D. Medková

Neumann problem for the Stokes system

In this paper we construct a solution $\mathbf{u} \in H^1(G)$, $p \in L^2(G)$ of the Neumann problem for the Stokes system

$$\Delta \mathbf{u} = \nabla p \quad \text{in} \quad G, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad G, \tag{1}$$

$$T(\mathbf{u}, p)\mathbf{n}^G \equiv [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]\mathbf{n}^G - p\mathbf{n}^G = \mathbf{g} \quad \text{on} \quad \partial G \tag{2}$$

using methods of hydrodynamical potential theory. Here G is a bounded domain with connected Lipschitz boundary in R^m, $\mathbf{n}^G$ is the outward unit normal vector of G, $\mathbf{u} = (u_1, \ldots, u_m)$ is a velocity field, p is a pressure.

For $\mathbf{\Psi} = [\Psi_1, \ldots, \Psi_m] \in H^{-1/2}(\partial G, R^m)$ denote by $E_G \mathbf{\Psi}$ the hydrodynamical single layer potential with density $\mathbf{\Psi}$ and by $Q_G \mathbf{\Psi}$ the corresponding pressure.

Put

$$K'_G \mathbf{\Psi}(\mathbf{x}) = \int\limits_{\partial G} \frac{m(x_j - y_j)(x_k - y_k)(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}^G(\mathbf{x})}{\omega_m |\mathbf{x} - \mathbf{y}|^{m+2}} \, d\mathbf{y}.$$

Then $\mathbf{u} = E_G \mathbf{\Psi}$, $p = Q_G \mathbf{\Psi}$ is a weak solution of the Neumann problem for the Stokes system with the boundary condition $\mathbf{g}$ if and only if $\frac{1}{2}\mathbf{\Psi} - K'_G \mathbf{\Psi} = \mathbf{g}$.

Denote $\mathcal{R}_m = \{\mathbf{f}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}; \mathbf{b} \in R^m, A$ is a skew symmetric matrix $(A^T = -A)\}$ the space of rigid body motions.

Theorem 1. *Fix $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$. Then there is a weak solution of the Neumann problem for the Stokes system (1), (2) with the boundary condition $\mathbf{g}$ if and only if*

$$\langle \mathbf{g}, \mathbf{w} \rangle = 0 \quad \forall \mathbf{w} \in \mathcal{R}_m. \tag{3}$$

Suppose now that $\mathbf{g}$ satisfies (3) and $\mathbf{\Psi}_0 \in H^{-1/2}(\partial G, R^m)$. For a nonnegative integer k put

$$\mathbf{\Psi}_{k+1} = [(1/2)I + K'_G]\mathbf{\Psi}_k + \mathbf{g}. \tag{4}$$

Then there is $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ such that $\mathbf{\Psi}_k \to \mathbf{\Psi}$ in $H^{-1/2}(\partial G, R^m)$ as $k \to \infty$. Moreover, there are constants $0 < q < 1$, $C > 0$ depending only on G such that

$$\|\mathbf{\Psi}_k - \mathbf{\Psi}\|_{H^{-1/2}(\partial G, R^m)} \leq C q^k \bigg(\|\mathbf{g}\|_{H^{-1/2}(\partial G, R^m)} + \|\mathbf{\Psi}_0\|_{H^{-1/2}(\partial G, R^m)} \bigg). \tag{5}$$

If we put $\mathbf{u} = E_G \mathbf{\Psi}$, $p = Q_G \mathbf{\Psi}$ then $\mathbf{u}$, p is a weak solution of the problem (1), (2).

Let now $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ be such that $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$. Then there exists a solution $\mathbf{u} \in H^1(G, R^m)$, $p \in L^2(G)$ of the problem (1), (2). Denote by $\tilde{\mathbf{u}}$ the trace of $\mathbf{u}$. Denote by $D_G \tilde{\mathbf{u}}$ the hydrodynamical double layer potential with density $\tilde{\mathbf{u}}$ and by $\Pi_G \tilde{\mathbf{u}}$ the corresponding pressure. Then

$$\mathbf{u}(\mathbf{x}) = E_G \mathbf{g}(\mathbf{x}) + D_G \tilde{\mathbf{u}}(\mathbf{x}), \tag{6}$$

$$p(\mathbf{x}) = Q_G \mathbf{g}(\mathbf{x}) + \Pi_G \tilde{\mathbf{u}}(\mathbf{x}). \tag{7}$$

If we denote by K_G the adjoint operator of K'_G then

$$\frac{1}{2}\tilde{\mathbf{u}} - K_G \tilde{\mathbf{u}} = E_G \mathbf{g} \quad \text{on } \partial G. \tag{8}$$

Theorem 2. *Let $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ be such that $\langle \mathbf{g}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \mathcal{R}_m$. Fix $\tilde{\mathbf{u}}_0 \in H^{1/2}(\partial G, R^m)$. For a nonnegative integer k put*

$$\tilde{\mathbf{u}}_{k+1} = [(1/2)I + K_G]\tilde{\mathbf{u}}_k + E_G \mathbf{g}. \tag{9}$$

Then there is $\tilde{\mathbf{u}} \in H^{1/2}(\partial G, R^m)$ such that $\tilde{\mathbf{u}}_k \to \tilde{\mathbf{u}}$ in $H^{1/2}(\partial G, R^m)$ as $k \to \infty$. Moreover, there are constants $0 < q < 1$, $C > 0$ depending only on G such that

$$\|\tilde{\mathbf{u}}_k - \tilde{\mathbf{u}}\|_{H^{1/2}(\partial G, R^m)} \le C q^k \left(\|\mathbf{g}\|_{H^{-1/2}(\partial G, R^m)} + \|\tilde{\mathbf{u}}_0\|_{H^{1/2}(\partial G, R^m)} \right). \tag{10}$$

The function $\tilde{\mathbf{u}}$ is a solution of the equation (8). If $\mathbf{u}$, p are given by (6), (7) in G, then $\mathbf{u}$, p is a weak solution of the problem (1), (2) and $\tilde{\mathbf{u}}$ is the trace of $\mathbf{u}$ on ∂G.

References

[1] D. Medková: Convergence of the Neumann series in BEM for the Neumann problem of the Stokes system. Acta Applicandae Mathematicae, to appear