1. Preliminaries and terminology

1.1. Line integral and its geometrical meaning

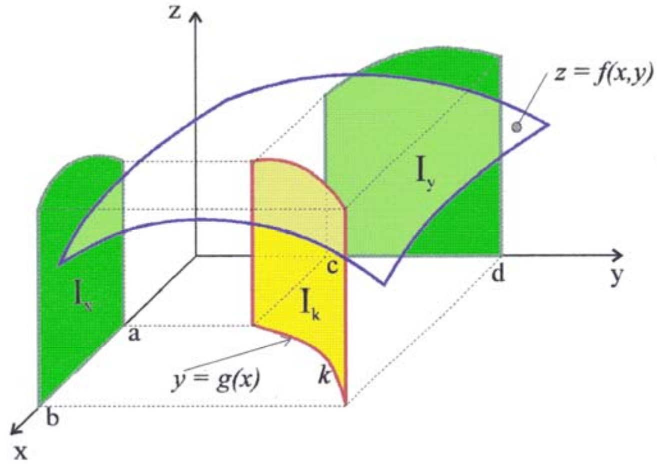

Fig. 1.

The planar curve k is described by the function $y = g(x)$, or by its inverse $x = \bar{g}(y)$.

$$I_k = \int_k f(x,y)\,\mathrm{d}s\,.$$ See Fig. 1.

$$I_x = \int_k f(x,y)\,\mathrm{d}x = \int_a^b f(x,g(x))\,\mathrm{d}x\,.$$ Projection into (xz) plane.

$$I_y = \int_k f(x,y)\,\mathrm{d}y = \int_c^d f(g(y),y)\,\mathrm{d}y\,.$$ Projection into (yz) plane.

Example 1. Show relations between the above integrals

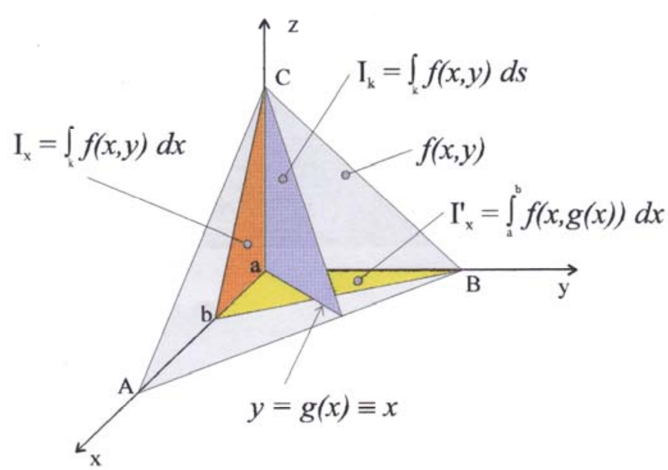

Fig. 2.

Let the function $f(x, y)$ is defined as follows

$$z = f(x, y) = ax + by + c\,.$$

From the 'boundary' conditions we get

$$A:\quad 0 = a + 0 + c$$

$$B:\quad 0 = 0 + b + c \qquad\qquad \Rightarrow \quad c = 1; \quad b = -1; \quad a = -1\ .$$

$$C:\quad 1 = 0 + 0 + c$$

Projection I_k to (x,z): $I_x = \int_k f(x, y)\,\mathrm{d}x$.

$$I_x = I'_x \qquad\qquad \text{see Fig. 2}$$

$$F(x, y) = f(x, g(x)) = -x - x + 1 = -2x + 1,$$

$$I_x = \int_k f(x, y)\,\mathrm{d}x = \int_a^b f(x, g(x))\,\mathrm{d}x = I'_x\,.$$

Example 2. Calculate the integrals for the following functions.

$$f(x, y) = \sqrt{1 - (x^2 + y^2)}\,,$$

$$g(x) = x\,,$$

$$F(x) = \sqrt{1 - 2x^2}\,.$$

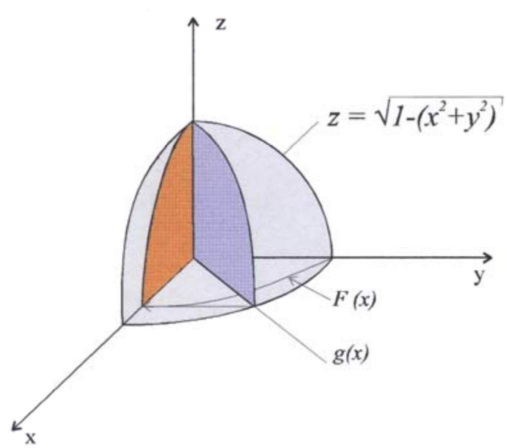

Fig. 3

1.2. Gradient

Consider a two-dimensional quantity Φ constituting a scalar field depending on x, y

$\Phi = \Phi(x, y)$.

The total differential shows how Φ differs with position

$$\mathrm{d}\Phi = \frac{\partial \Phi}{\partial x}\,\mathrm{d}x + \frac{\partial \Phi}{\partial y}\,\mathrm{d}y\,.$$

Let's introduce column vectors

$$\{\nabla\Phi\} = \begin{Bmatrix} \dfrac{\partial \Phi}{\partial x} \\ \dfrac{\partial \Phi}{\partial y} \end{Bmatrix},\ \{\mathrm{d}x\} = \begin{Bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{Bmatrix}.$$

The former vector is called the gradient of a scalar field $\Phi(x, y)$. Then, the total differential can be rewritten in the form of a scalar product

$$\mathrm{d}\Phi = \{\nabla\Phi\}^{\mathrm{T}}\,\{\mathrm{d}x\},$$

for which we can write

$$\mathrm{d}\Phi = \left|\{\nabla\Phi\}\right|\,\left|\{\mathrm{d}x\}\right|\,\cos\Theta\,.$$

The gradient is orthogonal to contour lines since if $\Phi = const.$, then $\mathrm{d}\Phi = 0$ and $\left|\{\nabla\Phi\}\right|\left|\{\mathrm{d}x\}\right|\cos\Theta = 0$ only if $\cos\Theta = 0$, which we have for $\Theta = \dfrac{\pi}{2}$.

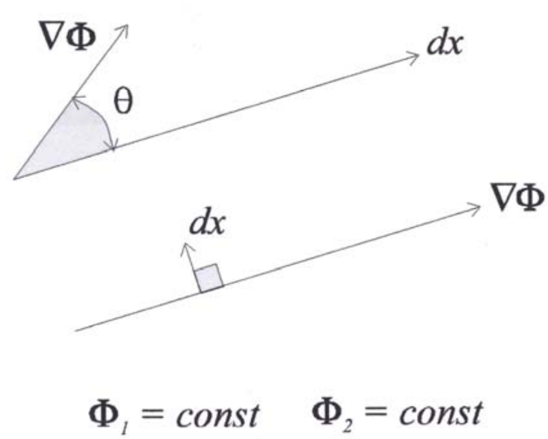

Fig. 4

1.3 Gauss Divergence Theorem

For functions $\Phi = \Phi(x, y),\ \ \Psi = \Psi(x, y)$ defined over the area A, consider the integrals

$$\int_A \frac{\partial \Psi(x, y)}{\partial x}\mathrm{d}A, \qquad \int_A \frac{\partial \Phi(x, y)}{\partial y}\mathrm{d}A\,.$$

Take the second one first in the counterclockwise direction along both boundaries.

$$\int_A \frac{\partial \Phi}{\partial y}\,\mathrm{d}A = \iint_A \frac{\partial \Phi}{\partial y}\,\mathrm{d}x\mathrm{d}y = \int_a^b \int_{f_1(x)}^{f_2(x)} \left(\frac{\partial \Phi}{\partial y}\mathrm{d}y \right) \mathrm{d}x = \int_a^b \left[\Phi(x,y)\right]_{f_1(x)}^{f_2(x)}\,\mathrm{d}x =$$

$$\int_a^b \left[\Phi(x, f_2(x)) - \Phi(x, f_1(x))\right] \mathrm{d}x =$$

$$= \int_a^b \Phi(x, f_2(x))\,\mathrm{d}x - \int_a^b \Phi(x, f_1(x))\,\mathrm{d}x = -\int_a^b (x, f_2(x))\,\mathrm{d}x - \int_a^b \Phi(x, f_1(x))\,\mathrm{d}x$$

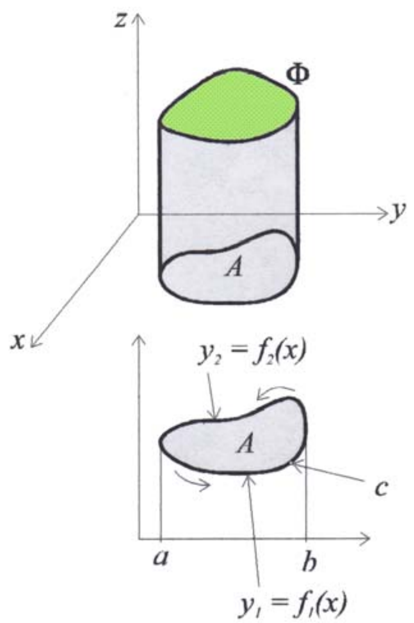

Fig. 5

We can conclude that $\displaystyle\int_A \frac{\partial \Phi(x,y)}{\partial y}\,\mathrm{d}A = -\oint_c \Phi(x,y)\,\mathrm{d}x$. (1.1)

Let the curve c is defined by two functions f_2 and f_1 respectively. Similarly for the other coordinate and for the other function $\Psi = \Psi(x,y)$ we get

$$\int_A \frac{\partial \Psi(x,y)}{\partial x}\,\mathrm{d}A = +\oint_c \Psi(x,y)\,\mathrm{d}y\,. \tag{1.2}$$

Again in the counterclockwise direction.

The equations (1.1), (1.2) form the Green theorem in the simplest case. Sometimes also called the Green-Ostrogradsky theorem or the theorem of Green-Gauss.

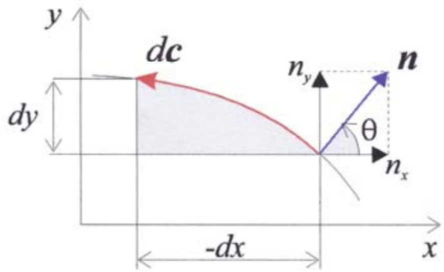

Fig. 6

Let us rewrite the equations (1.1), (1.2) by introducing vectors $\mathrm{d}\vec{c}$ and $\vec{n}$, such in such a way that

$$\vec{n} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}, \qquad |\vec{n}| = \sqrt{n_x^2 + n_y^2} = 1.$$

$$|\mathrm{d}\vec{c}| = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}\,,$$

$$|\mathrm{d}\vec{c}| = |\mathrm{d}x|\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = |\mathrm{d}y|\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \qquad \text{if } \mathrm{d}x \neq 0 \text{ or } \mathrm{d}y \neq 0.$$

Observing triangles in the previous figure one can write

$$\sin\Theta = \frac{-\,\mathrm{d}x}{|\mathrm{d}\vec{c}|} = \frac{n_y}{|\vec{n}|} \text{ (due to counterclockwise orientation of the curve),}$$

$$\cos\Theta = \frac{\mathrm{d}y}{|\mathrm{d}\vec{c}|} = \frac{n_x}{|\vec{n}|}, \text{ where } |\vec{n}| = 1.$$

From it follows

$$\mathrm{d}x = -n_y|\mathrm{d}\vec{c}|, \qquad \mathrm{d}y = n_x|\mathrm{d}\vec{c}|. \tag{1.3}$$

Substituting (1.3) into (1.1), (1.2), we get

$$\int_A \frac{\partial \Psi(x,y)}{\partial x}\,\mathrm{d}A = \oint_c \Psi(x,y) n_x\,\mathrm{d}c\,, \tag{1.4}$$

$$\int_A \frac{\partial \Phi(x,y)}{\partial y}\,\mathrm{d}A = \oint_c \Phi(x,y) n_y\,\mathrm{d}c\,. \tag{1.5}$$

Now define an arbitrary vector $\vec{q}$ by

$$\{q\} = \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = \begin{Bmatrix} \Psi \\ \Phi \end{Bmatrix}. \tag{1.6}$$

Adding (1.4), (1.5) and using (1.6) we get

$$\int_A \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right) \mathrm{d}A = \oint_c \left(q_x n_x + q_y n_y \right) \mathrm{d}c \,.$$

The divergence is defined

$$\mathrm{div}\,\vec{q} = \mathrm{div}\{q\} = \frac{\partial q_i}{\partial x_i} \,. \tag{1.7}$$

The scalar product can be expressed

$$\{q\}^{\mathrm{T}} \{n\} = q_i n_i \,.$$

So we can finally write

$$\int_A \mathrm{div}\{q\} \; \mathrm{d}A = \oint_c \; \{q\}^{\mathrm{T}} \{n\} \mathrm{d}c \,. \tag{1.8}$$

This is a so called **divergence theorem of Gauss**.

The Gauss-Green theorem can be seen as a two-dimensional counterpart of the integration by parts

$$\int u\, v' = u\, v - \int u' v \,. \tag{1.9}$$

The Gauss divergence theorem could be found in literature in different forms.

$$\int_V \mathrm{div}\{q\} \, \mathrm{d}V = \int_S \{q\}^{\mathrm{T}} \; \{n\} \, \mathrm{d}S \,. \tag{1.10}$$

The equivalent notations are as follows

$$\int_V \{\nabla\}^{\mathrm{T}} \{q\} \, \mathrm{d}V = \int_S \{q\}^{\mathrm{T}} \{n\} \, \mathrm{d}S \,, \quad \{\nabla\}^{\mathrm{T}} = \left\{ \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3} \right\} . \tag{1.11}$$

$$\int_V \mathrm{div}\,\vec{q} \; \mathrm{d}V = \int_S \vec{q} . \vec{n} \; \mathrm{d}S \,,$$

$$\int_V \frac{\partial q_i}{\partial x_i} \, \mathrm{d}V = \int_S q_i \, n_i \; \mathrm{d}S \,, \tag{1.12}$$

$$\int_V \frac{\partial f}{\partial x_i} \, \mathrm{d}V = \int_S f \, n_i \; \mathrm{d}S \,.$$

The Gauss divergence theorem for a tensor quantity is defined as follows

$$\int_V \frac{\partial T_{ij}}{\partial x_i} \, \mathrm{d}V = \int_S n_i \, T_{ij} \; \mathrm{d}S \,. \tag{1.13}$$

So the Green theorem represents the transformation of a volume integral into a surface integral (or vice versa) for quantities associated with a considered body having the volume V, bounded by the surface S. The outward normal n_i is defined at each point of the surface. The function appearing under the integral sign is real valued with the first continuous derivative within the body.

1.4. The generalization of 'per partes' integration (integration by parts)

According to Green divergence theorem we can write

$$\int_V \frac{\partial}{\partial x_i}(uv)\,\mathrm{d}V = \int_S (uv)\,n_i\,\mathrm{d}S\,. \tag{1.14}$$

The left hand side could be rewritten

$$\int_V \frac{\partial}{\partial x_i}(uv)\,\mathrm{d}V = \int_V \frac{\partial u}{\partial x_i}\,\mathrm{d}V + \int_V u\frac{\partial v}{\partial x_i}\,\mathrm{d}V\,. \tag{1.15}$$

Equalling the right hand sides of the last two equations and rearranging gives a formula

$$\int_V u\frac{\partial v}{\partial x_i}\,\mathrm{d}V = \int_S (uv)n_i\,\mathrm{d}S - \int_V \frac{\partial u}{\partial x_i}v\,\mathrm{d}V\,, \tag{1.16}$$

which reminds the integration by parts

$$\int_a^b uv'\,\mathrm{d}x = [u\,v]_a^b - \int_a^b u'v\,\mathrm{d}x\,. \tag{1.17}$$

It is of interest to remind the **Stokes theorem** which transforms the integral over the closed curve in space to the surface integral

$$\int_S \vec{n}.\left(\vec{\nabla}\times\vec{q}\right)\mathrm{d}S = \int_c \vec{t}.\vec{q}\,\mathrm{d}c\,, \tag{1.18}$$

$$\mathrm{curl}\,\vec{q} = \mathrm{rot}\,\vec{q} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ q_x & q_y & q_z \end{vmatrix}.$$

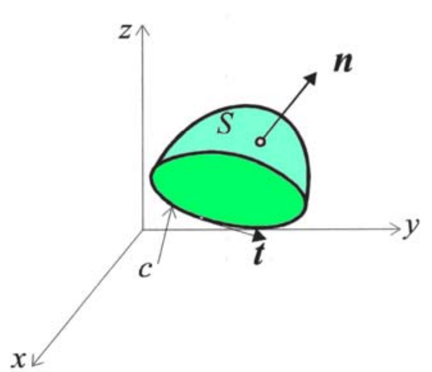

Fig. 7

1.5. Flux

Imagine a surface S in space and the continuum flowing by the velocity $\vec{v}$ through it. The volume of material flowing through $\mathrm{d}S$ in time $\mathrm{d}t$ is

$$\vec{v}\,.\,\vec{n}\ \mathrm{d}t\ \mathrm{d}S\,, \qquad \left[\mathrm{m}^3\right] \ldots \left[\mathrm{m/s}\right]\left[1\right]\left[\mathrm{s}\right]\left[\mathrm{m}^2\right].$$

The volume flux is defined

$$\int_S \vec{v}.\vec{n}\ \mathrm{d}S = \int_S v_i\, n_i\ \mathrm{d}S\,,\ \ \left[\mathrm{m}^3/\mathrm{s}\right]. \tag{1.19}$$

Similarly the mass flux is $\int_S \rho\ v_i\, n_i\ \mathrm{d}S\,,\left[\mathrm{kg}/\mathrm{s}\right].$ (1.20)

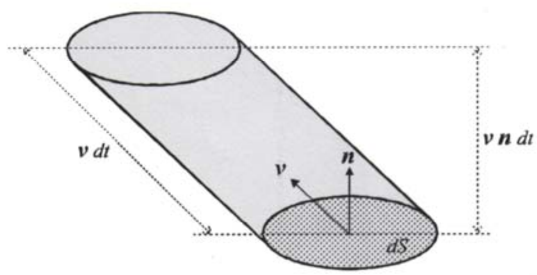

Fig. 8

Kinetic energy flux corresponds to $\frac{1}{2}mv^2$, so

$$\int_S \frac{1}{2}\rho\, v^2\left(\vec{v}.\,\vec{n}\right)\mathrm{d}S\,, \tag{1.21}$$

Dimensional check

$$\frac{\mathrm{m}^2}{\mathrm{s}^2}\,\frac{\mathrm{kg}}{\mathrm{s}} = \frac{\mathrm{kgm}^2}{\mathrm{s}^3} = \frac{\mathrm{kgm}}{\mathrm{s}^2}\,\mathrm{m}\,\frac{1}{\mathrm{s}} = \mathrm{Nm}\frac{1}{\mathrm{s}} = \left[\frac{1}{s}\right] \quad .$$

Generally

The flux of Φ through S is $\int_S \rho\,\Phi\, v_i\, n_i\ \mathrm{d}S\,,$ (1.22)

where Φ is a quantity (defined per unit mass) which is associated with particles.

Remark

Often we encounter a so called oriented surface defined by

$$d\vec{A} = \mathrm{d}\vec{S}\,\vec{n} \qquad \text{or} \qquad \mathrm{d}A_i = n_i\,\mathrm{d}S\,. \tag{1.23}$$

1.6. Material derivative

Imagine that the motion of a particle is defined by the *material description* in the form

$$x = x(a,t). \tag{1.24}$$

The particle velocity could simply be calculated by taking the partial time derivative with a variable a held constant

$$v = v(a,t) = \left(\frac{\partial x}{\partial t}\right)_a = \frac{\partial}{\partial t}x(a,t). \tag{1.25}$$

Similarly, the acceleration is

$$z = \left(\frac{\partial v}{\partial t}\right)_a = \frac{\partial}{\partial t}v(a,t) = \frac{\partial^2}{\partial t^2}x(a,t) = \dot{v}\,. \tag{1.26}$$

These are examples of material time derivatives in material description. Notice the different types of notation.

The material time derivative may be thought of as a time rate of change that would be masured by an observer travelling with the specific particle under study. The same physical phenomenon could be described by the spatial description. The velocity fields is

$$v = v(x,t). \tag{1.27}$$

Note

The studied phenomenon is supposed to be the same regardless of the formulation being applied, so we are tempted to use the same symbol for the variable describing it, even if it is defined by a different function. There are authors using different symbols for the same variables in material and spatial descriptions respectively.

The derivative, with spatial coordinate x held constant

$$\left(\frac{\partial v}{\partial t}\right)_x = \frac{\partial}{\partial t}v(x,t) \tag{1.28}$$

is called the local rate of change of v. It is the rate of change of an ideal velocity meter located at the fixed place x. This is not the same thing as the acceleration of the particle passing the place x just now.

Remark

For example, in a steady state flow the local rate of change is zero everywhere. This does not imply, however, that the acceleration of all particles is zero everywhere. Even in a steady state flow the velocity varies in general from point to point and a particle changes its velocity as it moves from one place of constant velocity to another place, having a different constant velocity.

If we want to calculate the particle acceleration from knowledge of spatial velocity description $v(x,t)$ we have to employ the chain rule of calculus

$$\left(\frac{\partial v}{\partial t}\right)_a = \left(\frac{\partial v}{\partial t}\right)_x + \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right)_a . \tag{1.29}$$

Since $\left(\frac{\partial x}{\partial t}\right)_a = v$ we can finally write

$$z = \left(\frac{\partial v}{\partial t}\right)_a = \left(\frac{\partial v}{\partial t}\right)_x + v\left(\frac{\partial v}{\partial x}\right). \tag{1.30}$$

Another name for the material derivative is the substantial derivative. There are other notations used in textbooks and references, as

$$\frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\mathrm{d}v}{\mathrm{d}t} = \dot{v} = \left(\frac{\partial v}{\partial t}\right)_a . \tag{1.31}$$

In vector notation we can write

$$\vec{z} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\partial \vec{v}}{\partial t} + \vec{v}.\nabla\vec{v} = \frac{\partial \vec{v}}{\partial t} + \vec{v}.\mathrm{grad}\,\vec{v}, \tag{1.32}$$

where $\nabla = \mathrm{grad} = \left\{\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right\}^{\mathrm{T}}$.

For any scalar Φ, vector $\vec{u}$ or tensor T quantities, the formulas of material derivatives are as follows

$$\dot{\Phi} = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\partial \Phi}{\partial t} + \vec{v}\,.\,\nabla f = \frac{\partial \Phi}{\partial t} + v_k \frac{\partial \Phi}{\partial x_k},$$

$$\{\dot{u}\} = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial \vec{u}}{\partial t} + \vec{v}\,.\,\nabla\vec{u} = \frac{\partial \vec{u}}{\partial t} + \vec{v}\,.\,\mathrm{grad}\,\vec{u},$$

$$\dot{u}_i = \frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial u_i}{\partial t} + v_k \frac{\partial u_i}{\partial x_k},$$

$$\dot{T}_{ij} = \frac{\mathrm{d}T_{ij}}{\mathrm{d}t} = \frac{\partial T_{ij}}{\partial t} + v_k \frac{\partial T_{ij}}{\partial x_k}.$$

2. Conservation laws

2.1. Conservation of mass

Assume that continuum with density ρ fills the volume V, bounded by surface S. The total mass contained in V is

$$m = \int_V \rho \, \mathrm{d}V . \tag{2.1}$$

It is assumed that the density $\rho = \rho(x_i, t)$ is a continuous function of space and time coordinates and that there is no flux through the surface S.

The mass of the considered body at configuration ${}^0C$ is equal to that at configuration ${}^tC$, i.e.

$$\int_{{}^0V} {}^0\rho \, \mathrm{d}{}^0V = \int_{{}^tV} {}^t\rho \, \mathrm{d}{}^tV , \tag{2.2}$$

$$\int_{{}^0V} \rho(a_j, t_0) \, \mathrm{d}{}^0V = \int_{{}^tV} \rho(x_j, t) \, \mathrm{d}{}^tV . \tag{2.3}$$

Realizing that $x_i = x_i(a_j, t)$ and substituting after the integral sign allows rewriting the right hand side of (2.3) in the form

$$\int_{{}^tV} \rho(x_j, t) \, \mathrm{d}{}^tV = \int_{{}^0V} \rho(x_i(a_j, t)) |J| \, \mathrm{d}{}^0V , \tag{2.4}$$

where the Jacobian of the transformation is the determinant of the deformation gradient

$$J = \det F_{ij} , \qquad F_{ij} = \frac{\partial x_i}{\partial a_j} . \tag{2.5}$$

Equations (2.3) and (2.4), written in short, give

$$\int_{{}^0V} {}^0\rho \, \mathrm{d}\,{}^0V = \int_{{}^0V} {}^t\rho \, |J| \, \mathrm{d}\,{}^0V . \tag{2.6}$$

Since the last equation must be valid for an arbitrary volume we can write

$${}^0\rho = {}^t\rho \, |J| . \tag{2.7}$$

But $|J| = J$, since $J > 0$.

Proof

The continuum in ${}^0V$ completely fills the space. The initial density is ${}^0\rho > 0$. At the initial configuration ${}^0C$, there is no deformation, so $\mathbf{F} = \mathbf{I}$ and consequently $J = 1$ which is greater than zero. The value of $J < 0$ in the process of deformation would mean that at a certain time $\in (t_0, t)$ the value of the Jacobian would become $J = 0$. For such a case there would be no one-to-one correspondence

$$x_i = x_i(a_j, t) \Leftrightarrow a_i = a_i(x_j, t) \tag{2.8}$$

which is a contradiction with initial assumption about the physical acceptability of deformation description.

The consequence of the conservation of mass is known as the continuity equation.

2.2. Lagrangian (material) form of the continuity equation can be written in different forms

$$\rho J = {}^0\rho = const\,, \text{ where } J = \det F_{ij}\,, \qquad F_{ij} = \frac{\partial x_i}{\partial a_j}\,, \tag{2.9a}$$

or $$\frac{\mathrm{D}}{\mathrm{D}t}(\rho J) = 0\,, \tag{2.9b}$$

or $${}^0\rho\, \mathrm{d}^0V = {}^t\rho\, \mathrm{d}^tV = {}^t\rho J\, \mathrm{d}^0V = const\,, \tag{2.9c}$$

or $$\frac{{}^0\rho}{{}^t\rho} = \frac{\mathrm{d}^tV}{\mathrm{d}^0V} = J = \det F_{ij}\,. \tag{2.9d}$$

Remark 1

Remember that the condition $J \neq 0$ *is necessary for the equivalence of material and spatial descriptions* $x_i = x_i(a_j, t) \Leftrightarrow a_i = a_i(x_j, t)$.

If the Jacobian of the above transformations $J = \det F_{ij} = 0$ *then the inverse function of* $x_i = x_i(a_j, t)$ *does not exist.*

Also, if $\det F_{ij} = 0$*, then, either* ${}^0\rho = 0$*, or* ${}^t\rho \to +\infty$.

Remark 2

Remember that $\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathrm{d}\mathbf{a}$*,* $\mathrm{d}\mathbf{a} = \mathbf{F}^{-1}\,\mathrm{d}\mathbf{x}$. *If* $\det\mathbf{F} = 0$*, then* $\mathbf{F}^{-1}$ *cannot be computed.*

2.3. Eulerian (spatial) form of the continuity equation

Again the total mass of a continuous medium of density ρ filling the volume V at time t is

$$M = \int_V \rho\, \mathrm{d}V\,.$$

The time rate of increase of the total mass in the volume V is

$$\frac{\partial M}{\partial t} = \int_V \frac{\partial \rho}{\partial t}\, \mathrm{d}V\,.$$

We assume that no mass is created inside the volume V. Then the time rate of mass must be equal to the rate of flux of the mass through the surface.

flux = rate of mass outflow = $\int_S \rho \vec{v} . \vec{n} \, \mathrm{d}S$

rate of mass inflow = $-\int_S \rho \vec{v} . \vec{n} \, \mathrm{d}S = -\int_V \mathrm{div}\left(\rho \vec{v}\right) \mathrm{d}V$

So $$\int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V = -\int_V \mathrm{div}\left(\rho \vec{\mathrm{v}}\right) \mathrm{d}V \, ,$$

or $$\int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V = -\int \left\{\nabla\right\}^{\mathrm{T}} \left\{ \overrightarrow{\rho v} \right\} \mathrm{d}V \, , \text{ where } \left\{\nabla\right\}^{\mathrm{T}} = \left\{ \frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial z} \right\}.$$

From it follows

$$\int \left(\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho \vec{\mathrm{v}}\right) \right) \mathrm{d}V = 0 \, .$$

This equation must be valid for any volume, so

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho \vec{\mathrm{v}}\right) = 0 \, ,$$

or $$\frac{\partial \rho}{\partial t} + \left\{\nabla\right\}^{\mathrm{T}} \left\{\rho v\right\} = 0 \, , \qquad (2.10)$$

or $$\frac{\partial \rho}{\partial t} + \frac{\partial\left(\rho v_i\right)}{\partial x_i} = 0$$

There are different forms of continuity equation in spatial description. The last equation could be rewritten using the rule for the derivative of a product

$$\frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0 \, . \qquad (2.11)$$

Realizing that the material derivative of density is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} \, .$$

The equation (2.11) could be simplified into

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \frac{\partial v_i}{\partial x_i} = 0 \, . \qquad (2.12a)$$

The equivalent formulas are

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \, \mathrm{div} \vec{v} = 0 \, , \qquad (2.12b)$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0, \tag{2.12c}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho D_{ii} = 0. \tag{2.12d}$$

Remark 1

The velocity gradient is $L_{ij} = \dfrac{\partial v_i}{\partial x_i}$*, the strain rate is its symmetrical part* $D_{ij} = \frac{1}{2}\left(L_{ij} + L_{ji}\right)$*. From it follows that* $L_{ii} = D_{ii}$.

Remark 2

If the material is incompressible, then $\rho = const$ *at any particle and*

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$$

so the incompressibility condition is

$$\frac{\partial v_i}{\partial x_i} = \operatorname{div} \vec{v} = D_{ii} = 0.$$

2.4. Conservation of linear momentum

For a particle of mass m we say that the rate of change of linear momentum is equal to the resultant force applied to a particle

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(m\vec{v}\right) = \vec{F}. \tag{2.13}$$

The validity of this principle is postulated in continuum mechanics.

Continuum form

Assume that at time t a given amount of mass is in volume V, bounded by surface S.
Denote b_i [N/kg] - body force (per unit mass) and

t_i [N/m^2] - surface traction, defined per a unit surface.

Based upon Newton's second law the rate of change momentum of a given amount of mass is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\int_V \rho \vec{v}\,\mathrm{d}V\right) = \int_S \vec{t}\,\mathrm{d}S + \int_V \rho \vec{b}\,\mathrm{d}V\,, \tag{2.14a}$$

$$\left[\frac{kg}{m^3}\frac{m}{s^2}m^3 = \frac{kgm}{s^2} = N\right] \quad \left[\frac{N}{m^2}m^2\right] \quad \left[\frac{kg}{m^3}\frac{N}{kg}m^3\right]$$

or
$$\int_S t_i\ \mathrm{d}S + \int_V \rho b_i\ \mathrm{d}V = \frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho v_i\ \mathrm{d}V\,. \tag{2.14b}$$

The relation between the stress vector and stress components is given by so called Cauchy relation

$$t_i = \sigma_{ji} n_j\,,$$

where t_i and σ_{ij} are the stress vector and Cauchy (true) stress tensor components respectively. The symbol n_j stands for the component of a normal.

Substituting the Cauchy relation into the surface integral in (2.14b) and using the divergence theorem gives

$$\int_S t_i\ \mathrm{d}S = \int_S \sigma_{ji} n_j\ \mathrm{d}S = \int_V \frac{\partial \sigma_{ji}}{\partial x_j}\ \mathrm{d}V \tag{2.15}$$

The former equality is due to Cauchy relation while the latter is due to divergence theorem of Gauss.

2.5. Interlude

What is the material time derivative of a volume integral in (2.14b)? It can be proved that

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho v_i\ \mathrm{d}V = \int_V \rho \frac{\mathrm{D}v_i}{\mathrm{D}t}\mathrm{d}V\,. \tag{2.16}$$

Proof in literature is based on Reynold's transport theorem + Gauss divergence theorem + continuity equation + definition of material derivative of ρ.

Using (2.15), (2.16) in (2.14b) we have

$$\int_V \left(\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i\right)\mathrm{d}V = \int_V \rho \frac{\mathrm{D}v_i}{\mathrm{D}t}\ \mathrm{d}V\,. \tag{2.17}$$

From the condition that it must hold for an arbitrarily chosen volume V we get

$$\frac{\partial \sigma_{ji}}{\partial x_i} + \rho b_i = \rho \frac{\mathrm{D}v_i}{\mathrm{D}t} \tag{2.18a}$$

which is called **the** *Cauchy equation of motion***.**

2.6. Cauchy equation of motion in another form is

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i = \rho \ddot{x}_i \,. \tag{2.18b}$$

If we introduce body forces per unit volume

$$\left[\mathrm{N/m^3}\right] \quad f_i = \rho b_i \quad \left[\frac{\mathrm{kg}}{\mathrm{m^3}}\frac{\mathrm{N}}{\mathrm{kg}}\right] \tag{2.19}$$

we obtain still another form of Cauchy relation of motion in the form

$$\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = \rho \ddot{x}_i \,. \tag{2.18c}$$

Remarks

Cauchy equations of motion represent 3 PDE for 6 unknown components of stress. Notice that σ_{ij} *is symmetric.*

These equations are written for a given spatial domain, for a collection of considered material particles - filling volume V *, bounded by surface* S*, considered at a configuration* ${}^{t}C$*. Derivatives are with respect to spatial coordinates.*

In special cases the acceleration could be neglected and equations (2.18) reduce to the equations of equilibrium

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \,. \tag{2.20}$$

These equations do not contain any kinematic variables. They do not generally suffice to determine the stress distribution since they are only three partial differential equations for six independent unknown stress components.
Additional equations must be considered, i.e.

a) displacement vs. strain relations - kinematic relations
b) stress vs., strain relations - constitutive equation.

2.7. Equation of motion in the reference state

It was already mentioned that the Cauchy equations of motion apply to the current deformed configuration ${}^{t}C$. The equations of motion could be transformed to referential configuration ${}^{0}C$ by means of the first and second Piola-Kirchhoff stress tensors.

It should be reminded that Piola-Kirchhoff tensors are useful stress measures which recalculate the actual stress at ${}^{t}C$ to the reference state, i.e. to the non-deformed configuration ${}^{0}C$.

It was already shown that

1st P.-K.: $\boldsymbol{\tau} = \dfrac{{}^{0}\rho}{\rho}\mathbf{F}^{-1}\,\boldsymbol{\sigma}$ or $\tau_{ji} = \dfrac{{}^{0}\rho}{\rho}F_{jr}^{-1}\,\sigma_{ri}$. (2.21)

2nd P.-K.: $\mathbf{S} = \dfrac{{}^{0}\rho}{\rho}\mathbf{F}^{-1}\,\boldsymbol{\sigma}\,\mathbf{F}^{-\mathrm{T}}$ or $S_{ji} = \dfrac{{}^{0}\rho}{\rho}F_{js}^{-1}\,\sigma_{sr}\,F_{ir}^{-1}$. (2.22)

The inverse relations are

for 1st P.-K.: $\boldsymbol{\sigma} = \dfrac{\rho}{{}^{0}\rho}\mathbf{F}\,\boldsymbol{\tau}$ or $\sigma_{ji} = \dfrac{\rho}{{}^{0}\rho}F_{jr}\,\tau_{ri}$. (2.23)

for 2nd P.-K.: $\boldsymbol{\sigma} = \dfrac{\rho}{{}^{0}\rho}\mathbf{F}\,\mathbf{S}\,\mathbf{F}^{\mathrm{T}}$ or $\sigma_{ji} = \dfrac{\rho}{{}^{0}\rho}F_{js}\,S_{sr}\,F_{ir}$, (2.24)

where $\mathbf{F} = \left[\dfrac{\partial x_i}{\partial a_j}\right]$, $\mathbf{F}^{-1} = \left[\dfrac{\partial a_i}{\partial x_j}\right]$. (2.25)

So the equation (2.15)

$$\int_S t_i\,\mathrm{d}S + \int_V \rho b_i\,\mathrm{d}V = \int_V \rho\ddot{x}_i\,\mathrm{d}V, \text{ where } t_i = \sigma_{ji}n_j$$

could be transformed to the referential configuration followingly

$$\int_{{}^{0}S} \tau_{ji}\,{}^{0}n_j\,\mathrm{d}\,{}^{0}S + \int_{{}^{0}V} {}^{0}\rho\,{}^{0}b_i\,\mathrm{d}\,{}^{0}V = \int_{{}^{0}V} {}^{0}\rho\ddot{x}_i\,\mathrm{d}\,{}^{0}V, \tag{2.26}$$

where ${}^{0}b_1 = b_i(a_j, t)$ using $x_i = x_i(a_j, t)$.

Notice that for the right-hand side we could write

$$\int_V \rho\frac{\mathrm{D}v}{\mathrm{D}t}\,\mathrm{d}V = \int_{{}^{0}V} {}^{0}\rho\frac{\mathrm{d}v_i}{\mathrm{d}t}\,\mathrm{d}\,{}^{0}V$$

since $\rho\,\mathrm{d}V = \rho J\,\mathrm{d}^{0}V = {}^{0}\rho\,\mathrm{d}^{0}V$ (see 2.9c)), where J is the Jacobian of the transformation $x_i = x_i(a_j, t)$.

$\ddot{x}_i = \dfrac{\mathrm{D}v_i}{\mathrm{D}t}$ – material derivative of v_i expressed in spatial coordinates $v_i = v_i(x_j, t)$

$\ddot{x}_i = \dfrac{\mathrm{d}v_i}{\mathrm{d}t}$ – material derivative of v_i expressed in material coordinates $v_i = v_i(a_j, t)$.

Using the divergence theorem

$$\int_{{}^{0}S} \tau_{ji}\,{}^{0}n_j\,\mathrm{d}\,{}^{0}S = \int_{{}^{0}V} \frac{\partial\tau_{ji}}{\partial a_j}\,\mathrm{d}\,{}^{0}V$$

and realizing that eq. (2.26) must hold for any volume we finally get

$$\frac{\partial \tau_{ji}}{\partial a_j} + {}^0\rho\ {}^0b_i = {}^0\rho\, \ddot{x}_i\,. \tag{2.27}$$

This is the Cauchy equation of motion expressed in referential coordinates by means of the first Piola-Kirchhoff strain tensor.

Using the relation between the first Piola-Kirchhoff and the second Piola-Kirchhoff

$$\tau = \mathbf{S}\,\mathbf{F}^{\mathrm{T}} \quad \text{or} \quad \tau_{ji} = S_{jr}\ F_{ir} \tag{2.28}$$

we could write the Cauchy equation of motion by means of the second Piola-Kirchhoff stress tensor in the form

$$\frac{\partial}{\partial a_j}\left(S_{jr}F_{ir}\right) + {}^0\rho\ {}^0b_i = {}^0\rho\, \ddot{x}_i\,. \tag{2.29}$$

2.8. Conservation of angular momentum

The law of conservation of angular momentum for a particle of the mass m is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(m\left(\vec{r}\times\vec{v}\right)\right) = \vec{r}\times\vec{F}\,, \text{ where } \vec{r} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}. \tag{2.30}$$

For continuum we could similarly write

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \left(\vec{r}\times\rho\vec{v}\right)\,\mathrm{d}V = \int_S \left(\vec{r}\times\vec{t}\right)\,\mathrm{d}S + \int_V \left(\vec{r}\times\rho\vec{b}\right)\,\mathrm{d}V\,. \tag{2.31}$$

For rewriting it into indicial notation we have to realize that the equivalent of vector product $\vec{c} = \vec{a}\times\vec{b}$ is $c_i = e_{ijk}a_jb_k$, where e_{ijk} is the Civita-Levi permutation symbol.

Note

$e_{ijk} = +1$ *for even permutation of indices, i.e.:* $1,2,3 - 2,3,1 - 3,1,2$,

$e_{ijk} = -1$ *for odd permutation of indices, i.e.:* $3,2,1 - 2,1,3 - 1,3,2$,

$e_{ijk} = 0$ *for repeating indices: 1,1,2 etc.*

So,

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V e_{rmn}\, x_m\, \rho\, v_n\ \mathrm{d}V = \int_S e_{rmn}\, x_m\, t_n\ \mathrm{d}S + \int_V e_{rmn}\, x_m\ \rho b_n\ \mathrm{d}V\,. \tag{2.32}$$

Substituting the Cauchy relation $t_n = \sigma_{jn}n_j$, using the divergence theorem for the surface integral and the conclusion (2.16) concerning the material time derivative of a volume integral, i.e.

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho v_i \,\mathrm{d}V = \int_V \rho \frac{\mathrm{D}v_i}{\mathrm{D}t}\,\mathrm{d}V$$

we could rewrite eq. (32) into the form

$$\int_V e_{rmn}\frac{\mathrm{D}}{\mathrm{D}t}(x_m v_n)\rho\,\mathrm{d}V = \int_V e_{rmn}\left(\frac{\partial(x_m\sigma_{jn})}{\partial x_j} + x_m\rho b_n\right)\mathrm{d}V\ .$$

Realizing that

$$\frac{\mathrm{D}x_m}{\mathrm{D}t} = v_m\,, \qquad \frac{\partial x_m}{\partial x_j} = \delta_{mj}\,,$$

$$\frac{\partial(x_m\sigma_{jn})}{\partial x_j} = \frac{x_m\partial\sigma_{jn}}{\partial x_j} + \delta_{mj}\sigma_{jn} = x_m\frac{\partial\sigma_{jn}}{\partial x_j} + \sigma_{mn}\,,$$

we have

$$\int_V e_{rmn}\left(v_m v_n + x_m\frac{\mathrm{D}v_n}{\mathrm{D}t}\right)\rho\,\mathrm{d}V = \int_V e_{rmn}\left[x_m\left(\frac{\partial\sigma_{jn}}{\partial x_j} + \rho b_n\right) + \sigma_{mn}\right]\mathrm{d}V$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

= 0, equation of motion, see eq. (2.18a)

and also

$$e_{rmn}v_m v_n = 0\,,$$

since e_{rmn} is skew-symmetric in m, n.

Example of double product evaluation shows the trick

$$e_{1mn}v_m v_n = \begin{bmatrix} e_{111} & e_{112} & e_{113} \\ e_{121} & e_{122} & e_{123} \\ e_{131} & e_{132} & e_{133} \end{bmatrix} : \begin{bmatrix} v_1v_1 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2v_2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3v_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} : \begin{bmatrix} v_1v_1 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2v_2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3v_3 \end{bmatrix} = v_2v_3 - v_3v_2 = 0\,.$$

And similarly for other indices. So what remains of eq. (2.32) is

$$\int_V e_{rmn}\sigma_{mn}\,\mathrm{d}V = 0 \quad \text{for arbitrary volume} \qquad (2.33)$$

$\Rightarrow$ $e_{rmn}\sigma_{mn} = 0$ only if σ_{mn} is symmetric, i.e.

$$\sigma_{mn} = \sigma_{nm} \qquad (2.34)$$

This establishes the symmetry of stress matrix without any assumptions of equilibrium or of uniformity of the stress distribution. The symmetry of the stress matrix is so called Cauchy's second law of motion.

2.9. Conservation of energy

If mechanical quantities only are considered the principle of conservation of energy for the continuum may be derived directly from the equation of motion.

Power input

Assume at first that only external surface traction t_i per unit area and body forces b_i per unit mass are doing work on the mass instantaneously occupying volume V , bounded by S . The power input is

$$P_{\text{input}} = \int_S t_i v_i \,\mathrm{d}S + \int_V \rho b_i v_i \,\mathrm{d}V \,, \tag{2.35}$$

where v_i are components of the velocity field. As before we express the components of tractions by means of stress components

$$t_i = \sigma_{ji} n_j$$

and use the Gauss divergence theorem for the transformation of the surface integral into volume integral and get

$$\int_S t_i v_i \,\mathrm{d}S = \int_S \sigma_{ji} n_j v_i \,\mathrm{d}S = \int_V \frac{\partial(\sigma_{ji} v_i)}{\partial x_j} \mathrm{d}V = \int_V \left(\frac{\partial \sigma_{ji}}{\partial x_j} v_i + \sigma_{ji} \frac{\partial v_i}{\partial x_j} \right) \mathrm{d}V \,. \tag{2.36}$$

Realizing that the velocity gradient if defined by

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \tag{2.37}$$

we obtain

$$P_{\text{input}} = \int_V \left[\left(\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i \right) v_i + \sigma_{ji} L_{ij} \right] \mathrm{d}V \tag{2.38}$$

but $\dfrac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i = \rho \dfrac{\mathrm{D}v_i}{\mathrm{D}t}$ by (2.18a)

The first term of eq. (2.38) on the right hand side could be rewritten as

$$\int_V \rho v_i \frac{\mathrm{D}v_i}{\mathrm{D}t}\,\mathrm{d}V = \int_V \rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2} v_i v_i\right)\mathrm{d}V = \frac{\mathrm{D}}{\mathrm{D}t}\int_V \frac{1}{2}\rho v_i v_i\,\mathrm{d}V \tag{2.39}$$

↑	↑	↑
Remember	material derivative	Kinetic energy
$\mathrm{d}v^2 = 2v\,\mathrm{d}v$	of volume integral	of the system
	see eq. (16)	

and represents the time rate of the kinetic energy of the system. Realizing that the stress tensor is symmetric $\sigma_{ij} = \sigma_{ji}$ and using (2.39) we could rewrite (2.38) into the form

$$P_{\text{input}} = \frac{\mathrm{D}}{\mathrm{D}t}\int_V \frac{1}{2}\rho v_i v_i\,\mathrm{d}V + \int_V \sigma_{ij} L_{ij}\,\mathrm{d}V\,. \tag{2.40}$$

The last term of (40) is a double dot product of stress and velocity gradient tensors. The velocity gradient tensor can be decomposed into symmetric and skew-symmetric parts

$$L_{ij} = D_{ij} + W_{ij}\,,$$

where, as explained before, D_{ij} and W_{ij} represent rate of deformation and spin tensors respectively. It could be shown easily that

$$\sigma_{ij} W_{ij} = 0\,, \qquad (\sigma_{ij} \text{ - symmetric, } W_{ij} \text{ - skew-symmetric}).$$

From it follows that

$$\sigma_{ij} L_{ij} = \sigma_{ij} D_{ij}\,. \tag{2.41}$$

The final form for the power input expression is

$$P_{\text{input}} = \frac{\mathrm{D}}{\mathrm{D}t}\int_V \frac{1}{2}\rho v_i v_i\,\mathrm{d}V + \int_V \sigma_{ij} \mathrm{D}_{\mathrm{ij}}\,\mathrm{d}V\,. \tag{2.42}$$

We can conclude that the power input is the sum of two volume integrals. The first one is the material time derivative of the *kinetic energy* of the system, while the second one contributes to the *internal energy*.

The scalar $\sigma : \mathbf{L}$ equals to $\sigma : \mathbf{D}$ and is called *stress power per unit volume*. Stress power does not contribute to the kinetic energy of the system. This result is due to Stokes (1851).

If both mechanical and non-mechanical energies are to be considered the principle of conservation of energy in its most general form must be used.

In this form the conservation principle states that the time rate of change of kinetic + internal energy is equal to sum of the rate of work + all other energies supplied to the continuum per unit time. Such energies may include thermal, chemical, electromagnetic energies.

In the following only *mechanical* and *thermal* energies are considered. Then the energy principle takes on the form of the first law of thermodynamics.

For our purposes we will consider a thermodynamic system chosen as a *closed system* not interchanging matter with surroundings.

The first law of thermodynamics relates the work done on the system and the heat transfer into the system to the change in energy of the system.

It is assumed that only energy transfers to the system are by

a) – mechanic work done on the system by surface tractions and body forces,

b) – heat transfer through the boundary,

c) – distributed internal heat sources.

Surface tractions and body forces and their contribution to power input to the system were already treated by previous paragraph are summarized by eq. (2.42).

3. Finite deformations, incremental decomposition and finite element discretization

Overview

Principle of virtual work relates the work done by internal and external forces due to prescribed virtual displacements

$$\delta\, {}^{t+\Delta t}U = \delta\, {}^{t+\Delta t}W \,. \tag{3.1}$$

On the left hand side we have the virtual strain energy of a system at ${}^{t+\Delta t}C$

$$\delta\, {}^{t+\Delta t}U = \int_{{}^{t+\Delta t}V} {}^{t+\Delta t}_{t+\Delta t}\sigma_{ij}\;\; \delta\, {}^{t+\Delta t}_{t+\Delta t}\varepsilon_{ij}\; \mathrm{d}\, {}^{t+\Delta t}V = \int_{{}^{0}V} {}^{t+\Delta t}_{0}S\; \delta\, {}^{t+\Delta t}_{0}E_{ij}\; \mathrm{d}^0 V \,. \tag{3.2}$$

Using the total Lagrangian approach, the incremental decompositions for the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor are

$${}^{t+\Delta t}_{0}S_{ij} = {}^{t}_{0}S_{ij} + \Delta S_{ij} \,, \tag{3.3}$$

$${}^{t+\Delta t}_{0}E_{ij} = {}^{t}_{0}E_{ij} + \Delta E_{ij} \,. \tag{3.4}$$

It was already shown that the increment of Green-Lagrange strain tensor is

$$\Delta \mathbf{E} = \frac{1}{2}\left(\Delta\mathbf{Z} + \Delta\mathbf{Z}^{\mathrm{T}}\right) + \frac{1}{2}\left(\Delta\mathbf{Z}^{\mathrm{T}}\,\mathbf{Z} + \mathbf{Z}^{\mathrm{T}}\,\Delta\mathbf{Z}\right) + \frac{1}{2}\left(\Delta\mathbf{Z}^{\mathrm{T}}\,\Delta\mathbf{Z}\right)$$

where $\mathbf{Z} \quad \cdots \quad Z_{\mathrm{ij}} = \dfrac{\partial\, {}^{t}u_i}{\partial\, {}^{0}x_j}$ is the material displacement gradient.

We can introduce the following notation which will consistently be used later

$$\begin{aligned}
\Delta\mathbf{E} &= \Delta\mathbf{E}^{\mathrm{L}} + \Delta\mathbf{E}^{\mathrm{N}} \\
\Delta\mathbf{E}^{\mathrm{L}} &= \Delta\mathbf{E}^{\mathrm{L1}} + \Delta\mathbf{E}^{\mathrm{L2}} \\
\Delta\mathbf{E}^{\mathrm{L1}} &= \frac{1}{2}\left(\Delta\mathbf{Z} + \Delta\mathbf{Z}^{\mathrm{T}}\right) \\
\Delta\mathbf{E}^{\mathrm{L2}} &= \frac{1}{2}\left(\Delta\mathbf{Z}^{\mathrm{T}}\,\mathbf{Z} + \mathbf{Z}^{\mathrm{T}}\,\Delta\mathbf{Z}\right) \\
\Delta\mathbf{E}^{\mathrm{N}} &= \frac{1}{2}\left(\Delta\mathbf{Z}^{\mathrm{T}}\,\Delta\mathbf{Z}\right).
\end{aligned} \tag{3.6}$$

Variation of eq. (3.4) gives

$$\delta\, {}^{t+\Delta t}_{0}E_{ij} = \delta\Delta E_{ij} \,. \tag{3.7}$$

Substituting eqs. (3.7) and (3.3) to (3.2) we get

$$\begin{aligned}
\delta\, {}^{t+\Delta t}U &= \int_{{}^{0}V} \left({}^{t}_{0}S_{ij} + \Delta S_{ij}\right) \delta\Delta E_{ij}\; \mathrm{d}^0 V = \\
&= \int_{{}^{0}V} \left({}^{t}_{0}S_{ij} + \Delta S_{ij}\right) \left(\delta\Delta E^{\mathrm{N}}_{ij} + \delta\Delta E^{\mathrm{N}}_{ij}\right) \mathrm{d}^0 V =
\end{aligned}$$

$$= \int_{^0V} \left({}_0^tS_{ij}\, \delta\Delta E_{ij}^{\mathrm{L}} + {}_0^tS_{ij}\, \delta\Delta E_{ij}^{\mathrm{N}} + \Delta S_{ij}\, \delta\Delta E_{ij}^{\mathrm{L}} + \Delta S_{ij}\, \delta\Delta E_{ij}^{\mathrm{N}} \right) \mathrm{d}^0V \,. \tag{3.8}$$

The increment of the second Piola-Kirchhoff stress tensor appearing in the third term of the integrand could be linearized

$$\Delta S_{ij} = {}_0C_{ijkl}\, \Delta E_{ij}^{\mathrm{L}} \tag{3.9}$$

and the last term in (3.8) could be neglected since it is one order less than other terms, so the virtual strain energy could be approximated by

$$\delta\, {}^{t+\Delta t}U = \delta U_{\mathrm{I}} + \delta U_{\mathrm{II}} + \delta U_{\mathrm{III}} \tag{3.10}$$

where

$$\delta U_{\mathrm{I}} = \int_{^0V} {}_0^tS_{ij}\, \delta\Delta E_{ij}^{\mathrm{L}}\, \mathrm{d}^0V \,, \tag{3.10a}$$

$$\delta U_{\mathrm{II}} = \int_{^0V} {}_0^tS_{ij}\, \delta\Delta E_{ij}^{\mathrm{N}}\, \mathrm{d}^0V \,, \tag{3.10b}$$

$$\delta U_{\mathrm{III}} = \int_{^0V} {}_0C_{ijkl}\, \Delta E_{kl}^{\mathrm{L}}\, \delta\Delta E_{ij}^{\mathrm{L}}\, \mathrm{d}^0V \,. \tag{3.10c}$$

This approximation implicitly assumes that the changes between the configurations ${}^tC$ and ${}^{t+\Delta t}C$ are small.

For finite element implementation of these ideas it is convenient to switch from the tensor to matrix notation. The process could be summarized in four steps

a) Instead of tensor $\Delta E_{ij}^{\mathrm{L}}$ we will use a column array $\{\Delta E^{\mathrm{L}}\}$ defined by

$$\left(\Delta \mathbf{E}^{\mathrm{L}}\right)^{\mathrm{T}} = \left\{ \Delta E_{11}^{\mathrm{L}}\ \Delta E_{22}^{\mathrm{L}}\ \Delta E_{33}^{\mathrm{L}}\ 2\Delta E_{12}^{\mathrm{L}}\ 2\Delta E_{23}^{\mathrm{L}}\ 2\Delta E_{31}^{\mathrm{L}} \right\}. \tag{3.11}$$

b) Instead of tensor $\Delta E_{ij}^{\mathrm{N}}$ we will use a column array $\{\Delta \widetilde{E}^{\mathrm{N}}\}$ defined by

$$\Delta\left(\widetilde{\mathbf{E}}^{\mathrm{N}}\right)^{\mathrm{T}} = \left\{ \Delta E_{11}^{\mathrm{N}}\ \Delta E_{12}^{\mathrm{N}}\ \Delta E_{13}^{\mathrm{N}}\ \Delta E_{21}^{\mathrm{N}}\ \Delta E_{22}^{\mathrm{N}}\ \Delta E_{23}^{\mathrm{N}}\ \Delta E_{31}^{\mathrm{N}}\ \Delta E_{32}^{\mathrm{N}}\ \Delta E_{33}^{\mathrm{N}} \right\}. \tag{3.12}$$

c) The second Piola-Kirchhoff stress tensor will have two appearances. Instead of ${}_0^tS_{ij}$ we will use either $\{{}_0^tS\}$ defined by

$$\left\{{}_0^tS\right\}^{\mathrm{T}} = \left\{ {}_0^tS_{11}\ {}_0^tS_{22}\ {}_0^tS_{33}\ {}_0^tS_{12}\ {}_0^tS_{23}\ {}_0^tS_{31} \right\} \tag{3.13}$$

or a two-dimensional array $\left[{}_0^t\widetilde{S}\right]$ defined by

$$\left[{}_0^t\widetilde{S}\right] = \begin{bmatrix} \left[{}_0^tS\right] & 0 & 0 \\ 0 & \left[{}_0^tS\right] & 0 \\ 0 & 0 & \left[{}_0^tS\right] \end{bmatrix}_{9\times 9} , \tag{3.14}$$

where

$$\left[{}_0^tS\right]=\begin{bmatrix} {}_0^tS_{11} & {}_0^tS_{12} & {}_0^tS_{13} \\ {}_0^tS_{21} & {}_0^tS_{22} & {}_0^tS_{23} \\ {}_0^tS_{31} & {}_0^tS_{32} & {}_0^tS_{33} \end{bmatrix}. \tag{3.15}$$

d) ${}_0C_{ijkl} \to \left[{}_0C\right]$.

In the matrix notation, the virtual strain energy components, equivalent to those appearing in eq. (3.10), are

$$\delta U_{\mathrm{I}} = \int_{{}^0V} \left\{\delta\Delta E^{\mathrm{L}}\right\}^{\mathrm{T}} \left\{{}_0^tS\right\} \mathrm{d}^0V\,, \tag{3.17a}$$

$$\delta U_{\mathrm{II}} = \int_{{}^0V} \left\{\delta\Delta\widetilde{E}^{\mathrm{N}}\right\}^{\mathrm{T}} \left[{}_0^t\widetilde{S}\right] \left\{\Delta\widetilde{E}^{\mathrm{N}}\right\} \mathrm{d}^0V\,, \tag{3.17b}$$

$$\delta U_{\mathrm{III}} = \int_{{}^0V} \left\{\delta\Delta E^{\mathrm{L}}\right\}^{\mathrm{T}} \left[{}_0C\right] \left\{\Delta E^{\mathrm{L}}\right\} \mathrm{d}^0V\,, \tag{3.17c}$$

And now, the finite element discretization enters the stage. The generalized displacements within a finite element are usually expressed by means of shape functions systematically arranged in $\mathbf{A}$ and by generalized nodal displacements $\mathbf{q}$ in the form

$$\mathbf{u} = \mathbf{A}_{3*\mathrm{LMAX}}\,\mathbf{q}_{\mathrm{LMAX}*1} \tag{3.18}$$

where LMAX is the number of D.O.F. for a particular element. The increments of displacements are

$$\Delta\mathbf{u} = \mathbf{A}\,\Delta\mathbf{q} \tag{3.19}$$

Knowing the shape functions appearing in $\mathbf{A}$ we can easily calculate the components of the material displacement gradient and then to express its increments. It will contain derivatives of shape functions and will depend on $\mathbf{q}$ and $\Delta\mathbf{q}$.

$${}_0^tZ_{ij} = \frac{\partial\, {}^tu_i}{\partial\, {}^0x_j} \qquad \Delta Z_{ij} = \frac{\partial\, \Delta u_i}{\partial\, {}^0x_j} \tag{3.20}$$

The results for linear part of strain increments – in accordance with eqs. (3.9) and (3.11) – could be expressed in the form

$$\Delta\mathbf{E}_{6*1}^{\mathrm{L}} = \Delta\mathbf{E}^{\mathrm{L1}} + \Delta\mathbf{E}^{\mathrm{L2}} = \left({}_0\mathbf{B}^{\mathrm{L1}} + {}_0\mathbf{B}^{\mathrm{L2}}\right)\Delta\mathbf{q} = {}_0\mathbf{B}_{6*\mathrm{LMAX}}^{\mathrm{L}}\,\Delta\mathbf{q}_{\mathrm{LMAX}*1} \tag{3.21}$$

where obviously

$${}_0\mathbf{B}^{\mathrm{L}} = {}_0\mathbf{B}^{\mathrm{L1}} + {}_0\mathbf{B}^{\mathrm{L2}}. \tag{3.22}$$

The lower left hand index zero emphasises that the derivatives appearing in these matrices are taken with respect to coordinates ${}^0x_i$.

Similarly for the nonlinear part of strain increments

$$\Delta\widetilde{\mathbf{E}}^{\mathrm{N}}_{9*1} = {}_{0}\mathbf{B}^{\mathrm{N}}_{9*\mathrm{LMAX}}\ \Delta\mathbf{q}_{\mathrm{LMAX}*1} \tag{3.23}$$

Now, the three contributions to the virtual strain energy (3.17a), (3.17b), (3.17c) could be formally rewritten. The first component is

$$\delta U_{\mathrm{I}} = \delta\Delta\mathbf{q}^{\mathrm{T}}\ {}^{t}_{0}\mathbf{F} \tag{3.24}$$

where a so called *vector of internal forces in nodes* is

$${}^{t}_{0}\mathbf{F} = \int_{{}^{0}V} \left({}_{0}\mathbf{B}^{\mathrm{L}}\right)^{\mathrm{T}}\ {}^{t}_{0}\mathbf{S}\ \mathrm{d}\,{}^{0}V\ . \tag{3.25}$$

For the second component, i.e. (3.17b), we can write

$$\delta U_{\mathrm{II}} = \delta\Delta\mathbf{q}^{\mathrm{T}}\ {}^{t}_{0}\mathbf{K}^{\mathrm{N}}\,\Delta\mathbf{q}\,, \tag{3.26}$$

where a so called *non-linear part of incremental stiffness matrix* is

$${}^{t}_{0}\mathbf{K}^{\mathrm{N}} = \int_{{}^{0}V} \left({}_{0}\mathbf{B}^{\mathrm{N}}\right)^{\mathrm{T}}\ {}^{t}_{0}\widetilde{\mathbf{S}}\ {}_{0}\mathbf{B}^{\mathrm{N}}\,\mathrm{d}^{0}V\ . \tag{3.27}$$

The third component of the virtual strain energy, i.e. (3.17c), can be discretized using (3.9) and (3.21). After some algebraic manipulations we get

$$\delta U_{\mathrm{III}} = \delta\Delta\mathbf{q}^{\mathrm{T}}\ {}^{t}_{0}\mathbf{K}^{\mathrm{L}}\,\Delta\mathbf{q}\ , \tag{3.28}$$

where

$${}^{t}_{0}\mathbf{K}^{\mathrm{L}} = \int_{{}^{0}V} \left({}_{0}\mathbf{B}^{\mathrm{L}}\right)^{\mathrm{T}}\ {}_{0}\mathbf{C}\ {}_{0}\mathbf{B}^{\mathrm{L}}\ \mathrm{d}^{0}V \tag{3.29}$$

is the *linear part* of *incremental stiffness matrix*.

Using eqs. (3.24), (3.26) and (3.28) and realizing that the virtual work done by external forces $\left\{{}^{t+\Delta t}R\right\}$ is

$$\delta\ {}^{t+\Delta t}W = \left\{\delta\Delta q\right\}^{\mathrm{T}} \left\{{}^{t+\Delta t}R\right\}, \tag{3.30}$$

we can conclude that in agreement with (1) we get

$$\delta\Delta\mathbf{q}^{\mathrm{T}}\left({}^{t}_{0}\mathbf{F} + {}^{t}_{0}\mathbf{K}^{\mathrm{L}}\,\Delta\mathbf{q} + {}^{t}_{0}\mathbf{K}^{\mathrm{N}}\,\Delta\mathbf{q}\right) = \delta\Delta\mathbf{q}^{\mathrm{T}}\ {}^{t+\Delta t}\mathbf{R}\,. \tag{3.31}$$

This equation must hold for any virtual diplacement, hence finally we have

$${}^{t}_{0}\mathbf{K}\,\Delta\mathbf{q} = {}^{t+\Delta t}\mathbf{R} - {}^{t}_{0}\mathbf{F}\,, \tag{3.32}$$

where

$${}^{t}_{0}\mathbf{K} = {}^{t}_{0}\mathbf{K}^{\mathrm{L}} + {}^{t}_{0}\mathbf{K}^{\mathrm{N}} \tag{3.33}$$

This system of algebraic equations constitutes the conditions of equilibrium. Solving the system gives the unknown increments of nodal displacements and the new displacements in the configuration ${}^{t+\Delta t}C$ could be calculated by

$$ {}^{t+\Delta t}\mathbf{q} = {}^{t}\mathbf{q} + \Delta\mathbf{q} \tag{3.34} $$

The new displacements, however, have to be taken as the first approximation only and refined subsequently in an iterative process. Introducing the iteration counter (i) we can rewrite eq. (3.32) as follows

$$ {}_{0}^{t}\mathbf{K}^{(i)}\, \Delta\mathbf{q}^{(i)} = {}^{t+\Delta t}\mathbf{R} - {}_{0}^{t+\Delta t}\mathbf{F}^{(i-1)} \quad \text{for} \quad i = 0, 1, 2\ \ldots \tag{3.35} $$

At the beginning of the process we set

$$ {}^{t+\Delta t}\mathbf{q}^{(0)} = {}^{t}\mathbf{q}, \quad {}_{0}^{t+\Delta t}\mathbf{F}^{(0)} = {}_{0}^{t}\mathbf{F} \quad \text{and} \quad {}_{0}^{t}\mathbf{K}^{(0)} = {}^{0}\mathbf{K}\ . \tag{3.36} $$

The **Newton Raphson iteration process** with **total Lagrangian approach** could be implemented as follows.

Let's assume that from $t = 0$ to $t = tmax$ there are kmax (same) loading steps. The maximum force, corresponding to final configuration at time tmax is ${}^{tmax}R = {}^{kmax}R$. If a linear increase of force between consecutive loading steps is considered, then the force corresponding to $k = th$ loading step is ${}^{k}R = k * {}^{tmax}R / kmax$.

for $k = 1$ to kmax do % loop for loading steps

$i = 0;$

if $k = 1$ then ${}_0^kK^{(0)} = {}^0K;$ ${}_0^kF^{(0)} = {}_0^0F = 0;$ ${}^kq^{(0)} = {}^0q;$

else ${}_0^kK^{(0)} = {}_0^{k-1}K^{(ilast)};$ ${}_0^kF^{(0)} = {}_0^{k-1}F^{(ilast)};$ ${}^kq^{(0)} = {}^{k-1}q^{(ilast)};$

end of if

intermediate load level is ${}^kR = {}^{tmax}R * k/kmax;$

satisfied= .false.

while .not. satisfied do % iteration loop

$i = i + 1;$

solve ${}_0^kK^{(i-1)}\, \Delta q^{(i)} = {}^kR - {}_0^kF^{(i-1)}; \quad \Rightarrow \quad \Delta q^{(i)}$

${}^kq^{(i)} = {}^kq^{(i-1)} + \Delta q^{(i)};$

calculate ${}_0^kS^{(i)};$ ${}_0^kK^{(i)} = f\left({}^kq^{(i)}, {}_0^kS^{(i)}\right);$ ${}_0^kF^{(i)} = g\left({}_0^kS^{(i)}\right);$

% Note: f() and g() are functions

ilast=i;

$$satisfied = \left(\frac{\left\|\Delta q^{(i)}\right\|}{\left\|{}^kq^{(\mathrm{ilast})}\right\|} < \varepsilon_1 \; .and. \; \frac{\left\|{}^kR - {}_0^kF^{(\mathrm{ilast})}\right\|}{\left\|{}^kR\right\|} < \varepsilon_2 \right)$$

end of while loop

end of for loop

4. What is the material derivative of volume integral?

Let φ be a function that is sufficiently smooth in a given volume V. Then

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \varphi\rho \,\mathrm{d}V = \int_V \frac{\mathrm{D}\varphi}{\mathrm{D}t}\rho \,\mathrm{d}V . \qquad (*)$$

Proof: From left-hand side of the equation (*), we can write

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \varphi\,\rho \,\mathrm{d}V = \frac{\partial}{\partial t}\int_V \varphi\,\rho \,\mathrm{d}V + \int_S \varphi\,\rho\,\vec{v}\,\mathrm{d}S . \qquad (4.1)$$

Using Gauss theorem we can transform surface integral to volume integral and get

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \varphi\rho \,\mathrm{d}V = \frac{\partial}{\partial t}\int_V \varphi\rho \,\mathrm{d}V + \int_V \mathrm{div}\left(\varphi\rho\vec{v}\right)\mathrm{d}V = \int_V \left[\frac{\partial}{\partial t}(\varphi\rho) + \mathrm{div}\left(\varphi\rho\vec{\mathrm{v}}\right)\right]\mathrm{d}V$$

(4.2)

For a divergence of product of scalar function α and general vector function $\vec{k}$, we can write

$$\mathrm{div}\left(\alpha\vec{\mathrm{k}}\right) = \mathrm{grad}\,\alpha.\vec{k} + \alpha\,\mathrm{div}\vec{k} \qquad (4.3)$$

Using this expression we can rewrite integrand of integral (2) and we obtain

$$\frac{\partial}{\partial t}(\varphi\rho) + \mathrm{div}\left(\varphi\rho\vec{\mathrm{v}}\right) = \rho\frac{\partial\varphi}{\partial t} + \varphi\frac{\partial\rho}{\partial t} + \varphi\,\mathrm{div}\left(\rho\vec{\mathrm{v}}\right) + \rho\vec{v}\,\mathrm{grad}\varphi =$$

$$= \rho\left[\frac{\partial\varphi}{\partial t} + \vec{v}\,\mathrm{grad}\varphi\right] + \varphi\left[\frac{\partial\varphi}{\partial t} + \mathrm{div}\left(\rho\vec{\mathrm{v}}\right)\right]. \qquad (4.4)$$

The second bracket of expression (4.4) is from equation of continuity equal to zero and first bracket from this expression is a definition relation for $\frac{\mathrm{D}\varphi}{\mathrm{D}t}$. After substituting back into integral (4.2) we finally get

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\varphi \,\mathrm{d}V = \int_V \left\{\rho\left[\frac{\partial\varphi}{\partial t} + \vec{v}\,\mathrm{grad}\varphi\right] + \varphi\left[\frac{\partial\rho}{\partial t} + \mathrm{div}\left(\rho\vec{\mathrm{v}}\right)\right]\right\}\mathrm{d}V =$$

$$= \int_V \left[\left(\rho\frac{\mathrm{D}\varphi}{\mathrm{D}t}\right) + \varphi.0\right]\mathrm{d}V = \int_V \frac{\mathrm{D}\varphi}{\mathrm{D}t}\rho \,\mathrm{d}V . \qquad (4.5)$$

5. Conjugate stress and strain measures

A stress is called conjugate to the strain if its scalar product with strain gives work. Stress and strain quantities giving mechanical work as their scalar product are *energetically conjugate*. Mechanical work per unit time is power, or rate of work, so we could also relate stress and strain rate quantities whose scalar product gives power. Such quantities could be called *power conjugate*.

The mechanical work of surface tractions and body forces at the current configuration ${}^{t}C$ is

$$ {}^{t}W = \int_{{}^{t}S} {}^{t}t_i \; {}^{t}u_i \; \mathrm{d}^{t}S + \int_{{}^{t}V} {}^{t}f_i \; {}^{t}u_i \; \mathrm{d}^{t}V \; . \tag{5.1}$$

All quantities are related to the current configuration. Let's omit the upper left index t for a moment. Using the Cauchy relation and the Gauss theorem we get

$$ W = \int_S \sigma_{ji} n_j u_i \, \mathrm{d}S + \int_V f_i u_i \, \mathrm{d}V = $$

$$ = \int_V \left[\frac{\partial (\sigma_{ji} u_i)}{\partial x_j} + f_i u_i \right] \mathrm{d}V = \int_V \left[\sigma_{ji} \frac{\partial u_i}{\partial x_j} + \left(\frac{\partial \sigma_{ji}}{\partial x_j} + f_i \right) u_i \right] \mathrm{d}V \; . \tag{5.2}$$

The second term in eq. (5.2) is equal to zero, since it is the equilibrium equation. Exploiting the fact that the stress tensor is symmetric, the mechanical work could be calculated by a double dot product of Cauchy (true) strain and infinitesimal strain at the current configuration ${}^{t}C$.

$$ W = \int_V \sigma_{ji} u_{ij} \, \mathrm{d}V = \int_V \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}V = \int_V \sigma : \varepsilon \, \mathrm{d}V \; . \tag{5.3}$$

We can conclude that the true stress and infinitesimal strain constitutes the energetically conjugate variables.

Remark

Remember that the infinitesimal strain could be calculated exactly, involving no approximation, from

$$ \boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{Z} + \mathbf{Z}^{\mathrm{T}} \right) = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^{\mathrm{T}} \right) - \mathbf{I} $$

since the deformation gradient ${}_{0}^{t}F_{ij} = \dfrac{\partial \, {}^{t}x_i}{\partial \, {}^{0}x_j}$ *could be expressed by means of the material displacement gradient* ${}_{0}^{t}Z_{ij} = \dfrac{\partial \, {}^{t}u_i}{\partial \, {}^{0}x_j}$ *in the form*

$$\mathbf{F} = \mathbf{Z} + \mathbf{I} .$$

Similarly for the mechanical power, or the rate of work gives

$$P = \dot{W} = \int_V \sigma_{ij} \dot{u}_{ij} \, \mathrm{d}V = \int_V \sigma_{ij} L_{ij} \, \mathrm{d}V = \int_V \sigma_{ij} \left(D_{ij} + W_{ij} \right) \mathrm{d}V =$$

$$= \int_V \sigma_{ij} \, D_{ij} \, \mathrm{d}V = \int_V \sigma : \mathbf{D} \, \mathrm{d}V , \tag{5.4}$$

so the power conjugate quantities are the true stress σ_{ij} and the velocity gradient L_{ij}. Using the fact that the spin tensor W_{ij} is skew-symmetric and its scalar product with symmetric stress tensor σ_{ij} gives zero, we can state that the true stress and the strain rate D_{ij} are also power conjugate quantities.

Using the definition of the first Piola-Kirchhoff stress tensor we could express the previous equation in the reference configuration ${}^0C$.

Substituting $\sigma_{ji} = \frac{\rho}{{}^0\rho} F_{jk} \, \tau_{ki}$ into eq. (5.4) we get

$$\int_V \sigma_{ji} v_{ij} \, \mathrm{d}V = \int_V \frac{\rho}{{}^0\rho} F_{jk} \, \tau_{ki} \, L_{ij} \, \mathrm{d}V = \int_{{}^0V} \tau_{ki} L_{ij} F_{jk} \, \mathrm{d}^0 V . \tag{5.5}$$

But

$$L_{ij} F_{jk} = \dot{F}_{ik} \tag{5.6}$$

since

$$\dot{F}_{ik} = \frac{\mathrm{D}}{\mathrm{D}t} F_{ik} = \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{\partial^t x_i}{\partial^0 x_k} \right) = \frac{\partial^t \dot{x}_i}{\partial^t x_j} \, \frac{\partial^t x_j}{\partial^0 x_k} = \frac{\partial^t v_i}{\partial^t x_j} \, \frac{\partial^t x_j}{\partial^0 x_k} = L_{ij} F_{jk} .$$

So the mechanical power in the reference configuration is expressed by

$$P = \int_{{}^0V} \tau_{ki} \, \dot{F}_{ik} \, \mathrm{d}^0 V = \int_{{}^0V} \tau_{ik} \, \dot{F}_{ik} \, \mathrm{d}^0 V = \int_{{}^0V} \tau : \dot{\mathbf{F}} \, \mathrm{d}^0 V . \tag{5.7}$$

which is a double dot product of the first Piola-Kirchoff stress tensor and the rate of deformation gradient tensor. These tensors form another suitable couple of power conjugate quantities.

Similarly for the second Piola-Kirchhoff stress tensor. Substituting

$$\boldsymbol{\sigma} = \frac{\rho}{{}^0\rho} \mathbf{F} \, \mathbf{S} \, \mathbf{F}^{\mathrm{T}} \quad \text{or} \quad \sigma_{ij} = \frac{\rho}{{}^0\rho} F_{ir} \, S_{rs} \, F_{js}$$

into eq. (5.4) we get

$$P = \int_V \sigma_{ij} \, D_{ij} \, \mathrm{d}V = \int_V \frac{\rho}{{}^0\rho} F_{ir} \, F_{js} \, S_{rs} \, D_{ij} \, \mathrm{d}V = \int_{{}^0V} F_{ir} \, F_{js} \, S_{rs} \, D_{ij} \, \mathrm{d}^0 V .$$

But

$$F_{ir}\,F_{js}\,D_{ij} = \dot{E}_{rs}$$

since for the Green-Lagrange strain tensor

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\,\mathbf{F} - \mathbf{I}\right)$$

we could express its time rate by

$$\dot{\mathbf{E}} = \frac{1}{2}\frac{\mathrm{D}}{\mathrm{D}t}\left(\mathbf{F}^{\mathrm{T}}\,\mathbf{F}\right) = \frac{1}{2}\left(\dot{\mathbf{F}}^{\mathrm{T}}\,\mathbf{F} + \mathbf{F}^{\mathrm{T}}\,\dot{\mathbf{F}}\right).$$

Using eq. (5.6) the previous equation can be rewritten into

$$\dot{\mathbf{E}} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\,\mathbf{L}^{\mathrm{T}}\,\mathbf{L} + \mathbf{F}^{\mathrm{T}}\,\mathbf{L}\,\mathbf{F}\right) = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\left(\mathbf{D} + \mathbf{W}\right)^{\mathrm{T}}\,\mathbf{F} + \mathbf{F}^{\mathrm{T}}\left(\mathbf{D} + \mathbf{W}\right)\mathbf{F}\right) =$$

$$= \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\left(\mathbf{D}^{\mathrm{T}} + \mathbf{W}^{\mathrm{T}}\right)\mathbf{F} + \mathbf{F}^{\mathrm{T}}\left(\mathbf{D} + \mathbf{W}\right)\mathbf{F}\right) =$$

$$= \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\left(\mathbf{D} - \mathbf{W}\right)\mathbf{F} + \mathbf{F}^{\mathrm{T}}\left(\mathbf{D} + \mathbf{W}\right)\mathbf{F}\right) = \mathbf{F}^{\mathrm{T}}\mathbf{D}\,\mathbf{F}\,.$$

since $\mathbf{D}$ is symmetric and $\mathbf{W}$ skew-symmetric. So the stress power in a reference configuration can also be expressed by

$$P = \int_V \sigma_{ij}\,D_{ij}\,\mathrm{d}V = \int_{{}^0V} S_{ij}\,\dot{E}_{ij}\,\mathrm{d}\,{}^0V = \int_{{}^0V} \mathbf{S} : \dot{\mathbf{E}}\,\mathrm{d}\,{}^0V$$

giving another couple of suitable quantities. It is obvious that in terms of mechanical wark we have

$$W = \int_V \sigma_{ij}\,\varepsilon_{ij}\,\mathrm{d}V = \int_{{}^0V} S_{ij}\,E_{ij}\,\mathrm{d}\,{}^0V\,.$$

Finally, let's find what role plays the Almansi strain tensor is these considerations. It can be proved that

$$P = \int_V \sigma : \mathbf{D}\,\mathrm{d}V = \int_V \sigma : \mathbf{A}^{\nabla}\,\mathrm{d}V\,,$$

where

$\mathbf{A}^{\nabla} = \mathbf{D}$ is a so called Rivlin-Ericksen rate of Almansi strain $\mathbf{A}$.

Proof

a) The Almansi strain tensor is defined by $\left(\mathrm{d}^t s\right)^2 - \left(\mathrm{d}^0 s\right)^2 = 2\,\mathrm{d}^t\mathbf{x}\,\mathbf{A}\,\mathrm{d}^t\mathbf{x}$.

b) The time rate of the previous expression is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\left(\mathrm{d}^t s\right)^2 - \left(\mathrm{d}^0 s\right)^2\right) = \frac{\mathrm{D}}{\mathrm{D}t}\left(\mathrm{d}^t s\right)^2 = 2\frac{\mathrm{D}}{\mathrm{D}t}\left(\mathrm{d}^t\mathbf{x}\,\mathbf{A}\,\mathrm{d}^t\mathbf{x}\right) =$$

$$= 2\left(\mathrm{d}^t\dot{\mathbf{x}}\ \mathbf{A}\ \mathrm{d}^t\mathbf{x} + \mathrm{d}^t\mathbf{x}\ \dot{\mathbf{A}}\ \mathrm{d}^t\mathbf{x} + \mathrm{d}^t\mathbf{x}\ \mathbf{A}\ \mathrm{d}^t\dot{\mathbf{x}}\right) =$$

But $\mathrm{d}^t\mathbf{x} = \mathbf{L}\ \mathrm{d}^t\mathbf{x}$, $\mathrm{d}^t\mathbf{x}^{\mathrm{T}} = \mathrm{d}^t\mathbf{x}^{\mathrm{T}}\ \mathbf{L}^{\mathrm{T}}$

so

$$= 2\ \mathrm{d}^t\mathbf{x}^{\mathrm{T}}\left(\mathbf{L}^{\mathrm{T}}\ \mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\ \mathbf{L}\right)\mathrm{d}^t\mathbf{x}\,.$$

c) We can also write

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\mathrm{d}^t s\right)^2 = \frac{\mathrm{D}}{\mathrm{D}t}\left(\mathrm{d}^t\mathbf{x}^{\mathrm{T}}\ \mathrm{d}^t\mathbf{x}\right) = 2\,\mathrm{d}^t\mathbf{x}^{\mathrm{T}}\ \frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}^t\mathbf{x} = 2\,\mathrm{d}^t\mathbf{x}^{\mathrm{T}}\ \mathrm{d}^t\mathbf{x} =$$

$$= 2\,\mathrm{d}^t\mathbf{x}^{\mathrm{T}}\ \mathbf{L}\ \ \mathrm{d}^t\mathbf{x} = 2\,\mathrm{d}^t\mathbf{x}^{\mathrm{T}}\left(\mathbf{D} + \mathbf{W}\right)\mathrm{d}^t\mathbf{x} = 2\,\mathrm{d}^t\mathbf{x}^{\mathrm{T}}\ \mathbf{D}\ \mathrm{d}^t\mathbf{x}$$

since

$2\,\mathrm{d}^t\mathbf{x}^{\mathrm{T}}\ \mathbf{W}\ \mathrm{d}^t\mathbf{x} = 0$ due to the skew-symmetry of $\mathbf{W}$ and the symmetry of $\mathrm{d}^t\mathbf{x}^{\mathrm{T}}\ \mathrm{d}^t\mathbf{x}$.

This way we have proved that

$$\mathbf{D} = \mathbf{A}^{\nabla} = \mathbf{L}^{\mathrm{T}}\ \mathbf{A} + \dot{\mathbf{A}} + \mathbf{A}\ \mathbf{L}\,.$$

6. Summary for conjugate strain and stress measures

Measure of strain	**Measure of stress**	**Their scalar product**
ε_{ij} - Cauchy (infinitesimal)	σ_{ij} - Cauchy (true) stress	work
E_{ij} - Green-Lagrange	S_{ij} - second Piola-Kirchhoff	work
$\dot{E}_{ij}$ - rate of Green-Lagrange	S_{ij} - second Piola-Kirchhoff	power, rate of work
$\dot{F}_{ij}$ - rate of deformation gradient	τ_{ij} - first Piola-Kirchhoff	power, rate of work
D_{ij} - strain rate	σ_{ij} - Cauchy (true) stress	power, rate of work
L_{ij} - velocity gradient	σ_{ij} - Cauchy (true) stress	power, rate of work
A_{ij}^{∇} - Rivlin-Eriksen rate of Almansi	σ_{ij} - Cauchy (true) stress	power, rate of work