana Petzeltová and Dalibor Pražák

Nonlinear Evolution Equations and Applications
Hammamet, March, 2012

GOAL: *Mathematical analysis and modeling of complete fluid systems*

- Fluid system conserves total mass and energy
- Fluid motion is dissipative, mechanical energy is irreversibly converted into heat
- Fluid system occupies an energetically insulated (bounded) physical space

Main topics

- Existence theory of global-in-time weak solutions for the complete Navier-Stokes-Fourier system
- Long-time behavior, stabilization to equilibria
- Singular limits, in particular, the passage from compressible to incompressible fluid motion
- Propagation of acoustic waves (dispersion)

Physical space

Eulerian coordinate system

- $\Omega \subset R^3$ - bounded regular (Lipschitz) domain
  $x \in \Omega$ - reference spatial position
- $t \in I = [0, T), \ T \leq \infty$ - time

State variables

- $\varrho = \varrho(t, x)$ - mass density
- $\mathbf{u} = \mathbf{u}(t, x)$ - velocity field
- $\vartheta = \vartheta(t, x)$ - (absolute) temperature

MASS CONSERVATION

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

BALANCE OF MOMENTUM - NEWTON'S SECOND LAW

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho \mathbf{f}$$

STOKES' LAW

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}$$

$\mathbb{T}$ - Cauchy stress, $\mathbb{S}$ - viscous stress, p - pressure, $\mathbf{f}$ - external force

Energy balance

Kinetic energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + p \right) \mathbf{u} - \mathbb{S} \cdot \mathbf{u} \right)$$

$$= \boxed{p \mathrm{div}_x \mathbf{u} - \mathbb{S} : \nabla_x \mathbf{u}} + \varrho \mathbf{f} \cdot \mathbf{u}$$

Internal energy balance

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \boxed{\mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}}$$

e - (specific) internal energy, $\mathbf{q}$ - the internal energy (heat) flux

Second Law - Entropy

Thermodynamic functions

$$p = p(\varrho, \vartheta),\ \ e = e(\varrho, \vartheta)$$

$$p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u} = -\frac{p(\varrho, \vartheta)}{\varrho}\Big(\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho \Big)$$

(internal energy balance) $\Longrightarrow$

$$\varrho \left(\left[\frac{\partial e}{\partial \varrho} - \frac{p}{\varrho^2} \right] \partial_t \varrho + \left[\frac{\partial e}{\partial \varrho} - \frac{p}{\varrho^2} \right] \mathbf{u} \cdot \nabla_x \varrho \right)$$

$$+\varrho \left(\left[\frac{\partial e}{\partial \vartheta} \right] \partial_t \vartheta + \left[\frac{\partial e}{\partial \vartheta} \right] \mathbf{u} \cdot \nabla_x \vartheta \right) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u}$$

Gibbs' relation

$s = s(\varrho, \vartheta)$ - (specific) entropy

$$\vartheta \frac{\partial s(\varrho,\vartheta)}{\partial \varrho} = \left[\frac{\partial e(\varrho,\vartheta)}{\partial \varrho} - \frac{p(\varrho,\vartheta)}{\varrho^2} \right], \ \vartheta \frac{\partial s(\varrho,\vartheta)}{\partial \vartheta} = \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta}$$

GIBBS' RELATION

$$\vartheta D s(\varrho,\vartheta) = D e(\varrho,\vartheta) + p(\varrho,\vartheta) D \left(\frac{1}{\varrho} \right)$$

Entropy equation

Entropy balance equation

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

Entropy production rate

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

Second Law $\Longrightarrow \ \sigma \geq 0$

Newton's law, Fourier's law

$\mathbb{S}$ linear in $\nabla_x \mathbf{u}$, $\mathbf{q}$ linear in $\nabla_x \vartheta$

Newton's law

$$\mathbb{S} = \mu \Big(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \Big) + \eta \mathrm{div}_x \mathbf{u} \ \mathbb{I}$$

μ - shear viscosity coefficient, η - bulk viscosity coefficient

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

κ - heat conductivity coefficient

Energetically closed systems

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = -\kappa \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Boundary behavior of velocity

No-slip

$$[\mathbf{u}]_{\rm tan}|_{\partial\Omega} = 0$$

No-stick

$$[\mathbb{S}\mathbf{n}]_{\rm tan}|_{\partial\Omega} = 0$$

Navier's boundary condition

$$[\mathbb{S}\mathbf{n}]_{\rm tan} + \beta[\mathbf{u}]_{\rm tan}|_{\partial\Omega} = 0$$

Total energy balance

Conservative driving force

$$\mathbf{f} = \nabla_x F, \ F = F(x)$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x$$

$$= \int_\Omega \left(\vartheta \sigma - \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \right) \ \mathrm{d}x = 0$$

ENTROPY BALANCE EQUATION

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

ENTROPY PRODUCTION RATE

$$\sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

TOTAL ENERGY BALANCE

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \, \mathrm{d}x = 0$$

weak + regular $\Longrightarrow$ strong

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

as soon as

- $$0 < \underline{\varrho} \leq \varrho(t,x) \leq \overline{\varrho},\ 0 < \underline{\vartheta} \leq \vartheta(t,x) \leq \overline{\vartheta}$$
- $$|\mathbf{u}(t,x)| \leq U$$
- $$\nabla_x \varrho \in L^2((0,T) \times \Omega; R^3)$$

Weak-strong uniqueness principle

Weak and strong solutions necessarilly coincide as long as the latter exists. Strong solutions are *unique* in the class of weak solutions.

Navier-Stokes-Fourier system - weak formulation

FIELD EQUATIONS - BALANCE LAWS

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \, \mathrm{d}x = 0$$

ENERGETICALLY INSULATING BOUNDARY CONDITIONS

Existence theory - main stumbling blocks

- Weak a priori bounds
- Absence of a priori bounds on $\nabla_x \varrho$
- Possible concentration phenomena for $\nabla_x \mathbf{u}$, $\nabla_x \vartheta$
- Possible occurrence of vacuum regions ($\varrho = 0$)

A priori bounds: Motto

Die Energie der Welt ist constant;
Die Entropie der Welt strebt einem Maximum zu

Rudolph Clausius, 1822-1888

Equilibria **minimize** the entropy production and **maximize** the entropy among all states of the same mass and energy

$$\mathbf{u}_{\rm static} \equiv 0, \ \ \vartheta_{\rm static} = \overline{\vartheta} > 0 - \text{constant}, \ \ \varrho_{\rm static} = \tilde{\varrho}(x)$$

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F \text{ in } \Omega$$

$$\int_\Omega \tilde{\varrho} \ {\rm d}x = M_0 \text{ (total mass)}$$

$$\int_\Omega \Big(\tilde{\varrho} e(\tilde{\varrho}, \overline{\vartheta}) - \tilde{\varrho} F \Big) \ {\rm d}x = E_0 \text{ (total energy)}$$

Thermodynamic stability

Positive compressibility

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$$

Positive specific heat at constant volume

$$\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

$$\liminf_{\varrho \to 0} \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0 \text{ for any fixed } \vartheta > 0$$

Ballistic free energy - relative entropy

Ballistic free energy

$$H(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta)$$

$$\frac{\partial^2 H(\varrho, \overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho} \frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho} > 0 \implies$$

- $\varrho \mapsto H(\varrho, \overline{\vartheta})$ is strictly convex

$$\frac{\partial H(\varrho, \vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta} (\vartheta - \overline{\vartheta}) \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} \implies$$

- $\vartheta \mapsto H(\varrho, \vartheta)$ attains its strict local minimum at $\overline{\vartheta}$

Relative entropy - coercivity

"Relative entropy"

$$H(\varrho,\vartheta) - \frac{\partial H(\tilde\varrho,\overline{\vartheta})}{\partial \varrho}(\varrho - \tilde\varrho) - H(\tilde\varrho,\overline{\vartheta})$$

$$\geq c(B)\Big(|\varrho - \tilde\varrho|^2 + |\vartheta - \overline{\vartheta}|^2 \Big)$$

provided ϱ, ϑ belong to a compact interval $B \subset (0,\infty)$

$$\geq c(B)\Big(1 + \varrho e(\varrho,\vartheta) + \varrho |s(\varrho,\vartheta)| \Big)$$

otherwise

---

as soon as $\tilde\varrho$, $\overline{\vartheta}$ belong to $\mathrm{int}[B]$

Principle of maximal entropy

$$\tilde{\varrho},\ \overline{\vartheta} \ \text{static state} \Rightarrow \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} = F + \text{const} \Rightarrow$$

$$0 \leq \int_{\Omega} \left(H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \right) \, \mathrm{d}x$$

$$= \int_{\Omega} \left(\boxed{\varrho e(\varrho, \vartheta) - \varrho F - \tilde{\varrho} e(\tilde{\varrho}, \overline{\vartheta}) + \tilde{\varrho} F} - \boxed{\overline{\vartheta} \varrho s(\varrho, \vartheta) + \overline{\vartheta} \tilde{\varrho} s(\tilde{\varrho}, \overline{\vartheta})} \right) \, \mathrm{d}x$$

as soon as

$$\int_{\Omega} \varrho \, \mathrm{d}x = \int_{\Omega} \tilde{\varrho} \, \mathrm{d}x$$

Principle of maximal entropy - conclusion

- Given the total mass and energy, there is a unique static state $\tilde{\varrho}$, $\overline{\vartheta}$
- The static state $\tilde{\varrho}$, $\tilde{\vartheta}$ maximizes the entropy among all admissible states ϱ, ϑ with the same total mass and energy

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho,\vartheta) - \frac{\partial H(\tilde\varrho,\overline{\vartheta})}{\partial\varrho}(\varrho-\tilde\varrho) - H(\tilde\varrho,\overline{\vartheta})\right)\ \mathrm{d}x$$

$$+\overline{\vartheta}\int_\Omega \sigma\ \mathrm{d}x = 0$$

$$\sigma \geq \frac{\mu}{\vartheta}\left|\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right|^2 + \frac{\kappa}{\vartheta^2}|\nabla_x\vartheta|^2$$

- Total dissipation balance yields *practically* all available a priori bounds
- Effective dependence of the transport coefficients μ, η, and κ on the absolute temperature ϑ is needed

Relative entropy revisited

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right)$$

$$= \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + H_{\tilde{\vartheta}}(\varrho, \vartheta) - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta}) \right) \, \mathrm{d}x$$

$$H_{\tilde{\vartheta}}(\varrho, \vartheta) \equiv \varrho e(\varrho, \vartheta) - \tilde{\vartheta} \varrho s(\varrho, \vartheta)$$

Hypotheses imposed on constitutive relations

WHAT IS NEEDED...

- integrability of all quantities in the weak formulation - hypotheses of coercivity imposed on thermodynamic functions p, e, s
- bounds on the spatial gradients of $\mathbf{u}$, ϑ - the transport coefficients μ, κ depend on the temperature
- compactness of the temperature field on the "vacuum" zones - introducing radiation pressure

Monoatomic gas:

$$p = \frac{2}{3}\varrho e \ \Rightarrow \ p = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right)$$

Third Law:

$$P(Z) \approx Z^{5/3} \text{ for } Z \to \infty$$

Radiation pressure:

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4$$

PRESSURE

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,$$

INTERNAL ENERGY

$$e(\varrho,\vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \boxed{\frac{a}{\varrho}\vartheta^4}$$

ENTROPY

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \boxed{\frac{4}{3}\frac{a}{\varrho}\vartheta^3}$$

Transport coefficients

Shear viscosity

$$0 < \underline{\mu}(1 + \vartheta^{\alpha}) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^{\alpha}),\ 1/2 \leq \alpha \leq 1$$

Bulk viscosity

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta^{\alpha})$$

Heat conductivity

$$0 < \overline{\kappa}(1 + \vartheta^{3}) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^{3})$$

A priori bounds

UNIFORM-IN-TIME L^p–BOUNDS:

$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega;R^3))$$

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega))$$

$$\vartheta \in L^\infty(0,T;L^4(\Omega))$$

Gradient bounds:

$$\mathbf{u} \in L^2(0,T;W^{1,q}(\Omega;R^3)),\ q = \frac{8}{5-\alpha}$$

$$\vartheta \in L^2(0,T;W^{1,2}(\Omega))$$

$$\log(\vartheta) \in L^2(0,T;W^{1,2}(\Omega))$$

Pressure bounds:

$$p(\varrho,\vartheta)\varrho^\beta \in L^1((0,T)\times\Omega) \text{ for a certain } \beta > 0$$

- Pressure bounds are obtained by multiplying the momentum equation on $\psi(t)\mathcal{B}(\varrho^\beta)$, where $\mathcal{B} \approx Div^{-1}$

Weak sequential stability

$$\varrho_\varepsilon \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^{5/3}(\Omega))$$

$$\vartheta_\varepsilon \to \vartheta \text{ weakly-(*) in } L^\infty(0,T;L^4(\Omega))$$

$$\text{and weakly in } L^2(0,T;W^{1,2}(\Omega))$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,q}(\Omega;R^3))$$

Main difficulties

- Compactness of convective terms $\varrho \mathbf{u}$, $\varrho \mathbf{u} \otimes \mathbf{u}$, $\varrho s \mathbf{u}$
- Pointwise convergence of temperature
- Pointwise convergence of density

[F.Murat, L.Tartar, 1975]

Lemma

Let

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } L^p,$$

$$\mathbf{w}_\varepsilon \to \mathbf{w} \text{ weakly in } L^q,$$

with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Let, moreover,

$$\mathrm{div}[\mathbf{v}_\varepsilon],\ \mathrm{curl}[\mathbf{w}_\varepsilon] \text{ be precompact in } W^{-1,s}$$

Then

$$\mathbf{v}_\varepsilon \cdot \mathbf{w}_\varepsilon \to \mathbf{v} \cdot \mathbf{w} \text{ weakly in } L^r.$$

Weak sequential stability of convective terms

$$\mathbf{v}_\varepsilon = [\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon], \ \mathbf{w}_\varepsilon = [u^i_\varepsilon, 0, 0, 0], \ i = 1, 2, 3$$

Aubin-Lions argument (Div-Curl lemma) $\Rightarrow$

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}$$

$$\overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u}$$

$$\overline{\varrho s(\varrho, \vartheta) \vartheta} = \overline{\varrho s(\varrho, \vartheta)} \vartheta$$

Pointwise convergence of temperature, I

GOAL: Use monotonicity of $s(\varrho, \vartheta)$ in ϑ

to show

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \Big) (\vartheta_\varepsilon - \vartheta) \ \mathrm{d}x \ \mathrm{d}t \to 0$$

$$\Rightarrow$$

$$\| \vartheta_\varepsilon \to \vartheta \|_{L^4} \to 0$$

STEP 1: Aubin-Lions argument (Div-Curl lemma) $\Rightarrow$

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(\vartheta_\varepsilon - \vartheta) \ \mathrm{d}x \ \mathrm{d}t \to 0$$

STEP 2: Renormalized equation of continuity [DiPerna and P.-L. Lions, 1989]

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho)\Big)\mathrm{div}_x\mathbf{u} = 0$$

STEP 3: Aubin-Lions argument (Div-Curl lemma):

$$\overline{b(\varrho)g(\vartheta)} = \overline{b(\varrho)}\ \overline{g(\vartheta)}$$

[J.M Ball 1989, P.Pedregal 1997]

Theorem

Let $\mathbf{v}_\varepsilon : Q \subset R^N \to R^M$ *be a sequence of vector fields bounded in* $L^1(Q; R^M)$.
Then there exists a subsequence (not relabeled) and a family of probability measures $\{\nu_y\}_{y \in Q}$ *on* R^M *such that:*
For any Carathéodory function $\Phi = \Phi(y, Z)$, $y \in Q$, $Z \in R^M$ *such that*

$$\Phi(\cdot, \mathbf{v}_\varepsilon) \to \overline{\Phi} \text{ weakly in } L^1(Q)$$

we have

$$\overline{\Phi}(y) = \int_{R^M} \Phi(y, Z) \, \mathrm{d}\nu_y(Z) \text{ for a.a. } y \in Q.$$

STEP 4: Since we already know from STEP 3 that

$$\nu[\varrho_\varepsilon \vartheta_\varepsilon] = \nu[\varrho_\varepsilon] \otimes \nu[\vartheta_\varepsilon],$$

Fundamental theorem yields the desired conclusion

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta)(\vartheta_\varepsilon - \vartheta) \ \mathrm{d}x \ \mathrm{d}t \to 0$$

Pointwise convergence of temperature

$$\vartheta_\varepsilon \to \vartheta \text{ a.a. on } (0, T) \times \Omega$$

STEP 1: Renormalized equation of continuity:

$$\partial_t(\varrho \log(\varrho)) + \mathrm{div}_x(\varrho \log(\varrho)\mathbf{u}) + \varrho \mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t(\overline{\varrho \log(\varrho)}) + \mathrm{div}_x(\overline{\varrho \log(\varrho)}\mathbf{u}) + \overline{\varrho \mathrm{div}_x \mathbf{u}} = 0$$

PROPAGATION OF DENSITY OSCILLATIONS

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right) \, \mathrm{d}x = - \int_\Omega \left(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u} \right) \, \mathrm{d}x$$

STEP 2: Effective viscous pressure [P.-L.Lions, 1998]

$$\overline{p(\varrho,\vartheta)b(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{b(\varrho)} = \overline{[\mathcal{R}:\mathbb{S}]b(\varrho)} - [\mathcal{R}:\mathbb{S}]\overline{b(\varrho)}$$

where

$$\mathcal{R}_{i,j} \equiv \partial_{x_i}\Delta^{-1}\partial_{x_j}$$

$$\mathcal{R}:\mathbb{S}$$

$$= \left[\mathcal{R}:\mathbb{S} - \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\mathrm{div}_x\mathbf{u}\right] + \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\mathrm{div}_x\mathbf{u}$$

Commutator lemma

[in the spirit of Coifman and Meyer]

Lemma

Let $w \in W^{1,r}(R^N)$, $\mathbf{V} \in L^p(R^N; R^N)$ *be given, where*

$$1 < r < N, \ 1 < p < \infty, \ \frac{1}{r} + \frac{1}{p} - \frac{1}{N} < 1.$$

The for any s *satisfying*

$$\frac{1}{r} + \frac{1}{p} - \frac{1}{N} < \frac{1}{s} < 1$$

there exists $\beta > 0$ *such that*

$$\| \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}] \|_{W^{\beta,s}(R^N,R^N)} \leq c \|w\|_{W^{1,r}} \|\mathbf{V}\|_{L^p}.$$

STEP 3: Effective viscous pressure revisited:

$$0 \leq \overline{p(\varrho,\vartheta)\varrho} - \overline{p(\varrho,\vartheta)}\varrho = (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\Big(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u}\Big)$$

yielding (combined with renormalized equation of continuity)

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho)$$

STEP 1: Renormalized equation of continuity:

$$\partial_t(\varrho L_k(\varrho)) + \mathrm{div}_x(\varrho L_k(\varrho)\mathbf{u}) + T_k(\varrho)\mathrm{div}_x\mathbf{u} = 0$$

$$\partial_t(\overline{\varrho L_k(\varrho)}) + \mathrm{div}_x(\overline{\varrho L_k(\varrho)}\mathbf{u}) + \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} = 0$$

$$T_k(\varrho) = \min\{\varrho, k\}$$

$$L_k(\varrho) = \begin{cases} \log(\varrho) \text{ for } 0 < \varrho \leq k \\ \text{constant otherwise} \end{cases}$$

Pointwise convergence of density - general case, II

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho) \right) \,\mathrm{d}x = \int_\Omega \left(T_k(\varrho)\mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} \right) \,\mathrm{d}x$$

$$+ \int_\Omega \left(\overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}} \right) \,\mathrm{d}x$$

STEP 2: Effective viscous flux revisited:

$$\overline{p(\varrho,\vartheta) T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}$$

$$= (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\left(\overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} \right)$$

yielding

Oscillations defect measure

$$\sup_{k\geq 1}\left[\limsup_{\varepsilon\to 0}\int_0^T\int_\Omega |T_k(\varrho_\varepsilon)-T_k(\varrho)|^q \ \mathrm{d}x\ \mathrm{d}t\right]<\infty$$

$$q=5/3+1=8/3$$

Boundedness of oscillations defect measure guarantees:

- The limit functions ϱ, $\mathbf{u}$ satisfy the renormalized equation of continuity
- $$\int_\Omega \left(T_k(\varrho)\mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} \right) \ \mathrm{d}x \to 0 \text{ for } k \to \infty$$

CONCLUSION - POINTWISE CONVERGENCE OF DENSITY

$$\overline{\varrho \log(\varrho)} = \lim_{k\to\infty} \overline{\varrho L_k(\varrho)} = \lim_{k\to\infty} \varrho L_k(\varrho) = \varrho \log(\varrho)$$

$$\varrho_\varepsilon \to \varrho \text{ a.a. on } (0,T) \times \Omega$$

Energy of a closed system is constant;
entropy of a closed system tends to a maximum

Rudolph Clausius, 1822-1888

Navier-Stokes-Fourier system

Field equations - balance laws

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x = 0$$

Energetically insulating boundary conditions

- N-S-F system is *conservative*: the total mass and energy are constants of motion
- Because of viscosity and heat conductivity, the N-S-F system is *dissipative*: a (part of) mechanical energy is irreversibly transformed to heat

CONSERVATIVE DRIVING FORCES

$$\mathbf{f} = \nabla_x F,\ \ F = F(x)$$

CONSERVED QUANTITIES

Total mass:

$$M = \int_\Omega \varrho \ \mathrm{d}x$$

Total energy:

$$E = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x$$

STEP 1: Boundedness of total energy $\Rightarrow$ boundedness of total entropy:

$$S(t) = \int_\Omega \varrho s(\varrho, \vartheta)(t, \cdot) \ \mathrm{d}x \leq S_\infty$$

STEP 2: Boundedness of total entropy $\Rightarrow$ finite integral of the dissipation rate:

$$\int_0^\infty \int_\Omega \left(\frac{\mu}{2\vartheta} \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right|^2 + \frac{\kappa}{\vartheta^2} |\nabla_x \vartheta|^2 \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \sigma[(0, \infty) \times \overline{\Omega}] \ \mathrm{d}t < \infty$$

STEP 3: The velocity field $\mathbf{u}$ as well as the temperature gradient vanish in the asymptotic limit $t \to \infty \Rightarrow$ any solution tends to a uniquely determined *static state*

$$\tilde{\varrho} = \tilde{\varrho}(x), \ \overline{\vartheta} > 0$$

STEP 4: Total dissipation balance:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \right) \ \mathrm{d}x$$

$$+ \overline{\vartheta} \int_\Omega \sigma \ \mathrm{d}x = 0$$

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \right) \ \mathrm{d}x \to 0 \text{ as } t \to \infty$$

CONCLUSION:

$$\mathbf{f} = \nabla_x F, \ F = F(x)$$

$$\varrho(t,\cdot) \to \tilde{\varrho} \text{ in } L^{5/3}(\Omega) \text{ as } t \to \infty$$

$$\vartheta(t,\cdot) \to \overline{\vartheta} \text{ in } L^{4}(\Omega) \text{ as } t \to \infty$$

$$(\varrho \mathbf{u})(t,\cdot) \to 0 \text{ in } L^{1}(\Omega; R^3) \text{ as } t \to \infty$$

Hypotheses:

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x > M_0,\ \ t > 0$$

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right)(t,\cdot)\ \mathrm{d}x < E_0,\ \ t > 0$$

$$\int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)\ \mathrm{d}x > S_0,\ \ t > 0$$

Conclusion:

$$\|\varrho(t,\cdot) - \tilde{\varrho}\|_{L^{5/3}(\Omega)} < \varepsilon,\ \ \|\vartheta(t,\cdot) - \overline{\vartheta}\|_{L^4(\Omega)} < \varepsilon \text{ for } t > T(\varepsilon)$$

$$\|\varrho\mathbf{u}(t,\cdot)\|_{L^1(\Omega;R^3)} < \varepsilon \text{ for } t > T(\varepsilon)$$

Uniform decay of density oscillations

$$d(t) = \int_\Omega \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right)(t, \cdot) \ \mathrm{d}x$$

$$\partial_t d(t) + \Psi(d(t)) \leq 0$$

General time-dependent driving forces

$$\mathbf{f} = \mathbf{f}(t,x),\ |\mathbf{f}(t,x)| \leq \overline{F}$$

EITHER

$$E(t) \equiv \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right)(t,\cdot)\ \mathrm{d}x \to \infty \text{ as } t \to \infty$$

OR

$$|E(t)| \leq E \text{ for a.a. } t > 0$$

In the case $E(t) \leq E$, each sequence of times $\tau_n \to \infty$ contains a subsequence such that

$$\mathbf{f}(\tau_n + \cdot, \cdot) \to \nabla_x F \text{ weakly-(*) in } L^\infty((0,1) \times \Omega),$$

where $F = F(x)$ may depend on $\{\tau_n\}$

STEP 1: Assume that $E(\tau_n) < E$ for certain $\tau_n \to \infty \Rightarrow$ total entropy remains bounded $\Rightarrow$ integral of entropy production bounded

STEP 2: For $\tau_n \to \infty$ we have $\nabla_x p(\varrho, \vartheta) \approx \varrho \mathbf{f}$, $\vartheta \approx \overline{\vartheta}$, meaning, $\mathbf{f} \approx \nabla_x F$

STEP 3: The energy cannot "oscillate" since bounded entropy *static solutions* have bounded total energy

Corollaries:

- $$\mathbf{f} = \mathbf{f}(x) \neq \nabla_x F$$
  $$\Rightarrow$$
  $$E(t) \to \infty$$

- $$\mathbf{f} = \mathbf{f}(t,x) \text{ (almost) periodic in time, } \mathbf{f} \neq \nabla_x F, \ F = F(x)$$
  $$\Rightarrow$$
  $$E(t) \to \infty$$

Rapidly oscillating driving forces

$$\mathbf{f} = \omega(t^\beta)\mathbf{w}(x), \mathbf{w} \in W^{1,\infty}(\Omega; R^3),\ \beta > 2$$

$$\omega \in L^\infty(R),\ \sup_{\tau>0} \left| \int_0^\tau \omega(t)\ \mathrm{d}t \right| < \infty$$

$$(\varrho\mathbf{u})(t,\cdot) \to 0 \text{ in } L^1(\Omega; R^3) \text{ as } t \to \infty$$

$$\varrho(t,\cdot) \to \overline{\varrho} \text{ in } L^{5/3}(\Omega) \text{ as } t \to \infty$$

$$\vartheta(t,\cdot) \to \overline{\vartheta} \text{ in } L^4(\Omega) \text{ as } t \to \infty$$

- Possible extension for $\mathbf{f}(t,x) = t^\alpha \omega(t^\beta)\mathbf{w}(x)$

Motto:

However beautiful the strategy,
you should occasionally look at the results

Sir Winston Churchill, 1874-1965

$$X \approx \frac{X}{X_{\rm char}}$$

$$\text{Mach number } {\rm Ma} = \frac{|\mathbf{u}|_{\rm char}}{\sqrt{p_{\rm char}/\varrho_{\rm char}}}$$

$$\text{Froude number } {\rm Fr} = \frac{|\mathbf{u}|_{\rm char}}{\sqrt{|x|_{\rm char}/|\nabla_x F|_{\rm char}}}$$

Incompressibility: ${\rm Ma} \approx \varepsilon \to 0$

Stratification: ${\rm Fr} \approx \varepsilon^{\alpha/2} \to 0$

Characteristic numbers:

Symbol	Definition	Name
Sr	$\text{length}_{\text{ref}}/(\text{time}_{\text{ref}}\text{velocity}_{\text{ref}})$	Strouhal number
Ma	$\text{velocity}_{\text{ref}}/\sqrt{\text{pressure}_{\text{ref}}/\text{density}_{\text{ref}}}$	Mach number
Re	$\text{density}_{\text{ref}}\text{velocity}_{\text{ref}}\text{length}_{\text{ref}}/\text{viscosity}_{\text{ref}}$	Reynolds number
Fr	$\text{velocity}_{\text{ref}}/\sqrt{\text{length}_{\text{ref}}\text{force}_{\text{ref}}}$	Froude number
Pe	$\text{pressure}_{\text{ref}}\text{length}_{\text{ref}}\text{velocity}_{\text{ref}}/(\text{temperature}_{\text{ref}}\text{heat conductivity}_{\text{ref}})$	Péclet number

Material vs. geometric scaling

- Material scaling concerns *material* properties of the fluid: viscosity, pressure etc.
- Geometric scaling compares the relative amplitude of purely geometric quantities: velocity, time, length
- Different scaling procedures may lead to the same resulting system of equations

Scaled Navier-Stokes-Fourier system:

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2} \nabla_x p(\varrho, \vartheta) = \frac{1}{\mathrm{Re}} \mathrm{div}_x \mathbb{S} + \frac{1}{\mathrm{Fr}^2} \varrho \nabla_x F$$

$$\mathrm{Sr}\ \partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \frac{1}{\mathrm{Pe}} \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma \geq \frac{1}{\vartheta} \left(\frac{\mathrm{Ma}^2}{\mathrm{Re}} \mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}} \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\mathrm{Ma}^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \varrho F \right) = 0$$

Acoustically hard boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}]_{\tan} = 0$$

$$\mathrm{Ma} = \varepsilon, \ \mathrm{Fr} = \sqrt{\varepsilon}$$

STRATEGY:

1. Existence theory for the primitive Navier-Stokes-Fourier system
2. Uniform bounds independent of the singular parameter
3. Passage to the limit - analysis of acoustic waves
4. Identification of the limit system

Scaled Navier-Stokes-Fourier system revisited

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \text{ in } (0,T)\times\Omega, \ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \boxed{\frac{1}{\varepsilon^2}\nabla_x p(\varrho,\vartheta)} = \mathrm{div}_x\mathbb{S} + \boxed{\frac{1}{\varepsilon}\varrho\nabla_x F} \text{ in } (0,T)\times\Omega$$

$$[\mathbb{S}\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma \text{ in } (0,T)\times\Omega$$

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\boxed{\frac{\varepsilon^2}{2}} \varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \boxed{\varepsilon\varrho F} \right) \ \mathrm{d}x = 0$$

$$\sigma \geq \frac{1}{\vartheta}\left(\boxed{\varepsilon^2\mathbb{S}:\nabla_x\mathbf{u}} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta} \right) \geq 0$$

Total dissipation balance revisited

$$\int_{\Omega}\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\frac{1}{\varepsilon^2}\left(H(\varrho,\vartheta)-\partial_\varrho H(\tilde{\varrho}_\varepsilon,\overline{\vartheta})(\varrho-\tilde{\varrho}_\varepsilon)-H(\tilde{\varrho}_\varepsilon,\overline{\vartheta})\right)\right)(\tau,\cdot)$$

$$+\frac{\overline{\vartheta}}{\varepsilon^2}\int_0^\tau\int_\Omega \sigma =$$

$$\int_{\Omega}\left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2+\frac{1}{\varepsilon^2}\left(H(\varrho_0,\vartheta_0)-\partial_\varrho H(\tilde{\varrho}_\varepsilon,\overline{\vartheta})(\varrho_0-\tilde{\varrho}_\varepsilon)-H(\tilde{\varrho}_\varepsilon,\overline{\vartheta})\right)\right)$$

$$\nabla_x p(\tilde{\varrho}_\varepsilon,\overline{\vartheta})=\varepsilon\tilde{\varrho}_\varepsilon\nabla_x F,\ \int_\Omega \tilde{\varrho}_\varepsilon\ \mathrm{d}x=\int_\Omega \varrho_0\ \mathrm{d}x,\ \tilde{\varrho}_\varepsilon\approx\overline{\varrho}$$

Ill-prepared initial data

$$\varrho_0 \approx \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^1 \cap L^\infty(\Omega),\ \int_\Omega \varrho^{(1)}_{0,\varepsilon}\ \mathrm{d}x = 0$$

$$\vartheta_0 \approx \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^1 \cap L^\infty(\Omega),\ \int_\Omega \vartheta^{(1)}_{0,\varepsilon}\ \mathrm{d}x = 0$$

$$\mathbf{u}_0 \approx \mathbf{u}_{0,\varepsilon},\ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega; R^3)$$

Uniform bounds

Bounds uniform in time

$$\left\{ \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\}_{\varepsilon > 0} \text{ bounded in } L^\infty(0, T; L^2 \oplus L^q(\Omega)), \ q < 2$$

$$\left\{ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\}_{\varepsilon > 0} \text{ bounded in } L^\infty(0, T; L^2 \oplus L^q(\Omega)), \ q < 2$$

$$\left\{ \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \right\}_{\varepsilon > 0} \text{ bounded in } L^\infty(0, T; L^1(\Omega))$$

$$\left\{ \frac{\sigma_\varepsilon}{\varepsilon^2} \right\}_{\varepsilon > 0} \text{ bounded in } \mathcal{M}^+([0, T] \times \overline{\Omega})$$

Integral bounds

$$\left\{ \frac{\sigma_\varepsilon}{\varepsilon^2} \right\}_{\varepsilon>0} \text{ bounded in } \mathcal{M}^+([0,T]\times\overline{\Omega})$$

$$\Longrightarrow$$

$$\{\nabla_x \mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2((0,T)\times\Omega; R^{3\times 3})$$

$$\left\{ \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right\}_{\varepsilon>0} \text{ bounded in } L^2((0,T)\times\Omega; R^3)$$

Convergence

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } L^\infty(0,T;L^2 \oplus L^q(\Omega))$$

$$\vartheta_\varepsilon \to \overline{\vartheta} \text{ in } L^\infty(0,T;L^2 \oplus L^q(\Omega))$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3))$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega))$$

Target problem - Oberbeck-Boussinesq system

$$\mathrm{div}_x \mathbf{U} = 0$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x (\mathbf{U} \otimes \mathbf{U}) \Big) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S} + r \nabla_x F \text{ in } (0,T) \times \Omega$$

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\overline{\varrho} c_p \Big(\partial_t \Theta + \mathrm{div}_x (\Theta \mathbf{U}) \Big) - \mathrm{div}_x (G \mathbf{U}) - \mathrm{div}_x (\kappa \nabla_x \Theta) = 0 \text{ in } (0,T) \times \Omega$$

$$G = \beta F, \ \nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$r + \alpha \Theta = 0, \ \alpha > 0$$

Helmholtz decomposition

$$\varrho_\varepsilon \mathbf{u}_\varepsilon = \boxed{\underbrace{\mathbf{w}_\varepsilon}_{\text{solenoidal part}} + \underbrace{\nabla_x \Phi_\varepsilon}_{\text{acoustic part}}}, \ \mathrm{div}_x(\mathbf{w}_\varepsilon) = 0, \ \mathbf{w}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Lions-Aubin argument $\Longrightarrow$

$$\mathbf{w}_\varepsilon \to \mathbf{U} \text{ in } L^2_{\mathrm{loc}}((0,T) \times \Omega)$$

GOAL:

$$\nabla_x \Phi_\varepsilon \to 0 \text{ in } L^2_{\mathrm{loc}}((0,T) \times \Omega)$$

Lighthill's acoustic equation ($F = 0$)

$$\boxed{\varepsilon \partial_t Z_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon} = \varepsilon \mathrm{div}_x \mathbf{F}^1_\varepsilon$$

$$\boxed{\varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x Z_\varepsilon} = \varepsilon \left(\mathrm{div}_x \mathbb{F}^2_\varepsilon + \nabla_x F^3_\varepsilon + \frac{A}{\varepsilon^2 \omega} \nabla_x \Sigma_\varepsilon \right)$$

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$Z_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega} \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) + \frac{A}{\varepsilon\omega} \Sigma_\varepsilon, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$< \Sigma_\varepsilon; \varphi > = < \sigma_\varepsilon; I[\varphi] >$$

$$I[\varphi](t,x) = \int_0^t \varphi(z,x)\ \mathrm{d}z \text{ for any } \varphi \in L^1(0,T; C(\overline{\Omega}))$$

Helmholtz decomposition

$$\mathbf{V}_\varepsilon = \mathbf{w}_\varepsilon + \nabla_x \Phi_\varepsilon, \ \mathrm{div}_x(\mathbf{w}_\varepsilon) = 0, \ \mathbf{w}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Wave equation

$$\varepsilon \partial_t Z_\varepsilon + \Delta_x \Phi_\varepsilon = \varepsilon G^1_\varepsilon \text{ in } (0,T) \times \Omega$$

$$\varepsilon \partial_t \Phi_\varepsilon + \omega Z_\varepsilon = \varepsilon G^2_\varepsilon \text{ in } (0,T) \times \Omega$$

$$\nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

- No-stick boundary conditions needed!

Problems on large domains

$$\Omega \approx \Omega_\varepsilon$$

$$\partial \Omega_\varepsilon = \Gamma_0 \cup \Gamma_\infty^\varepsilon \ \Gamma_0 \text{ regular and compact}$$

$$\varepsilon \ \mathrm{dist}[\Gamma_\infty^\varepsilon; \Gamma_0] \to \infty \text{ as } \varepsilon \to 0$$

Wave equation revisited

$$\varepsilon \partial_t Z_\varepsilon + \Delta_x \Phi_\varepsilon = \varepsilon G^1_\varepsilon \text{ in } (0,T) \times \Omega$$

$$\varepsilon \partial_t \Phi_\varepsilon + \omega Z_\varepsilon = \varepsilon G^2_\varepsilon \text{ in } (0,T) \times \Omega,$$

$$\nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Ω exterior domain $\partial\Omega = \Gamma_0$

$$\varepsilon\partial_t r_\varepsilon - A[\Phi_\varepsilon] = \varepsilon G^1(A)[h^1_\varepsilon]$$
$$\varepsilon\partial_t \Phi_\varepsilon + r_\varepsilon = \varepsilon G^2(A)[h^2_\varepsilon]$$

$$A[v] = -\omega\Delta_x v,\ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0$$

A is a non-negative self-adjoint operator on the Hilbert space $L^2(\Omega)$

$$h^1_\varepsilon, h^2_\varepsilon \in L^2(0,T;L^2(\Omega))$$

- Functions $G^i = G^i(Z)$ may be **singular** for $Z \to 0$ and $Z \to \infty$

Example:

$$\mathrm{div}_x \mathbf{F},\ \ \mathbf{F} \in L^2(0,T;L^2(\Omega;R^3))$$

$$\Longrightarrow$$

$$\mathrm{div}_x \mathbf{F} = A^{-1/2}[h],\ \ h \in L^2(0,T;L^2(\Omega))$$

$$\Longrightarrow$$

$$G(Z) = Z^{-1/2}$$

$$
\begin{aligned}
\Phi_\varepsilon(t,\cdot) = \exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{A}\right)\left[\frac{1}{2}\Phi_{0,\varepsilon} + \frac{\mathrm{i}}{2\sqrt{A}}[r_{0,\varepsilon}]\right] \\
+ \exp\left(-\mathrm{i}\frac{t}{\varepsilon}\sqrt{A}\right)\left[\frac{1}{2}\Phi_{0,\varepsilon} - \frac{\mathrm{i}}{2\sqrt{A}}[r_{0,\varepsilon}]\right] \\
+ \int_0^t \exp\left(\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{A}\right)\left[\frac{1}{2}G^2(A)[h^2_\varepsilon](s) + \frac{\mathrm{i}G^1(A)}{2\sqrt{A}}[h^1_\varepsilon(s)]\right]\ \mathrm{d}s \\
+ \int_0^t \exp\left(-\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{A}\right)\left[\frac{1}{2}G^2(A)[h^2_\varepsilon](s) - \frac{\mathrm{i}G^1(A)}{2\sqrt{A}}[h^1_\varepsilon(s)]\right]\ \mathrm{d}s
\end{aligned}
$$

Acoustic propagator

$$h \mapsto \exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right)[h]$$

Since $\nabla_x \mathbf{u}_\varepsilon$ is integrable it is enough to control only:

$$\underbrace{\varphi}_{\text{cut-off in physical space}} \quad \exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right) \quad \underbrace{F(-\Delta_N)}_{\text{cut-off in frequency space}} \quad [h]$$

Dispersive estimates

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right)[h]; F(-\Delta_N)[\varphi] \right\rangle \right|^2 \mathrm{d}t \le \omega(\varepsilon, \varphi, F)\|h\|^2_{L^2(\Omega)}$$

$\omega(\varepsilon, \varphi, F) \to 0$ as $\varepsilon \to 0$ for any fixed $\varphi \in C^\infty_c(\Omega)$, $F \in C^\infty_c(0, \infty)$

Theorem [Kato, 1965]

Theorem

Let C be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R, \ v \in \mathcal{D}(C^*), \ \|v\|_X = 1} \| C \circ R_H[\lambda] \circ C^*[v] \|_X < \infty.$$

Then

$$\sup_{w \in X, \ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \| C \exp(-\mathrm{i} t H)[w] \|_X^2 \ \mathrm{d}t \leq \Gamma^2.$$

Apply Kato's theorem to

$$X = L^2(\Omega)$$

$$H = \sqrt{-\Delta_N},\ C = \varphi \circ F(-\Delta_N),\ F \in C_c^\infty(0,\infty),\ \varphi \in C_0^\infty(\Omega)$$

$$\varphi F(-\Delta_N) \frac{1}{\sqrt{-\Delta_N} - \lambda} F(-\Delta_N)\varphi = \varphi F(-\Delta_N) \frac{\sqrt{-\Delta_N} + \lambda}{(-\Delta_N) - \lambda^2} F(-\Delta_N)\varphi$$

Limiting absorption principle

$$\sup_{\mu \in C,\ 0<\alpha\leq \mathrm{Re}[\mu]\leq\beta<1,\ 0<|\mathrm{Im}[\mu]|<1} \left\| \varphi \frac{1}{(-\Delta_N) - \mu} \varphi \right\|_{[L^2;L^2]} < c < \infty$$

Theorem

Let X be a Hilbert space, $H : \mathcal{D}(H) \subset X \to X$ a self-adjoint operator, $C : X \to X$ a compact operator, and P_c the orthogonal projection onto H_c, where

$$X = H_c \oplus \mathrm{cl}_X \Big\{ \mathrm{span}\{ w \in X \mid w \textit{ an eigenvector of } H \} \Big\}.$$

Then

$$\left\| \frac{1}{\tau} \int_0^\tau \exp(-\mathrm{i}tH) C P_c \exp(\mathrm{i}tH) \ \mathrm{d}t \right\|_{\mathcal{L}(X)} \to 0 \textit{ for } \tau \to \infty.$$

Apply RAGE theorem to

$X = L^2(\Omega)$, $H = \sqrt{-\Delta_N}$, $C = \varphi^2 F(-\Delta_N)$, with $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, $F \in C_0^\infty(0,\infty)$, $0 \leq F \leq 1$:

$$\int_0^T \left\langle \exp\left(-\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right)\varphi^2 F(-\Delta_N)\exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right)X; Y\right\rangle \ \mathrm{d}t$$

$$= \varepsilon \int_0^{T/\varepsilon} \left\langle \exp\left(-\mathrm{i}t\sqrt{-\Delta_N}\right)\varphi^2 F(-\Delta_N)\exp\left(\mathrm{i}t\sqrt{-\Delta_N}\right)X; Y\right\rangle \ \mathrm{d}t$$

$$\leq \omega(\varepsilon)\|X\|_{L^2(\Omega)}\|Y\|_{L^2(\Omega)},$$

where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$

For $Y = F(-\Delta_N)[X]$ we deduce that:

$$\int_0^T \left\| \varphi F(-\Delta_N) \exp\left(\mathrm{i} \frac{t}{\varepsilon} \sqrt{-\Delta_N} \right) [X] \right\|^2_{L^2(\Omega)} \ \mathrm{d}t$$

$$\leq \omega(\varepsilon) \|X\|^2_{L^2(\Omega)} \text{ for any } X \in L^2(\Omega)$$

$$\omega(\varepsilon) \to 0 \text{ for } \varepsilon \to 0$$

- Kato's theorem yields $\omega(\varepsilon) = \varepsilon$

General approach via spectral measures

$$\left\langle \exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N} \right)[h]; G(-\Delta_N)[\varphi] \right\rangle$$
$$= \int_0^\infty \exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{\lambda} \right) G(\lambda)\tilde{h}(\lambda)\ \mathrm{d}\mu_\varphi$$

μ_φ - spectral measure associated to φ

$\tilde{h} \in L^2(0,\infty;\mathrm{d}\mu_\varphi)$, $\|\tilde{h}\|_{L^2_{\mu_\varphi}} \leq \|h\|_{L^2(\Omega)}$

Conclusion

RAGE theorem: μ_φ does not charge points (point spectrum of $-\Delta_N$ is empty) $\Rightarrow \omega(\varepsilon, \cdot) \to 0$ as $\varepsilon \to 0$

Kato's theorem: μ_φ is Lipschitz with respect to the classical Lebesgue measure:

$$\mu_\varphi(I) \leq c(\delta)|I| \text{ for any interval } I \subset (\delta, 1/\delta).$$

$\Rightarrow \omega(\varepsilon, \cdot) \approx \varepsilon$. This property holds if $-\Delta_N$ satisfies Limiting Absorption Principle.

Intermediate results:

$$\mu_\varphi(I) \leq c(\delta)|I|^\alpha \text{ for any interval } I \subset (\delta, 1/\delta).$$

Decay via RAGE theorem:

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon} \right) [\Psi], F(A)\varphi \right\rangle \right|^2 \ \mathrm{d}t$$

$$= \int_0^T \int_0^\infty \int_0^\infty \exp\left(\mathrm{i} \left(\sqrt{x} - \sqrt{y} \right) \frac{t}{\varepsilon} \right) \times$$

$$\times F(x) F(y) \tilde{\Psi}(x) \ \overline{\tilde{\Psi}}(y) \ \mathrm{d}\mu_\varphi(x) \ \mathrm{d}\mu_\varphi(y) \ \mathrm{d}t$$

$$\leq e \int_0^\infty \int_0^\infty \left(\int_{-\infty}^\infty \exp\left(-(t/T)^2 \right) \exp\left(\mathrm{i} \left(\sqrt{x} - \sqrt{y} \right) \frac{t}{\varepsilon} \right) \ \mathrm{d}t \right) \times$$

$$\times F(x) F(y) \Psi(x) \overline{\tilde{\Psi}(y)} \ \mathrm{d}\mu_\varphi(x) \ \mathrm{d}\mu_\varphi(y)$$

$$\leq eT\sqrt{\pi}\int_0^\infty\int_0^\infty \exp\left(-\frac{T^2|\sqrt{x}-\sqrt{y}|^2}{4\varepsilon^2}\right)\times$$

$$\times|\tilde{\Psi}(x)|\;|\tilde{\Psi}(y)|\;|F(x)|\mathrm{d}\mu_\varphi(x)\;|F(y)|\mathrm{d}\mu_\varphi(y)$$

by Cauchy-Schwartz inequality

$$\leq \omega^2(\varepsilon,F,\varphi)\|\Psi\|^2_{L^2(\Omega)}$$

$$\omega^4(\varepsilon,F,\varphi)\leq$$

$$c\int_0^\infty\int_0^\infty \exp\left(-\frac{T^2|\sqrt{x}-\sqrt{y}|^2}{2\varepsilon^2}\right)\;|F(x)||F(y)|\mathrm{d}\mu_\varphi(x)\;\mathrm{d}\mu_\varphi(y)$$

Conclusion via RAGE theorem

$$\omega(\varepsilon, F, \varphi) \to 0 \text{ as } \varepsilon \to 0$$

$$\Longleftrightarrow$$

μ_φ does not charge points

$$\Longleftrightarrow$$

the point spectrum of A is empty

Following Y.Last [1996]:

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon} \right) [\Psi], F(A)[\varphi] \right\rangle \right|^2 \ \mathrm{d}t$$

$$\leq eT\sqrt{\pi} \int_0^\infty |\Psi(x)|^2 \left(\int_0^\infty \exp\left(-\frac{|\sqrt{x} - \sqrt{y}|^2}{\varepsilon^2} \frac{T^2}{4} \right) \mathrm{d}\mu_\varphi(y) \right) \times$$

$$\times F^2(x)\ \mathrm{d}\mu_\varphi(x)$$

$$\int_0^\infty \exp\left(-\frac{|\sqrt{x}-\sqrt{y}|^2}{\varepsilon^2}\frac{T^2}{4}\right) \mathrm{d}\mu_\varphi(y)$$

$$= \sum_{n=0}^\infty \int_{\varepsilon n \leq |\sqrt{y}-\sqrt{x}| < \varepsilon(n+1)} \exp\left(-\frac{|\sqrt{x}-\sqrt{y}|^2}{\varepsilon^2}\frac{T^2}{4}\right) \mathrm{d}\mu_\varphi(y)$$

$$\leq \sup_{n\geq 0} \int_{\varepsilon n \leq |\sqrt{y}-\sqrt{x}| < \varepsilon(n+1)} 1 \mathrm{d}\mu_\varphi(y) \sum_{n=0}^\infty \exp\left(-\frac{n^2 T^2}{4}\right)$$

for $x \in \mathrm{supp}[F]$

Stone's formula:

$$\mu_\varphi(a,b)$$

$$= \lim_{\delta \to 0+} \lim_{\eta \to 0+} \int_{a+\delta}^{b-\delta} \left\langle \left(\frac{1}{A - \lambda - \mathrm{i}\eta} - \frac{1}{A - \lambda + \mathrm{i}\eta} \right) \varphi, \varphi \right\rangle \ \mathrm{d}\lambda$$

Limiting absorption principle:

$$\left[\begin{array}{c} \textit{Operators} \\ \mathcal{V} \circ (A - \lambda \pm \mathrm{i}\eta)^{-1} \circ \mathcal{V} : L^2(\Omega) \to L^2(\Omega), \\ \mathcal{V}[v] = (1 + |x|^2)^{-s/2},\ s > 1 \\ \textit{are bounded uniformly for } \lambda \in [a, b],\ 0 < a < b,\ \eta > 0, \end{array} \right]$$

Operator A satisfies Limiting Absorption Principle

$$\Longrightarrow$$

$$\mu_\varphi[I] \leq c_\delta |I| \text{ for any interval } I \subset [\delta, 1/\delta],\ \delta > 0$$

$$\Longrightarrow$$

$$\omega(\varepsilon, F, \varphi) \leq \sqrt{\varepsilon} c(F, \varphi)$$

Uniform decay for varying domains

- $$\Omega_\varepsilon = R^3 \setminus K_\varepsilon, \ \ K_\varepsilon \subset \{ |x| \leq r \}$$
- $$\Omega \subset \Omega_{\varepsilon_2} \subset \Omega_{\varepsilon_1} \ \text{for} \ \varepsilon_2 > \varepsilon_1$$
- $$|\Omega_\varepsilon \setminus \Omega| \to 0 \ \text{as} \ \varepsilon \to 0$$
- $$\Omega_\varepsilon \ \text{satisfy the uniform cone condition}$$

$$\Longrightarrow \ \omega(\varepsilon, F, \varphi) \approx \sqrt{\varepsilon}$$

Stone's formula $\Rightarrow$

$$\mu_{\varepsilon,\varphi}(a,b) = \int_a^b \left\langle \left(w^-_{\varepsilon,\lambda} - w^+_{\varepsilon,\lambda} \right), \varphi \right\rangle_{L^2(\Omega_\varepsilon)} \, \mathrm{d}\lambda, \ 0 < a < b,$$

where $w^{\pm}_{\varepsilon,\lambda}$ solve

$$\Delta w^{\pm}_{\varepsilon,\lambda} + \lambda w^{\pm}_{\varepsilon,\lambda} = \varphi \text{ in } \Omega_\varepsilon, \ \nabla_x w^{\pm}_{\varepsilon,\lambda} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,$$

Sommerfeld radiation condition

$$\lim_{r\to\infty} r \left(\partial_r \pm \mathrm{i}\sqrt{\lambda} \right) w^{\pm}_{\varepsilon,\lambda} = 0, \ r = |x|$$

Reduction to bounded domain case:

$$\mathrm{supp}[\varphi] \subset \{|x| \leq R\}$$

$$w^{\pm}_{\varepsilon,\lambda_\varepsilon}(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m(\theta, \phi) \text{ for } |x| = 2R$$

(r, θ, ϕ) polar coordinates

$$w^{\pm}_{\varepsilon,\lambda_\varepsilon}(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m(\theta, \phi) \frac{h_l^{(1)}(\pm\sqrt{\lambda} r)}{h_l^{(1)}(\pm\sqrt{\lambda} 2R)} \text{ for all } x \in R^3 \setminus \overline{B}_{2R},$$

Y_l^m spherical harmonics of order l

,

$h_l^{(1)}$ spherical Bessel functions

"Perforated" domains

$$\Omega_\varepsilon = \Omega \setminus \sum_{i=1}^{M} B_{\delta(\varepsilon)}[x_i]$$

$$B_\delta[x_i] = \{|x - x_i| \leq \delta\}$$

$$\delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0$$

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho} \nabla_x F$$

Anelastic constraint

$$\mathrm{div}_x (\tilde{\varrho} \mathbf{U}) = 0$$

$$A = \frac{1}{\tilde{\varrho}} \mathrm{div}_x (\tilde{\varrho} \nabla_x \mathbf{U})$$