MOTION OF MULTIPLE CYLINDERS IN POTENTIAL FLOW OF IDEAL FLUID

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Introduction

Flow past multiple cylinders moving in ideal incompressible fluid is investigated. Previous approaches dealing with multiple cylinders are presented in Pashaev and Yilmaz [1] and Crowdy [2, 3]. Pashaev and Yilmaz [1] deal with a point vortex with more than two fixed cylinders at arbitrary positions. Crowdy [2, 3] developed a general method to calculate a uniform potential flow past a stationary multi-cylinder configuration. His solution uses complex analysis and involves the Schottky-Klein prime function.

The present contribution generalises the images method that has been used previously for calculation of flow past two cylinders only. The principles of the images method are described in Milne-Thompson [4]. This method is based on the construction of an image of a dipole within a cylinder. Let us consider a cylinder with centre at point C and a dipole at point D of strength μ, see Figure 1. An image of dipole D within the cylinder is a dipole at the inverse point D', $ff' = a^2$. The image dipole D' is oriented in the mirror direction to D and attains strength $\mu' = \mu a^2/f'^2$. In the resulting flow past the dipole and its image the surface of the cylinder represents a streamline.

The problem of motion of two cylinders due to its linearity is usually split into two sub-cases: the motion of the first cylinder when the second one is immobile and the motion of the second one with an immobile first cylinder. The flow potential in the first sub-case can be expressed as a potential of an infinite sequence of dipoles. The first dipole is located in the centre of the first cylinder and corresponds to the motion of the first cylinder in an unbounded volume of fluid. This results in violation of the condition of impermeability on the surface of the second, immobile cylinder. To restore the impermeability conditions an image of the first dipole is introduced in the second cylinder – the second dipole. However, the flow generated by the second dipole reciprocally violates the boundary conditions on the surface of the first cylinder. For their recovery an image of the second dipole is introduced in the first cylinder – the third dipole. The next dipoles are introduced similarly in an alternate way. The strengths of the dipoles diminish and their total potential converges to the resultant flow potential. The second sub-case, when the first cylinder is immobile and the second one is moving, is solved analogously.

Method

With an increasing number ($N > 2$) of cylinders moving in parallel in an unbounded fluid the images method used is still more complicated as each introduced dipole simultaneously violates the boundary conditions on more than one cylinder. Let N cylinders of radii a_i, $1 \leq i \leq N$, located at the points with radius vectors $\vec{r}_i$ move in an ideal fluid with the velocities $\vec{v}_i$.

The solution of this problem is given by the potential of an infinite sequence of groups of dipoles. The first group consists of dipoles located in the centres of cylinders and having

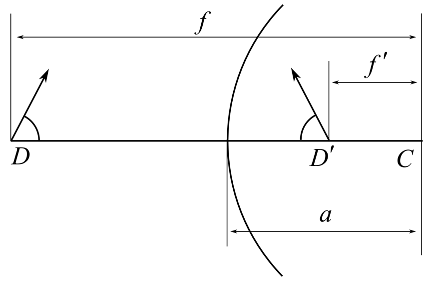

Figure 1: Image of a dipole within a cylinder.

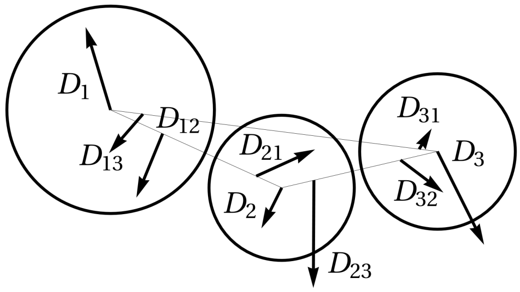

Figure 2: First D_i and second D_{ij} groups of dipoles, $N = 3$. Dipoles are denoted by the arrows, their lengths are proportional to the strengths of dipoles (for good orientation the lengths of arrows of the second group are magnified five times).

potentials

$$\varphi_i(\vec{r}) = -\frac{a_i^2 \vec{v}_i \cdot (\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^2}, \quad 1 \le i \le N, \quad N > 2. \tag{1}$$

The dipoles of the first group correspond to the motions of each cylinder in the absence of the others, their strengths equal $\mu_i = a_i^2 v_i$.

Let us take into account one of the cylinders. The dipoles belonging to the first group, which are located in other cylinders, violate the boundary condition on the wall of the chosen cylinder. Analogously the same is valid for each of the cylinders. To remove this adverse action of the dipoles of the first group the second group is introduced consisting of the images in each cylinder of the dipoles of the first group located in other cylinders, for illustration see figure 2.

For simplification of notation an operator of imaging in cylinder i is introduced, A_i , acting on potential of a dipole of strength μ_0, located at point $\vec{r}_0$ and directed along unit vector $\vec{e}_0$

$$\mathrm{A}_i\left(\varphi_0\left(\vec{r}\right)\right) = \mathrm{A}_i\left(\frac{\mu_0 \vec{e}_0 \cdot (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^2}\right) =$$

$$\begin{cases} \frac{\mu_0 a_i^2}{|\vec{r}_i - \vec{r}_0|^2}\left(\vec{e}_0 - 2(\vec{r}_i - \vec{r}_0)\frac{\vec{e}_0 \cdot (\vec{r}_i - \vec{r}_0)}{|\vec{r}_i - \vec{r}_0|^2}\right) \cdot \\ \frac{\vec{r} - \vec{r}_i + (\vec{r}_i - \vec{r}_0) a_i^2 / |\vec{r}_i - \vec{r}_0|^2}{|\vec{r} - \vec{r}_i + (\vec{r}_i - \vec{r}_0) a_i^2 / |\vec{r}_i - \vec{r}_0|^2|^2}, & |\vec{r}_i - \vec{r}_0| > a_i, \\ 0, & |\vec{r}_i - \vec{r}_0| < a_i. \end{cases} \tag{2}$$

Then potentials of the dipoles of the second group are denoted

$$\varphi_{ij} = \mathrm{A}_i\left(\varphi_j\right), \quad 1 \le i, j \le N. \tag{3}$$

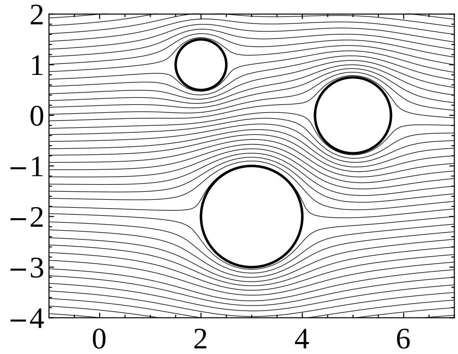

Figure 3: Streamlines for uniform flow past three stationary cylinders.

Now, among all dipoles only the ones of the second group violate boundary conditions on the cylinders. The third group of dipoles removes this action of the second group. Potentials of the dipoles of the third group are calculated as

$$\varphi_{ijk} = \mathrm{A}_i \left(\mathrm{A}_j \left(\varphi_k \right) \right), \quad 1 \le i, j, k \le N. \tag{4}$$

Potentials of the consecutive groups of dipoles are determined analogously. Each consecutive group in the generalised sense represents an image of the previous group. In total the first M groups contain $N\left((N-1)^M - 1\right) / (N-2)$ dipoles. The resulting flow potential is a sum of potentials of all groups

$$\varphi = \sum_i \varphi_i + \sum_{i,j} \varphi_{ij} + \sum_{i,j,k} \varphi_{ijk} + \dots, \quad 1 \le i, j, k, \dots \le N. \tag{5}$$

Comparison

The applicability of the approach presented is checked using two flow visualisations. The first example is taken from Crowdy [2] where streamlines of a uniform flow are calculated and plotted past three stationary cylinders. Figure 3 depicts this flow pattern calculated by the present method. For calculation there were used 9 groups of dipoles.

The second example is taken from Finn et al. [5], see Figure 4. For calculation there were used 7 groups of dipoles. Positions, radii and directions of motion correspond to those in Figure 2 in Finn et al. [5]. The authors did not specify the exact velocities of the cylinders (except for one that is motionless), hence the velocities equal in absolute value were chosen. Though the present flow lacks for the outer cylinder presented in [5], some flow features, such as stagnation points and flow between the cylinders are determined.

Conclusions

The present method is a natural generalisation of the conventional method of images. It is suitable for calculation of the potential flow past several arbitrarily moving parallel circular cylinders. The method uses groups of dipoles for the consecutive images. The potentials of the dipoles of each group are determined with use of the imaging operator from potentials of the dipoles of the preceding group. In the case of two cylinders the generalised images method projects to the classical one and the first M groups contain $2M$ dipoles.

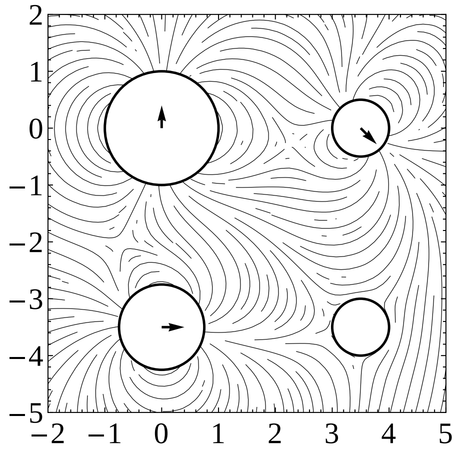

Figure 4: Streamlines for flow past four moving cylinders.

The method addresses a problem considered also in [2, 3]. The present method is less demanding for mathematical background and thus is simpler for engineering calculations. Nevertheless, the flow patterns obtained by both methods compare favourably.

Acknowledgements
This work has been supported by the Institutional Research Plan AV0Z20600510.

References

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