Relative entropy methods in the mathematical theory of complete fluid systems

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Mathematical model

State variables

Mass density

$$\varrho = \varrho(t, x)$$

Absolute temperature

$$\vartheta = \vartheta(t, x)$$

Velocity field

$$\mathbf{u} = \mathbf{u}(t, x)$$

Thermodynamic functions

Pressure

$$p = p(\varrho, \vartheta)$$

Internal energy

$$e = e(\varrho, \vartheta)$$

Entropy

$$s = s(\varrho, \vartheta)$$

Transport

Viscous stress

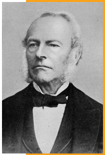
$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Heat flux

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta)$$

Field equations

Claude Louis Marie Henri Navier [1785-1836]



George Gabriel Stokes [1819-1903]

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum balance

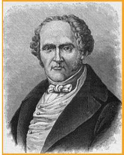
$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

Entropy production

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma = (\geq) \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

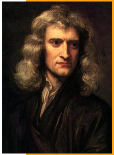
Constitutive relations



François Marie Charles Fourier
[1772-1837]

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta$$

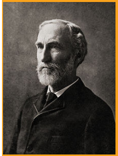


Isaac Newton
[1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}$$

Gibbs' relation



Willard Gibbs
[1839-1903]

Gibbs' relation:

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamics stability:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Boundary conditions

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No-slip

$$\mathbf{u}_{\tan}|_{\partial\Omega} = 0$$

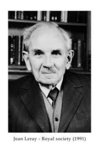
No-stick

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

A bit of history of global existence for large data



Jean Leray [1906-1998] Global existence of weak solutions for the incompressible Navier-Stokes system (3D)

Olga Aleksandrovna Ladyzhenskaya [1922-2004] Global existence of classical solutions for the incompressible 2D Navier-Stokes system

Pierre-Louis Lions[*1956] Global existence of weak solutions for the compressible barotropic Navier-Stokes system (2,3D)

and many, many others...

Weak solutions to the complete system

- Equation of continuity holds in the sense of distributions (renormalized equation also satisfied)
- Momentum balance holds in the sense of distributions
- Entropy production equation holds in the sense of distributions, entropy production rate satisfies the inequality
- The system is augmented by

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x = 0$$

Relative entropy (energy)

Dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = A(t, u(t)), \ u(t) \in X, \ u(0) = u_0$$

Relative entropy

$U : t \mapsto U(t) \in Y$ a "trajectory" in the phase space $Y \subset X$

$$\mathcal{E}\left\{ u(t) \middle| U(t) \right\}, \ \mathcal{E} : X \times Y \to R$$

Basic properties

Positivity(distance)

$\mathcal{E}\left\{u\,|U\right\}$ is a "distance" between u, and U, meaning $\mathcal{E}(u|U) \geq 0$ and $\mathcal{E}\left\{u|U\right\} = 0$ only if $u = U$

Lyapunov function

$\mathcal{E}\left\{u(t)|\tilde{U}\right\}$ is a Lyapunov function provided $\tilde{U}$ is an equilibrium

$$t \mapsto \mathcal{E}\left\{u(t)\Big|\tilde{U}\right\} \text{ is non-increasing}$$

Gronwall inequality

$$\mathcal{E}\left\{u(\tau)\Big|U(\tau)\right\} \leq \mathcal{E}\left\{u(s)\Big|U(s)\right\} + c(T)\int_s^\tau \mathcal{E}\left\{u(t)\Big|U(t)\right\}\,\mathrm{d}t$$

if U is a solution of the same system (in a "better" space) Y

Applications

Stability of equilibria

Any solution ranging in X stabilizes to an equilibrium belonging to Y (to be proved!)

Weak-strong uniqueness

Solutions ranging in the "better" space Y are unique among solutions in X.

Singular limits

Stability and convergence of a family of solutions u_ε corresponding to A_ε to a solution $U = u$ of the limit problem with generator A.

Navier-Stokes-Fourier system revisited

Total energy balance (conservation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right) \ \mathrm{d}x = 0$$

Total entropy production

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho s(\varrho,\vartheta) \ \mathrm{d}x = \int_\Omega \sigma \ \mathrm{d}x \geq 0$$

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta) - \varrho F \right) \ \mathrm{d}x + \int_\Omega \sigma \ \mathrm{d}x = 0$$

Equilibrium (static) solutions

Equilibrium solutions minimize the entropy production

$$\mathbf{u} \equiv 0,\ \vartheta \equiv \Theta > 0 \text{ a positive constant}$$

Static problem

$$\nabla_x p(\tilde{\varrho}, \Theta) = \tilde{\varrho} \nabla_x F$$

Total mass and energy are constants of motion

$$\int_{\Omega} \tilde{\varrho} \ \mathrm{d}x = M_0,\ \int_{\Omega} \tilde{\varrho} e(\tilde{\varrho}, \Theta) - \tilde{\varrho} F \ \mathrm{d}x = E_0$$

Total dissipation balance revisited

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(\tilde{\varrho}, \Theta)}{\partial \varrho} (\varrho - \tilde{\varrho}) - H_\Theta(\tilde{\varrho}, \Theta) \right) \ \mathrm{d}x$$

$$+ \int_\Omega \sigma \ \mathrm{d}x = 0$$

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho \Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \Big)$$

Coercivity of the ballistic free energy

$$\partial^2_{\varrho,\varrho} H_\Theta(\varrho, \Theta) = \frac{1}{\varrho}\partial_\varrho p(\varrho, \Theta)$$

$$\partial_\vartheta H_\Theta(\varrho, \vartheta) = \varrho(\vartheta - \Theta)\partial_\vartheta s(\varrho, \vartheta)$$

Coercivity

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ is convex}$$

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ attains its global minimum (zero) at } \vartheta = \Theta$$

Relative entropy

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)$$
$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Applications

Existence

Dissipative (weak) solutions exist (under certain constitutive restrictions) globally in time for any choice of the initial data.

Unconditional stability of the equilibrium solutions

Any (weak) solution of the Navier-Stokes-Fourier system stabilizes to an equilibrium (static) solution for $t \to \infty$.

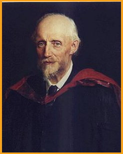
Weak-strong uniqueness

Weak and strong solutions emanating from the same initial data coincide as long as the latter exists. Strong solutions are unique in the class of weak solutions.

Singular limit in the incompressible, inviscid regime

Solutions of the Navier-Stokes-Fourier system converge in the limit of low Mach and high Reynolds and Péclet number to the Euler-Boussinesq system.

Ernst Mach [1838-1916]



Osborne Reynolds [1842-1912]

Jean Claude Eugène Péclet [1793-1857]

Scaled Navier-Stokes-Fourier system

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Balance of momentum

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta)} = \boxed{\varepsilon^a} \mathrm{div}_x \mathbb{S}$$

Entropy production

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \boxed{\varepsilon^b} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+a}} \mathbb{S} : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Target system

INCOMPRESSIBILITY

$$\mathrm{div}_x \mathbf{v} = 0$$

EULER SYSTEM

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0$$

TEMPERATURE TRANSPORT

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0$$

Basic assumption

The incompressible Euler system possesses a strong solution $\mathbf{v}$ *on a time interval* $(0, T_{\rm max})$ *for the initial data* $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$.

Prepared data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3),\ \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] \in W^{k,2}(\Omega; R^3),\ k > \frac{5}{2}$$

Boundary conditions

Navier's complete slip condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Convergence

$$b > 0, \ 0 < a < \frac{10}{3}$$

$$\mathrm{ess} \sup_{t \in (0,T)} \| \varrho_\varepsilon(t,\cdot) - \overline{\varrho} \|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon c$$

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \ \mathbf{v} \ \text{in} \ \boxed{L^\infty_{\rm loc}((0,T]; L^2_{\rm loc}(\Omega; R^3))}$$

and weakly-(*) in $L^\infty(0,T; L^2(\Omega; R^3))$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T \ \text{in} \ \boxed{L^\infty_{\rm loc}((0,T]; L^q_{\rm loc}(\Omega; R^3)), \ 1 \leq q < 2},$$

and weakly-(*) in $L^\infty(0,T; L^2(\Omega))$

Uniform bounds

The uniform bounds independent of ε are obtained by means of the choice

$$r = \overline{\varrho},\ \Theta = \overline{\vartheta},\ \mathbf{U} = 0$$

in the relative entropy inequality:

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2 + L^{5/3}(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \sqrt{\varrho} \mathbf{u}_\varepsilon \right\|_{L^2(\Omega; R^3)} \leq c$$

Stability

MAIN IDEA OF THE PROOF

Take

$$r_\varepsilon = \overline{\varrho} + \varepsilon R_\varepsilon, \ \Theta_\varepsilon = \overline{\vartheta} + \varepsilon T_\varepsilon, \ \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality

Acoustic equation

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta T_\varepsilon) = 0$$

Transport equation

$$\partial_t (\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0$$

Dispersion of acoustic waves

$-\Delta_N$ Neumann Laplacian

$$\partial^2_{t,t}\Phi - \omega\Delta_N\Phi = 0$$

Hypotheses imposed on Ω

- Limiting absorption principle. The operator Δ_N satisfies the limiting absorption principle in Ω:

$$\varphi \circ [-\Delta_N^{-1} - \lambda \pm \mathrm{i}\delta]^{-1} \circ \varphi,\ \varphi \in C_c^\infty(\Omega) \text{ bounded in } L^2(\Omega)$$

  for λ belonging to compact subintervals of $(0,\infty)$, $\delta > 0$.
- There is a compact set B such that Δ_N satisfies the Strichartz estimates on $D = \Omega \cup B$.
- The operator Δ_N satisfies the local energy decay.

Strichartz estimates and local energy decay

$$\|\Phi\|_{L^p(R;L^q(D))} \leq c\left(\|\Phi(0)\|_{H^\gamma(D)} + \|\partial_t \Phi(0)\|_{H^{\gamma-1}(D)} \right)$$

$$2 \leq q < \infty, \ \frac{2}{p} \leq \left(1 - \frac{2}{q} \right), \ \gamma = \frac{3}{2} - \frac{3}{q} - \frac{1}{p}$$

$$\int_{-\infty}^{\infty} \left(\|\chi \Phi(t,\cdot)\|^2_{H^\gamma(D)} + \|\chi \partial_t \Phi(t,\cdot)\|^2_{H^{\gamma-1}(D)} \right) \ \mathrm{d}t$$
$$\leq c\left(\|\Phi(0)\|_{H^\gamma(D)} + \|\partial_t \Phi(0)\|_{H^{\gamma-1}(D)} \right),$$

$\chi \in C_c^\infty(D)$.