Inviscid incompressible limits for the full Navier-Stokes-Fourier system

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Viscous, compressible, and heat conducting fluid in motion:

- mass density $\varrho = \varrho(t,x)$
- absolute temperature $\vartheta = \vartheta(t,x)$
- velocity field $\mathbf{u} = \mathbf{u}(t,x)$

Asymptotic limit for

- low Mach number (incompressible regime)
- high Reynolds number (inviscid limit)
- high Pèclet number (low heat conductivity)

Scaled Navier-Stokes-Fourier system

EQUATION OF CONTINUITY

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

BALANCE OF MOMENTUM

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta)} = \boxed{\varepsilon^a} \mathrm{div}_x \mathbb{S}$$

ENTROPY PRODUCTION

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \boxed{\varepsilon^b} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+a}} \mathbb{S} : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Thermodynamic functions

pressure $p = p(\varrho, \vartheta)$, internal energy $e = e(\varrho, \vartheta)$, entropy $s = s(\varrho, \vartheta)$

Gibbs' equation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Constitutive relations

Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Relative entropy

BALLISTIC FREE ENERGY

$$H_\Theta(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta) \Big)$$

$$\partial^2_{\varrho,\varrho} H_\Theta(\varrho,\Theta) = \frac{1}{\varrho}\partial_\varrho p(\varrho,\Theta)$$

$$\partial_\vartheta H_\Theta(\varrho,\vartheta) = (\vartheta - \Theta)\frac{1}{\vartheta}\partial_\vartheta e(\varrho,\vartheta)$$

RELATIVE ENTROPY

$$\mathcal{E}_\varepsilon \left(\varrho,\vartheta,\mathbf{u} \Big| r,\Theta,\mathbf{U} \right) =$$

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^2}\left(H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta) \right) \right]$$

RELATIVE ENTROPY INEQUALITY

$$\left[\mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon^{b-2} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}_\varepsilon(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}_\varepsilon(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \frac{1}{\varepsilon^2} \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \frac{1}{\varepsilon^2} \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \, \varepsilon^b \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \frac{1}{\varepsilon^2} \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Target system

INCOMPRESSIBILITY

$$\mathrm{div}_x \mathbf{v} = 0$$

EULER SYSTEM

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0$$

TEMPERATURE TRANSPORT

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0$$

BASIC ASSUMPTION

The incompressible Euler system possesses a strong solution $\mathbf{v}$ *on a time interval* $(0, T_{\mathrm{max}})$ *for the initial data* $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$.

Prepared data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3),\ \mathbf{v}_0 \in W^{k,2}(\Omega; R^3),\ k > \frac{5}{2}$$

Boundary conditions

Navier's complete slip condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Convergence

$$b > 0, \ 0 < a < \frac{10}{3}$$

$$\mathrm{ess} \sup_{t \in (0,T)} \| \varrho_\varepsilon(t, \cdot) - \overline{\varrho} \|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon c$$

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \ \mathbf{v} \ \text{in} \ \boxed{L^\infty_{\rm loc}((0,T]; L^2_{\rm loc}(\Omega; R^3))}$$

and weakly-(*) in $L^\infty(0,T; L^2(\Omega; R^3))$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T \ \text{in} \ \boxed{L^\infty_{\rm loc}((0,T]; L^q_{\rm loc}(\Omega; R^3)), \ 1 \leq q < 2},$$

and weakly-(*) in $L^\infty(0,T; L^2(\Omega))$

Uniform bounds

The uniform bounds independent of ε are obtained by means of the choice

$$r = \overline{\varrho},\ \Theta = \overline{\vartheta},\ \mathbf{U} = 0$$

in the relative entropy inequality:

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2 + L^{5/3}(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \sqrt{\varrho} \mathbf{u}_\varepsilon \right\|_{L^2(\Omega; R^3)} \leq c$$

Linearization

$$\varepsilon\partial_t \boxed{\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right)} + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\varepsilon\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \nabla_x \left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} \right) = \varepsilon \mathbf{f}_1$$

$$\partial_t \left(\overline{\varrho}\partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho}\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right)$$

$$+\mathrm{div}_x \left[\left(\overline{\varrho}\partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho}\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right) \mathbf{u}_\varepsilon \right] = \varepsilon f_2$$

Main idea of the proof

Take

$$r_\varepsilon = \overline{\varrho} + \varepsilon R_\varepsilon, \ \Theta_\varepsilon = \overline{\vartheta} + \varepsilon T_\varepsilon, \ \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality

Acoustic equation

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta T_\varepsilon) = 0$$

Transport equation

$$\partial_t (\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0$$

Dispersion of acoustic waves

$-\Delta_N$ Neumann Laplacian

$$\partial^2_{t,t}\Phi - \omega\Delta_N\Phi = 0$$

Hypotheses imposed on Ω

- Limiting absorption principle. The operator Δ_N satisfies the limiting absorption principle in Ω:

$$\varphi \circ [-\Delta_N^{-1} - \lambda \pm \mathrm{i}\delta]^{-1} \circ \varphi,\ \ \varphi \in C_c^\infty(\Omega) \text{ bounded in } L^2(\Omega)$$

  for λ belonging to compact subintervals of $(0,\infty)$, $\delta > 0$.
- There is a compact set B such that Δ_N satisfies the Strichartz estimates on $D = \Omega \cup B$.
- The operator Δ_N satisfies the local energy decay.

Strichartz estimates and local energy decay

$$\|\Phi\|_{L^p(R;L^q(D))} \leq c\Big(\|\Phi(0)\|_{H^{\gamma,2}(D)} + \|\partial_t \Phi(0)\|_{H^{\gamma-1,2}(D)} \Big)$$

$$2 \leq q < \infty,\ \frac{2}{p} \leq \left(1 - \frac{2}{q}\right),\ \gamma = \frac{3}{2} - \frac{3}{q} - \frac{1}{p}$$

$$\int_{-\infty}^{\infty} \Big(\|\chi \Phi(t,\cdot)\|^2_{H^\gamma(D)} + \|\chi \partial_t \Phi(t,\cdot)\|^2_{H^{\gamma-1}(D)} \Big)\ \mathrm{d}t$$
$$\leq c\Big(\|\Phi(0)\|_{H^{\gamma,2}(D)} + \|\partial_t \Phi(0)\|_{H^{\gamma-1,2}(D)} \Big),$$

$\chi \in C_c^\infty(\Omega)$.