The σ-ideal generated by H-sets*

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Abstract

It is consistent with the axioms of set theory that the circle $\mathbb{T}$ can be covered by $\aleph_1$ many closed sets of uniqueness while a much larger number of H-sets is necessary to cover it. In the proof of this theorem, the descriptive set theoretic phenomenon of overspill appears, and it is reformulated as a natural forcing preservation principle that persists through the operation of countable support product.

1 Introduction

Let $\mathbb{T}$ be the unit circle, understood as the factor topological group $\mathbb{R}/2\pi\mathbb{Z}$ with addition. The workers in harmonic analysis have introduced several concepts of smallness for subsets of $\mathbb{T}$, among others the H-sets, sets of uniqueness, and sets of extended uniqueness. Here, an H-set is a set $A \subset \mathbb{T}$ such that there is a nonempty open interval $O \subset \mathbb{T}$ and an infinite set $b \subset \omega$ such that $nA \cap O = 0$ for every number $n \in b$ [8]. A set $A \subset \mathbb{T}$ is a set of uniqueness if the only trigonometric series which adds up to zero off A is the zero series, and a set $A \subset \mathbb{T}$ is a set of extended uniqueness if it has null mass for every Rajchman measure. Every H-set is a set of uniqueness, and every set of uniqueness is a set of extended uniqueness. I will prove

Theorem 1.1. *Suppose that the Generalized Continuum Hypothesis holds and $\kappa \geq \aleph_1$ is a regular cardinal. Then there is a cardinal preserving forcing extension in which $\mathfrak{c} = \kappa$, the unit circle is covered by $\aleph_1$ many closed sets of uniqueness while it cannot be covered by fewer than κ many H-sets.*

The statement of the theorem is perhaps less interesting than the tools needed in the argument. The forcing used is not surprising: it is the countable support product of κ many copies of the poset P_{H_σ} of Borel H_σ-positive

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sets ordered by inclusion, where H_σ is the σ-ideal on $\mathbb{T}$ σ-generated by H-sets. However, the requisite properties of the forcing are verified not through combinatorial or fusion arguments of any explicit form; instead, they are obtained through descriptive set theoretic methods together with a reference to one seemingly irrelevant hard result in harmonic analysis [6]. The main parts of the argument are of immediate interest to a specialist in harmonic analysis without any training or interest in forcing.

The plan of attack is the following. First, I will prove that the quotient forcing P_{H_σ} is bounding, which is (modulo the fact that the σ-ideal H_σ is σ-generated by closed sets) equivalent to the statement that every analytic subset of $\mathbb{T}$ either can be covered by countably many H-sets, or contains a compact subset that cannot be so covered. The treatment of definable product forcing in [12] then shows that the countable support product of many copies forcing P_{H_σ} is proper and bounding, among other things. The poset P_{H_σ} turns out to be reminiscent of the poset introduced by Saharon Shelah [9, Proposition 1.10] and studied by Otmar Spinas [11].

Second, I will show that the σ-ideal H_σ has the overspill property, that is to say, every analytic collection of compact subsets of $\mathbb{T}$ containing all countable compact sets contains also an H_σ-positive compact set. This is the main distinction between H-sets and closed sets of uniqueness that will be used in this paper. The overspill property in descriptive set theory serves primarily as a construction principle; however, I will show that a natural hereditary version of the overspill property is also a sophisticated forcing preservation property connected with a certain type of fusion arguments. Moreover, the property is maintained under suitable countable support products.

Finally, everything comes together. The countable support product of κ many copies of the poset P_{H_σ} is proper, cardinal-preserving, bounding and has the rectangular Ramsey-type property, and as a consequence it forces that the circle $\mathbb{T}$ cannot be covered by fewer than κ many H-sets. The product also has a version of the overspill property. Now, the collection of closed sets of uniqueness does not have the overspill property: there is an analytic collection of compact sets in U_σ, which by a deep result of Loomis [6] contains all countable compact sets. This means that in the product extension, the ground model coded closed sets of uniqueness still cover the circle $\mathbb{T}$.

The paper is laid out in the following way. In the first section, I introduce the overspill property, reformulate it as a forcing preservation principle, relate it to several classical forcing preservation properties, and show that it persists under the operation of the countable support product. The treatment follows the lines familiar from [12, Secton 3.10]: there is an infinite two player fusion-type game such that the overspill property is equivalent to nonexistence of a winning strategy for the bad player. In the case of a definable forcing (such as P_{H_σ} or its products) the game is determined, thus the good player has a winning strategy, and these strategies allow a straightforward treatment of the countable support product as well as other forcing operations. In the second section, I will prove the requisite overspill, covering, and definability properties of the ideal H_σ. The proofs are straightforward, and the main theorem follows as an immediate corollary when the treatment of the countable support product from the first section is factored in. Finally, in the last section, I will tie several loose ends; in particular, I will show that the circle $\mathbb{T}$ can be consistently covered by $\aleph_1$ many H-sets while the continuum is large, and it is also consistent that the circle cannot be covered by κ many closed sets of uniqueness while there are sets of size $\aleph_1$ of positive outer Lebesgue mass and of second category.

The notation in the paper follows the set theoretic standard of [3]. As a canonical reference for harmonic analysis I use [4], for descriptive set theory [5], for definable forcing [12]. If I is a σ-ideal on a Polish space X then P_I denotes the partial order of Borel I-positive sets ordered by inclusion. For a Polish space X, $K(X)$ denotes its hyperspace, i.e. the space of compact subsets of X with the Vietoris topology. A subset $I \subset K(X)$ is hereditary if it is closed under taking subsets: $K \subset L \in I$ implies $K \in I$. A closure of a set A in a topological space is denoted by a bar: $\bar{A}$. A σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every Polish space Y and every analytic set $D \subset Y \times X$ the set $\{y \in Y :$ the vertical section D_y of D above y is in the ideal $I\}$ is coanalytic.

2 The overspill property

In order to prove Theorem 1.1, I must find a way to distinguish the σ-ideals generated by H-sets and closed sets of uniqueness in a way significant enough to survive the various forcing manipulations necessary. It turns out that the necessary concept has been present in descriptive set theory for a long time:

Definition 2.1. A collection I of subsets of a compact Polish space X has the *overspill property* if there is no analytic set $A \subset K(X)$ such that $K_\omega(X) \subset A \subset I$. Here, $K_\omega(X)$ is the collection of countable compact subsets of X.

For example, the σ-ideal of countable subsets of an uncountable compact Polish space X has the overspill property, since $K_\omega(X) = I \cap K(X)$ is a coanalytic set which is not analytic. The σ-ideal σ-generated by porous sets on a compact metric space has the overspill property [7], and the σ-ideal H_σ σ-generated by H-sets will be proved in the next section to have the overspill property as well. Numerous other examples will become apparent later. The nonexamples include the σ-ideal of sets of Lebesgue null mass or in fact null capacity for any fixed capacity, or the σ-ideal of meager sets, simply because $I \cap K(X)$ in these cases turns out to be a Borel subset of $K(X)$. A more significant nonexample is the σ-ideal generated by closed sets of uniqueness, and this is exactly the point on which the proof of Theorem 1.1 hinges.

The overspill property is typically used in descriptive set theory as a powerfull construction principle. As a single application, [7] proved that there is a non-σ-porous compact subset of $\mathbb{T}$ which at the same time is a set of uniqueness. This is simply because there is an analytic collection of compact sets of uniqueness and contains all countable compact sets, while the σ-ideal of σ-porous sets has the overspill property and so cannot contain this analytic collection as a subset. The main point in this paper is that in fact the overspill property is in fact a forcing preservation property associated with a certain type of fusion. In order to isolate the connection, a game theoretic reformulation of the overspill property will come handy.

Definition 2.2. Let X be a compact metric space with a fixed countable topology basis $\mathcal{O}$ closed under finite unions and intersections, and a fixed metric. Let $I \subset K(X)$ be a coanalytic collection of compact subsets of X, containing all countable compact sets and closed under subsets. The game $G(I)$ (or $G(I, X)$ if the underlying space is not clear from the context) is played between Player I and II for infinitely many rounds. In the n-th round of the game $G(I)$, Player I produces a countable compact set C_n and Player II responds with a basic open set $O_n \in \mathcal{O}$. The players must conform to the rules $K_n \subset O_n$, $K_{n+1} \subset O_n$, and $\bar{O}_{n+1} \subset O_n$. Player I wins if the *result of the play*, the intersection $\bigcap_n O_n = \bigcap_n \bar{O}_n$ does not belong to the collection I.

Clearly, I can and will assume that Player I is playing so that $K_0 \subset K_1 \subset \dots$, and Player II is playing so that every point of O_n is within 2^{-n}-distance of a point in K_n. That way, the result of the play is equal to the closure of the union $\bigcup_n K_n$.

Theorem 2.3. *Suppose that X is a compact metric space and I is a hereditary subset of $K(X)$. Then*

1. *I has the overspill property if and only if Player II has no winning strategy in the game $G(I)$;*
2. *if I is coanalytic then the game $G(I)$ is determined.*

Proof. To prove (1), suppose first that Player II has a winning strategy σ in the game $G(I)$. I must produce an analytic collection of compact subsets of X which is a subset of I and contains all countable compact subsets of X. By tree induction build a countable tree T of partial finite plays according to the strategy σ ending with a move of Player II, such that if $t \in T$ is a node and $O \in \mathcal{O}$ is a basic open set that strategy σ can produce in the next round after t is followed with some challenge of Player I, then there is an immediate successor $s \in T$ of the node t that indeed ends with the strategy σ playing the set O. Now, if $b \in [T]$ is a branch through the tree T, it is an infinite play against the strategy σ, so Player II won and the end result of it is in the collection I. Consider the set $A = \{C \in K(X) :$ for some branch $b \in [T]$, K is covered by the end result of the play $b\}$. This is an analytic collection of compact sets, and since I is closed under subsets, $A \subset I$. Moreover, all countable compact sets belong to A: if $C \in K(X)$ is countable, then by induction on n build nodes $t_n \in T$ so that C is a subset of the last move in the play t_n. The induction step is possible to perform, since C is a legal move of Player I in the next round past t_n and it must induce the strategy to answer with a set which is still a superset of C. The construction of tre tree T guarantees that there is an immediate successor t_{n+1} of t_n whose last move is still a superset of C as desired. In the end, the end result of the play $\bigcup_n t_n$ is a superset of C and shows that $C \in A$.

On the other hand, suppose that $A \subset K(X)$ is an analytic collection containing all countable compact sets, and $A \subset I$. I must produce a winning strategy σ for Player II. Let $g : \omega^\omega \to K(X)$ be a continuous function such that $A = \mathrm{rng}(g)$. Player II will win by producing, along with the moves of the game, sequences $t_n \in \omega^n$ so that $0 = t_0 \subset t_1 \subset$ and for every number $n \in \omega$, the following statement (*) holds: for every countable compact subset $K \subset O_n$ there is a point $y \in \omega^\omega$ extending t_n such that $K \subset g(y)$. That way, the end result $L \subset X$ of the play will be a subset of $g(y)$ where $y = \bigcup_n t_n$, and as $g(y) \in I$ and I is closed under subsets, $L \in I$ and Player II won. To see that $L \subset g(y)$, observe that L is the closure of $\bigcup_n K_n$; thus, if some point of L did not belong to the compact set $g(y)$, already some point of some K_n together with its whole open neigborhood would not belong to $g(y)$, and by the continuity of the function g there would have to be a number $m > n$ such that for every point $y' \in \omega^\omega$ with $t_m \subset y'$, the set $g(y')$ is disjoint from that open neighborhood, contradicting (*) at m.

It is necessary to prove that Player II can maintain (*) at every stage of the play. (*) holds at 0 by the assumptions on the set A no matter what the open set O_0 is. Now suppose that (*) holds at round n and Player I produces a set K_{n+1}. I must show that there is an open set O containing K_n and a number $i \in \omega$ such that (*) holds with $O_{n+1} = O$ and $t_{n+1} = t_n^\frown i$. Suppose for contradiction that this is not the case. Choose inclusion decreasing basic open sets $\langle P_i : i \in \omega \rangle$ such that each of them is a legal move for Player II at this stage and $K_{n+1} = \bigcap_i P_i$. None of them fulfills (*) with any j, so there will be countable compact sets $L_i \subset P_i$ such that for every $j \in i$ and every point $y \in \omega^\omega$ with $t_n^\frown j \subset y$ the inclusion $L_i \subset g(y)$ fails. Now, the closure L of the union $\bigcup_i L_i$ contains only the points in $\bigcup_i L_i$ and points in K_n, so in particular L is a countable compact subset of O_n. By the induction hypothesis, there must be a point $y \in \omega^\omega$ extending t_n such that $L \subset g(y)$. This, however, contradicts the choice of the set L_i where i is any number than the first entry of the sequence y past t_n!

(2) of the theorem is proved by a standard unraveling argument. Since the collection I is coanalytic, there is a continuous function $g : \omega^\omega \to K(X)$ whose image is the complement of I. Consider the game $G'(I)$ which is slightly more difficult than $G(I)$ for Player I. The game $G'(I)$ proceeds in the same way as the previous one, except in some rounds, Player I also indicates a natural number i_n. Player I wins he indicated infinitely many numbers, thereby creating a sequence $y \in \omega^\omega$, and $g(y) \subset \bigcap_n O_n$. Thus, if Player I wins in a play of the game $G'(I)$, then he also won the associated play of the game G: the set $g(y)$ is not in I, and as the collection I is closed under subsets, the set $\bigcap_n O_n$ cannot belong to I either. In the wide tree of all possible plays of the game $G'(I)$, the plays in which Player I wins forms a G_δ set, and the game $G'(I)$ is therefore determined. I will show that winning strategies for both players in the new game translate to winning strategies in the old game.

It is clear that if Player I has a winning strategy in the game $G'(I)$, then the same strategy, merely omitting the additional moves, will be his winning strategy in the game $G(I)$. Now suppose that σ' is a winning strategy for Player

II in the game $G'(I)$. To get a winning strategy for this player in the original game, note that σ' can be easily improved not to depend on the choices of the numbers i_n as long as these numbers are smaller than the index of the round at which thay are played. Simply at each round consider the finitely many possibilities for such choices of these numbers in the previous round and play the intersection of all sets that σ' advises to play against each. I claim that this improved strategy σ is in fact winning for Player II in the original game $G(I)$. Indeed, if there is a play p in the game $G(I)$ against this strategy in which Player II loses, then the result L of that play cannot be in I and there is a point $y \in \omega^\omega$ such that $g(y) = L$. Consider the play p' against the strategy σ' in which Player I plays the same compact sets as in p and produces the point y in such a way that each number on it is added at a round with index larger than that number. The definition of the strategy σ implies that the moves of the strategy σ' in p' will be supersets of the corresponding moves of the strategy σ in p, therefore the moves of Player I in p' are legal and the result L' of the Play p' will be a superset of $L = g(y)$, resulting in Player I's victory. This of course contradicts the choice of the strategy σ'. □

As one simple corollary of the theorem, note that the overspill property is closed under unions of finitely many coanalytic hereditary sets: finitely many winning strategies for Player I can be combined by just taking unions of moves in each.

The nature of winning strategies for the two players may be subject to an interesting discussion. As the most trivial example for Player I, he has a winning strategy if I is the collection of countable compact subsets of the Cantor space $X = 2^\omega$. He will win by playing finite sets C_n such that $C_0 \subset C_1 \subset \dots$ such that for every number n and every point $x \in C_n$ there is another point $x \neq y \in C_{n+1}$ such that x, y agree on the first n positions. In the end, the result of the play must contain the closure of the set $\bigcup_n C_n$, which is perfect, therefore uncountable and winning for Player I. Note the similarity between this winning strategy and the fusion arguments for the Sacks forcing (which is isomorphic to a dense subset of P_I).

As the most trivial example for Player II, he has a winning strategy if $I =$the Lebesgue null sets. He will simply make use of the fact that every countable set is null and at the n-th move, he will cover the move K_n with an open set of mass $\leq 2^{-n}$. In this way, the result of the play will be Lebesgue null and therefore winning for Player II.

Thus, in the case of $I = K_\omega(2^\omega)$, Player I will win with just playing finite sets. This leads to a rather trivial version of the overspill property, and the existence of such winning strategies is equivalent to the weak Sacks property of the quotient forcing P_I, as shown below. The less trivial examples use infinite compact sets as moves for Player I's winning strategy, and that includes the σ-ideal σ-generated by porous sets as well as the primary concern of this paper– the σ-ideal generated by the H-sets. I do not know an example in which Player I has a winning strategy and that strategy must necessarily use sets of infinite Cantor-Bendixson rank.

The winning strategies for Player I in the overspill game certainly remind the alert reader of various forcing fusion arguments. To exploit this parallel, I will need to consider a natural hereditary version of the overspill property. All examples of overspill property in this paper in fact have the hereditary version as well, even though in some circumstances such as the σ-ideal of σ-porous sets [7], this requires a nontrivial modification of the overspill proof.

Definition 2.4. A σ-ideal I on a Polish space X has the *hereditary overspill property* if for every compact set $K \subset X$, $K \notin I$, the σ-ideal $I \restriction K$ has the overspill property.

Theorem 2.5. *Let I be a σ-ideal on a Polish space X such that the quotient forcing P_I is proper and bounding. The following are equivalent:*

1. *I has the hereditary overspill property;*
2. *for every Polish space Y and every analytic set $A \subset K(Y)$ containing all countable compact sets, P_I forces Y to be covered by the ground model elements of the set A.*

Proof. On one hand, if I does not have the hereditary overspill property, then there is a compact I-positive set $C \subset X$ such that I on C does not have the overspill property, and therefore there is an analytic collection $A \subset K(C)$ containing all countable compact subsets of C, all of whose elements are in the σ-ideal I. Clearly, $C \Vdash \dot{x}_{gen}$ does not belong to any ground model coded I-small sets and so in particular to any ground model coded elements of A, and (2) fails.

On the other hand, if (1) holds, Y, A are as in (2), and $B \in P_I$ is a condition and $\dot{y}$ is a P_I-name for an element of the space Y, I must find an element of A such that a condition stronger than B forced $\dot{y}$ to this element of A. By the bounding property of the quotient P_I, strengthening B if necessary I may assume that B is compact and that there is a continuous function $f : B \to Y$ such that $B \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$ [12, Theorem 3.3.2]. Use the hereditary overspill property to thin out B further if necessary to make sure that the ideal I restricted to B has the overspill property. Let $A' = \{C \subset B$ compact: there is $K \in A$ such that $f''C \subset K\}$. This is an analytic collection of compact subsets of C which contains all countable compact sets, since an image of countable set is countable and A contains all countable compact subsets of Y. The overspill property yields an I-positive set $C \subset K$ with $C \in A'$. There is $K \in A$ such that $f''C \subset K$, and $C \Vdash \dot{y} \in \dot{K}$ as required. □

Corollary 2.6. *If I is a σ-ideal on a Polish space with the hereditary overspill property such that the quotient forcing P_I is proper and bounding. Then P_I preserves Baire category.*

In other words, P_I forces that the set of ground model elements of the unit interval is still non-meager. Restated without the forcing relation, there is no I-positive compact set K and a Borel set $D \subset K \times [0, 1]$ such that the horizontal sections of D are in the σ-ideal I while the vertical sections of the complement are meager. In the particular case under investigation in this paper, the σ-ideal H_σ, this follows already from the fact that it is σ-generated by closed sets.

Proof. By [1, Theorem 2.2.4], Baire category preservation is equivalent to P_I not adding an eventually different real. Since P_I is assumed to be bounding, this is equivalent to not adding a bounded eventually different real. So let $f \in \omega^\omega$ be a function and let Y be the space $\Pi_n f(n)$. Let A be the set of those compact sets $L \subset Y$ for which there is a function $g \in \omega^\omega$ such that g infinitely often meets every element of L. If I show that L is an analytic subset of $K(Y)$ and contains every countable compact subset of Y, the theorem will apply to show that P_I forces every function dominated by f to belong to a ground model coded set in A, and therefore infinitely many times equal to a ground model function as desired.

Now, it is clear that L contains every countable compact set, since an obvious diagonalization argument yields a function infinitely many times equal to every element of the compact set. The analyticity of A is slightly more challenging. I will show that for a compact set $L \in K(Y)$, the following are equivalent:

- $L \in A$;
- for every $n \in \omega$ there is a larger $m \in \omega$ and a function $g : [n, m] \to \omega$ such that every element of L has nonempty intersection with g.

Clearly, the second item yields an analytic, in fact G_δ, description of the set A. The second item also easily implies the first, since the finite functions it provides can be pieced together to give a function in ω^ω to which every element of L is inifinitely many times equal. On the other hand, if the second item fails for some $n \in \omega$ and $g \in \omega^\omega$ is a function, it is not difficult to show that there must be an element $h \in L$ such that g and h agree only at some entries below n, and therefore $L \notin A$. To find h, for every $m \in \omega$ larger than m use the failure of the second item to find a function h_m which disagrees with g on all values between n and m, and use the compactness of the set L to find an accumulation point $h \in L$ of the set $\{h_m : m \in \omega\}$. The function h has the required properties. □

Corollary 2.7. *If I is a σ-ideal on a Polish space with the hereditary overspill property such that the quotient forcing P_I is proper and bounding, then P_I does not add a random real.*

Proof. Of course, this follows from the previous corollary since the random forcing does not preserve the Baire category. Still, it is curious to see the precision of the complexity arguments at work. Let Y be the unit interval equipped with the Lebesgue measure λ. The set $A = \{C \in K(Y) : \lambda(C) = 0\}$ is analytic, in fact G_δ, and it contains all countable compact sets. Thus, in the P_I extension, the ground model coded elements of the set A still cover the unit interval, so every real in the unit interval belongs to a ground model coded compact sets and therefore cannot be random. □

Corollary 2.8. *If I is a σ-ideal on a Polish space with the hereditary overspill property such that the quotient forcing P_I is proper and bounding, then in the P_I-extension, the circle $\mathbb{T}$ is covered by ground model coded closed sets of uniqueness.*

Proof. This is the main point of this paper. The collection of closed sets of uniqueness is coanalytic in $K(\mathbb{T})$, so the theorem cannot be applied directly to it. However, it has a suitable analytic, in fact $G_{\delta\sigma}$, subcollection U' containing all countable compact sets. The subcollection is defined for example in [4, Section IV.2, Proposition 8]; the fact that every countable compact sets belong to it was proved by Loomis [4, Section V.5, Theorem 5], [6]. Thus, in fact, P_I forces that the circle is covered by ground model elements of this analytic collection. □

A very common special type of overspill occurs if Player I has winning strategies that only use finite sets as moves. While from the descriptive set theoretic point of view this situation is perhaps somewhat trivial, the forcing point of view offers an interesting reformulation, which yields a great number of σ-ideals with the hereditary overspill property.

Definition 2.9. A forcing P is said to have *the weak Sacks property* if for every function $f \in \omega^\omega$ in the P-extension there is a ground model infinite set $a \subset \omega$ and a ground model function g with domain a such that for every $n \in a$, $|g(n)| \leq 2^n$ and $f(n) \in g(n)$.

The weak Sacks property is an obvious weakening of Sacks property which requires $a = \omega$ [1, Definition 6.3.37]. It clearly implies the bounding property, and in a suitably definable case, its conjunction with adding no independent reals is in fact equivalent to the conjunction of the bounding property and P-point preservation [13]. The main point here is

Theorem 2.10. *Let I be a σ-ideal on a Polish space X such that the poset P_I is proper and bounding, and the set $I \cap K(X)$ is coanalytic. Then the following are equivalent:*

1. *P_I has the weak Sacks property;*
2. *every I-positive Borel set has an I-positive compact subset C such that Player I has a winning strategy in the game $G(I, C)$ which uses only finite sets as moves.*

Proof. (2) immediately implies (1). Suppose that $B \in P_I$ is a condition and $\dot{y}$ a name for a point in the Baire space ω^ω. Since the forcing P_I is bounding, there is a compact I-positive set $C \subset B$ and a continuous function $f : C \to \omega^\omega$ such that $C \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$ and Player I has a winning strategy σ in the game $G(I, C)$ that uses only finite sets as moves. Now consider the counterplay against the strategy σ in which Player II at round n finds a number $m = m_n$ such that $2^m > |K_n|$ and plays an open set O_n covering K_n on which the continuous function $x \mapsto f(x)(m)$ takes fewer than 2^m many values, collected in some set $g(m)$ of size $< 2^m$. In the end, the result of the play is an I-positive compact set $D \subset C$, and, writing $a = \{m_n : n \in \omega\}$, it forces $\forall n \in a\ \dot{y}(n) \in \check{g}(n)$ as desired.

The other direction is more difficult. Suppose that (2) fails below some I-positive Borel set $B \subset X$. Use the bounding property to thin out B if necessary so that all open sets from the countable basis of the space X are relatively clopen in B and B is compact. Since (2) fails, it must be the case that Player II has a winning strategy σ in the game G similar to $G(I, B)$ except Player I is allowed to play finite sets only in the game G. Now, by induction on $n \in \omega$ build increasing finite sets e_n of finite plays of the game G in which Player II follows the strategy σ and, whenever $t \in e_n$ is a play with the last move the strategy σ made in it a certain open set O, whenever $K \subset O \cap B$ is a set of size 2^n then there is a one round extension s of t in the set e_{n+1} such that the last move of strategy σ in s contains K as a subset.

In order to see how to make the induction step, choose $t \in e_n$ and note that the set $(O \cap B)^{2^n}$ is compact, and the set $U = \{P^{2^n}$:there is a move $K \in [O \cap B]^{2^n}$ of Player I that provokes the strategy σ to play $P\}$ covers it, since every set of size 2^n will provoke σ's answer that covers it. A compactness argument will yield a finite subcover of U, which will lead immediately to the construction of the finite set e_{n+1} on the next stage of induction.

Once the induction is complete, consider the function f defined on the set B so that $f(x)(n) =$ the intersection of the collection of those open sets used as last moves of plays in the set e_n to which x belongs. I claim that the name $\dot{f}(\dot{x}_{gen})$ violates the weak Sacks property: there is no condition $C \subset B$, with an infinite set $a \subset \omega$ and a function g on a such that for every $n \in a$, $|g(n)| < 2^n$ and $C \Vdash \dot{f}(\dot{x}_{gen})(n) \in \check{g}(n)$. Suppose for contradiction that such C, a, g exist and thin out C so that for every $x \in C$ and every $n \in a$, $f(x)(n) \in g(n)$. Let $n_i : i \in \omega$ enumerate the set a in an increasing order and by induction on i build plays $t_i \in e_{n_i}$ so that $t_0 \subset t_1 \subset \dots$, t_{i+1} is a one move extension of t_i, and its last move still contains C as a subset. If this succeeds, then in the end the result of the play $\bigcup_i t_i$ contains C as a subset and Player I won, contradicting the assumption that σ was a winning strategy for Player I. The induction step is simple: given t_i, find a set $K \subset C$ of size $2^{n_{i+1}}$ such that the values $f(x)(n_{i+1})$ for $x \in K$ exhaust all possibilities in C. Note that there are fewer than $2^{n_{i+1}}$ possibilities for this value at the set C since they are controlled by the function g. By the construction of the set $e_{n_{i+1}}$, there must be a one round extension t_{i+1} of t_i such that the last move O on it contains K as a subset. But then, O also contains C as a subset: for every point $x \in C$, there is $x' \in K$ such that $f(x)(n) = f(x')(n)$, and by the definition of the function f, $x \in f(x)(n) = f(x')(n) \subset O$ as desired! ☐

This theorem yields many examples of σ-ideals with the hereditary overspill property, since Sacks or weak Sacks property are fairly common in the realm of definable forcing. Thus, the σ-ideal σ-generated by Borel subsets of 2^ω consisting of pairwise non-modulo-finite-equal sequences has the overspill property, since the quotient forcing is proper and has the Sacks property [12, Section 4.7.1].

With a reformulation of overspill as a forcing preservation property, a question immediately arises whether it persists under the usual forcing operations. The game characterization of overspill leads to preservation theorems for the countable support product of definable forcings. There is a similar preservation theorem for the countable support iteration with essentially identical proof; as it is not needed in this paper, I omit it.

Theorem 2.11. *Let $I_n : n \in \omega$ be $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals on respective compact metric spaces $X_n : n \in \omega$, such that the quotient forcings are proper and bounding. If each of the ideals I_n has the hereditary overspill property, then so does their product ideal $\Pi_n I_n$.*

Theorem 2.12. *Let κ be a cardinal and $\{I_\alpha : \alpha \in \kappa\}$ be $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals on Polish spaces with the hereditary overspill property such that the quotient P_{I_α} are proper and bounding for every $\alpha \in \kappa$, and let P be the countable support product of the posets $\{P_{I_\alpha} : \alpha \in \kappa\}$. Then for every Polish space Y and every analytic set $A \subset K(Y)$ containing all countable compact sets, P forces Y to be covered by the ground model elements of the set A.*

This theorem has a minor strengthening expressed in terms of a suitable cardinal invariant. Define the *overspill number* $\mathfrak{os}$ as the supremum of the cardinal numbers $\min\{|B| : B \subset A, \bigcup B = X\}$ as X ranges over all Polish spaces and A ranges over all analytic subsets of $K(X)$ containing all countable compact sets. It is immediate that $\mathfrak{os}$ is not smaller than the dominating number–just choose $X = \omega^\omega$ and $A = K(X)$. It is also true that $\mathfrak{os}$ is not smaller than the uniformity of the meager ideal using the analytic family from Corollary 2.6. Since, as proved above, the Sacks property implies overspill, which in turn implies the preservation of covering by analytic families containing all countable sets, it seems plausible that $\mathfrak{os}$ is not greater than the cofinality of the null ideal, but I do not have a proof of that statement. The preservation theorem 2.12 can be improved to state the following:

Theorem 2.13. *Suppose that the Generalized Continuum Hypothesis holds, κ is a cardinal, and $I_\alpha : \alpha \in \kappa$ are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals on Polish spaces with the hereditary overspill property such that the quotient forcings are proper and bounding. Then the countable support product of the posets $\{P_{I_\alpha} : \alpha \in \kappa\}$ forces $\mathfrak{os} = \aleph_1$.*

Proof. For the proof of Theorem 2.11, it is first necessary to make sense of the product ideal $I = \Pi_n I_n$. This is the ideal on $\Pi_n X_n$ generated by those Borel sets which do not contain a box of the form $\Pi_n B_n$ where $B_n \subset X_n$ is a Borel I_n-positive set for every number n. Since the posets entering the product are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, proper, bounding, and Baire category preserving by Corollary 2.6, this is indeed a σ-ideal by [12, Theorem 5.2.6], it is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, the full support product is proper, bounding, and preserves Baire category, and it is naturally isomorphic to a dense subset of the quotient poset P_I.

I will start with the product of two ideals I_0 and I_1. Let I be the product ideal on the space $X_0 \times X_1$, and let $B \subset X_0 \times X_1$ be a Borel I-positive set. Thinning out if necessary, I may assume that in fact $B = C_0 \times C_1$ for some compact sets $C_0 \subset X_0, C_1 \subset X_1$ which are positive in the respective ideals and where Player I has a winning strategy σ_0, σ_1 in the respective overspill games by Theorem 2.3. I will find a winning strategy σ for Player I in the game $G(I, B)$. To specify the game completely, fix the basis on the product space which is in a nactural sense product of the bases on the spaces entering the product, so every set in the product basis is a finite union of products of sets in the bases on the spaces X_0, X_1 respectively. The n-th move K_n of the strategy σ is simply the product of the moves $K_n^0 \times K_n^1$ that the strategies σ_0, σ_1 would produce in related plays of their respective games. Note that K_n, as a product of countable sets, is again countable and if O_n is a basic open set covering K_n, then there are basic open sets O_n^0, O_n^1 on X_0, X_1 respectively such that $K \subset O_n^0 \times O_n^1 \subset O_n$. Thus, I may assume that Player II in fact plays such product sets and apply the strategies σ_0, σ_1 to the moves O_n^0, O_n^1 to get the sets K_{n+1}^0, K_{n+1}^1. In the end, the result of the play on the product is a product of the results of the plays on each of the two coordinates, and so positive in the product ideal. This confirms that Player I always wins if he sticks to the product strategy.

The easy diagonalization argument necessary for the case of the product of infinitely many ideals is left to the reader.

□

Proof. The proof of Theorem 2.12 is just a routine massaging of the previous argument. Consider the κ product of posets $\{P_{I_\alpha} : \alpha \in \kappa\}$. Suppose for contradiction that Y is a Polish space and $A \subset K(Y)$ is an analytic set containing all countable compact sets. Let $\dot{y}$ be a P-name for an element of Y and $p \in P$ is a condition; I must find a set $C \in A$ and a condition $q \leq p$ which forces $\dot{y} \in \dot{C}$. Let M be a countable elementary submodel of a large enough structure containing $p, \dot{x}$. A standard argument shows that there are I_α-positive compact sets $K_\alpha : \alpha \in \kappa \cap M$ such that the product $L = \Pi_{\alpha \in \kappa \cap M}$ consists of M-generic sequences only for the product forcing meeting the condition p, and the function $g : L \to Y$ given by $g(\vec{x}) = \dot{y}/\vec{x}$ is continuous. Let $A' \subset K(L)$ be the collection of all compact subsets of L whose images are covered by sets in A; this is an analytic collection of sets containing all countable sets. By the hereditary overspill property of the product ideal $\Pi_{\alpha \in \kappa \cap M} I_\alpha$, there are compact sets $\{K'_\alpha : \alpha \in \kappa \cap M\}$ such that $K'_\alpha \subset K_\alpha$ and $L' = \Pi_{\alpha \in \kappa \cap M} K'_\alpha \in A'$. The g-image of L' is then covered by some set $C \in A$, and a review of the definitions shows that L' is a condition below p that forces $\dot{y} \in \dot{C}$ as desired. □

Proof. The additional degree of difficulty in Theorem 2.13 compared to Theorem 2.12 lies in the possibility that new Polish spaces and new analytic collections of compact sets are added by the product, and in theory they could send the cardinal $\mathfrak{os}$ up. I will prove the key claim that rules out this possibility, and leave the further routine details to the reader. The claim is used to prove the overspill property of the product poset $(\Pi_{\alpha\in a}P_{I_\alpha})^V$ in the model $V[G\cap\Pi_{\alpha\in b}P_{I_\alpha}]$, where $a,b\subset\kappa$ are disjoint countable sets.

Claim 2.14. *Suppose that I is a coanalytic collection of compact subsets of a compact metric space X, closed under subsets, with the overspill property. Suppose that $V[G]$ is a bounding extension of V. Then in $V[G]$, I^* has the overspill property, where I^* is the collection of those compact sets which do not have a ground model coded compact subset which is not in I.*

Note that in $V[G]$, I^* is typically properly larger than I, so the statement of the claim is nontrivial. To prove the claim, suppose that I^* does not have the overspill property in $V[G]$, so Player II has a winning strategy $\sigma^*\in V[G]$ in the game $G(I^*)$ by Theorem 2.3. By the same theorem, Player I also has a winning strategy $\sigma\in V$ in the game $G(I)$ in the ground model. Let $a\in V$ be the countable set of all finite sequences of countable compact sets that the strategy σ can possibly produce against some counterplays by Player II. As $V[G]$ is a bounding extension of V, there is a ground model function $h:a\to\mathcal{P}(\mathcal{O})$ that assigns to every sequence $t\in a$ a finite set $h(t)$ such that the move the strategy σ^* dictates, Player I having played t, is in this finite set. Now, in the ground model V consider the play p of the game $G(I)$ in which Player I observes the strategy σ and at each intermediate stage t, Player II plays a set which is a subset of the intersection of all those elements of $h(t)$ which are basic open subsets of X covering the last move of Player I. In the extension, consider the play p^* in which Player II follows the strategy σ^* and Player I plays the same sequence of compact sets as in p. The choice of the function h implies that p^* is indeed a legal play against the strategy σ^*. The resulting set in both of these plays is the closure of the union of the sets that Player I played. It is certainly coded in the ground model, since P is in the ground model, and it is not in I, since σ was a winning strategy for Player I in the ground model. Thus, Player II lost the play p^*, contradicting the assumption that the strategy σ^* was winning. □

The attentive reader should not fail to notice how overspill fits into the doctrine of [12, Section 3.10]. Many forcing properties can be restated as the bad player not having a winning strategy in a certain game. If the forcing in question is suitably definable, then the game in question is determined, and the winning strategies for the good player can serve as a tool for proving preservation theorems for product or iteration.

3 The forcing associated with H-sets

Let H_σ be the σ-ideal on the circle $\mathbb{T}$ generated by H-sets. The quotient poset P_{H_σ} of Borel H_σ-positive sets ordered by inclusion is proper and preserves Baire category by the virtue of the generating sets being closed [12, Theorem 4.1.2]. We will show that the poset is also bounding, and the σ-ideal has the hereditary overspill property and a simple definition. The main theorem of the paper will then follow by a rather straightforward application of the work of the previous section.

Theorem 3.1. *The σ-ideal H_σ*

1. *has the hereditary overspill property;*
2. *has the covering property–every analytic set either is in H_σ or contains a compact subset not in H_σ;*
3. *is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

The second item is equivalent to the bounding property of the quotient by [12, Theorem 3.3.2].

In order to show that the σ-ideal H_σ on the circle $\mathbb{T}$ has the covering property, I will first introduce an abstract class of σ-ideals, prove that they all have the properties listed in the theorem, then show that the properties are closed under suitable countable unions of σ-ideals, and finally, I will show that the σ-ideal H_σ can be generated by a countable union of the σ-ideals in the abstract class.

Definition 3.2. Let X be a Polish space and $\vec{C} = \langle C_n : n \in \omega\rangle$ be a sequence of Borel subsets of X. The σ-ideal $I_{\vec{C}}$ on the space X associated with the sequence $\vec{C}$ is σ-generated by the sets $\bigcap_{n\in b} C_n$ as b ranges over all infinite subsets of ω.

Example 3.3. Let $X = 2^\omega$ and let $\vec{C}$ enumerate all sets of the form $\{x \in X : t \subset x\}$ as t ranges over all finite binary sequences. The generators of the σ-ideal $I_{\vec{C}}$ have all size at most one, and therefore the σ-ideal consists of exactly the countable sets.

Example 3.4. Let $X = 2^\omega$ and let $\vec{C}$ enumerate all open sets of the form $\{x \in X : x(n) = b\}$, as n ranges over all natural numbers and b ranges over the set $\{0, 1\}$. Then the generators of the derived σ-ideal $I = I_{\vec{C}}$ are those compact subsets $A \subset 2^\omega$ such that $\bigcap A$ is infinite. In [11], Otmar Spinas proved that every analytic I-positive set contains a compact I-positive subset of a quite specific form. The computation yielded an isomorphism of a poset introduced by Shelah with a dense subset of the poset of analytic I-positive sets ordered by inclusion modulo the ideal I.

It is quite possible that the list of σ-ideals of the form $I_{\vec{C}}$ does not go very far beyond this short list in the sense that their quotient posets must necessarily have properties close to the quotients of the two examples above. From the forcing point of view, it is important to observe that increasing the Polish topology of the underlying space X if necessary, one may enter the situation in which the sets on the $\vec{C}$-sequence are closed while the Borel structure does not change; therefore, the quotient forcing P_I is proper and preserves Baire category by [12, Theorem 4.1.2].

Theorem 3.5. *Every analytic set either belongs to $I_{\vec{C}}$ or contains a compact $I_{\vec{C}}$-positive subset. The σ-ideal $I_{\vec{C}}$ has the hereditary overspill property, and it is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

Proof. Use [5, Section 13.A] to find a larger Polish topology on the space X which yields the same Borel structure and in which all the sets on the sequence $\vec{C}$ are clopen. All topological notions from now on will refer to the new topology unless stated otherwise. Note that if $K \subset X$ is a compact set in the new topology then it is compact in the old one, and the two relative topologies on X coincide, as a newly closed subset of K is compact in the new topology and therefore compact in the old one.

Note that every $I_{\vec{C}}$-positive set must meet the complements of all but finitely many sets on the sequence $\vec{C}$. A nonempty compact set $K \subset X$ such that for every open set O, $K \cap O$ is either empty or else it meets the complements of all but finitely many sets on the sequence $\vec{C}$, is $I_{\vec{C}}$-positive. To see this, note that none of the closed σ-generators of the σ-ideal can cover any nonempty relatively open subset of K, thus the σ-generators must be meager in K.

For the first assertion, let $A \subset X$ be an analytic set; I must produce an I-positive set $K \subset A$. Find a closed subset of ω^ω such that the set A is its image under a continuous function g. Let $T \subset \omega^{<\omega}$ be a tree such that the closed set is equal to $[T]$. Thinning out the tree T and the set A if necessary, we may assume that for every node $t \in T$, $g''[T \restriction t] \notin I$. I will find a function $h \in \omega^\omega$ so that, writing T_h for the tree of all sequences of T that are pointwise dominated by h, $g''[T_h] \notin I_{\vec{C}}$. Since $[T_h]$ and its continuous images are compact, this will complete the proof. The proof uses the following simple claim:

Claim 3.6. *For every node $t \in T$ there is $i \in \omega$ such that the sets $g''\{y \in [T] : t \subset y\}$ and $g''\{y \in [T] : t \subset y$ and the first entry of y past t is below $i\}$ meet complements of the same sets on the sequence $\vec{C}$.*

Proof. This is immediate. Just find any number j such that $t^\frown j \in T$ and note that the set $g''\{y \in T : t^\frown j \subset y\}$ is $I_{\vec{C}}$-positive and so meets all but finitely many setson the sequence $\vec{C}$. For the finitely many exceptions, find $i > j$ large enough so that if there is $y \in [T]$ with $t \subset Y$ and $g(y)$ in one of the exceptional sets, then there is such a y which has the first entry past t smaller than i. This number i will work. □

A repeated use of this claim will yield a function $h \in \omega^\omega$ such that for every node $t \in T$ pointwise dominated by h, the sets $g''\{y \in [T] : t \subset y\}$ and $g''\{y \in [T] : t \subset y$ and the first value of y past t is smaller than the corresponding value of $h\}$ meet the same sets on the sequence $\vec{C}$. I claim that this function h works as desired.

To see this, first observe that for every node $t \in T$ pointwise dominated by h, the sets $g''\{y \in [T] : t \subset y\}$ and $g''\{y \in [T] : t \subset y, y \leq h\}$ meet complements of the same sets on the sequence $\vec{C}$. For if $y \in [T]$ extends t and $g(y) \in\notin C_n$ for some number $n \in \omega$, inductively on m one can construct successive extensions $s_m \in T$ of t pointwise dominated by h, and points $y_m \in [T]$ extending them so that $g(y_m) \notin C_n$, and in the end the point $z \in [T]$, $z = \bigcup_m s_m$ will be pointwise dominated by h and $g(z) \notin C_n$ since the set C_n is open and the function g is continuous. This means that every relatively open subset of the compact set

$g''[T_h]$ meets the complements of all but finitely many sets on the sequence $\vec{C}$ and so $g''[T_h] \notin I_{\vec{C}}$ as desired.

To prove the hereditary overspill property, let A be a compact $I_{\vec{C}}$-positive set and thin it out if necessary to make sure that its intersections with opens sets are either empty or $I_{\vec{C}}$-positive. I will construct a winning strategy for Player I in the game $G(I, A)$. The following simple claim parallel to Claim 3.6 will provide the moves for the strategy:

Claim 3.7. *Every $I_{\vec{C}}$-positive set has a countable compact subset which meets the complements of exactly the same sets on the sequence $\vec{C}$.*

In fact, the countable closed set will be just a converging sequence together with its limit.

Proof. Thinning out the original set A if necessary, I may assume that the intersections of A with open sets of X are either I-positive or empty. Let $x \in A$ be an arbitrary point. Build an increasing sequence of numbers $\langle n_i : i \in \omega \rangle$ by induction so that the I-positive set $A \cap \bar{B}(x, 2^{-n_i})$ meets complements of all sets on the sequence $\vec{C}$ indexed by numbers larger than n_{i+1}. For every number n with $n_{i+1} < n \leq n_{i+2}$ find a point $x_n \in A \cap \bar{B}(x, 2^{-n_i}) \setminus C_n$. The points x_n chosen in this way converge to x and together meet the complements of all but finitely many of the sets on the sequence $\vec{C}$. Add perhaps finitely many more points of K to meet the other finitely many open sets on the sequence if possible, and add x itself; the result will be the desired countable compact subset of A. $\square$

Now I am ready to describe the winning strategy for Player I in the overspill game. Fix a complete metric for the Polish space X with the new topology. Thin out K if necessary to make sure that all intersections of K with open sets are either I-positive or empty. In the game $G(K, I_{\vec{O}})$, Player I proceeds so that after the round m, he finds finitely many open balls $\{B_i : i \in k\}$ of radius $< 2^{-m}$ whose closure is a subset of P_m and which cover the set K_m. For each of the balls, he uses the claim to find a countable compact set $L_i \subset K \cap \bar{B}_i$ that meets the complements of the same sets on the sequence $\vec{C}$ as the intersection $K \cap \bar{B}_i$ itself, and he plays $K_{m+1} = K_m \cup \bigcup_i L_i$; his first move K_0 is an arbitrary nonempty countable closed subset of K.

In so playing, Player I must have won, since the result of the play, the closure L of $\bigcup_m K_m$ has the property that every nonempty relatively open subset of it meets the complements of all but finitely many sets on the sequence $\vec{C}$.

Finally, the complexity of the σ-ideal $I_{\vec{C}}$ is easy. Note that the status of σ-ideal as $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ depends only on the Borel structure of the underlying Polish space and not on the topology, so I can as well work with the new topology. There, the σ-ideal is σ-generated by closed sets and every analytic positive set has a compact positive subset, thus the asumptions of [12, Theorem 3.8.9] are satisfied and it is enough to show that the set of positive compact sets is analytic. A compact set K is $I_{\vec{C}}$-positive if it contains a nonempty compact subset $L \subset K$ (obtained by removing from K all open neighborhoods in which K is small) such that for every open set O, $K \cap O$ is either empty or it meets complements of all but finitely many sets on the sequence $\vec{C}$, and this is an analytic statement. $\square$

Theorem 3.8. *Let X be a Polish space an $\{I_n : n \in \omega\}$ be a countable collection of σ-ideals on X σ-generated by closed sets. Let I be the σ-ideal σ-generated by $\bigcup_n I_n$.*

1. *if for every $n \in \omega$, every I_n-positive analytic set contains a compact I_n-positive subset, then the same holds for the σ-ideal I;*
2. *if, in addition, every ideal I_n is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, then so is I;*
3. *if, further, each of the σ-ideals $I_n : n \in \omega$ has the hereditary overspill property, then so does I.*

Proof. To begin, observe that if $K \subset X$ is a nonempty compact set such that for every open set $O \subset X$, if $K \cap O$ is nonempty then it is I_n-positive for every $n \in \omega$, then K is in fact I-positive. To see this, let $\{C_i : i \in \omega\}$ be a collection of closed sets in $\bigcup_n I_n$; re-indexing if necessary I may assume that $C_i \in I_n$ for some $n \leq i$. By induction on $i \in \omega$ find numbers n_i and open sets $O_i \subset X$ of decreasing diameter such that the closure of O_{i+1} is a subset of O_i, $O_i \cap K \neq 0$, and the closure of O_i is disjoint from C_i. This is easy to do by the assumption on the set K and the intesection $\bigcap_i Q_i$ will contain a singleton $x \in K$ which does not belong to the set $\bigcup_i C_i$, proving the I-positivity of the set K. Note also that every compact I-positive set can be thinned out to a compact set as above by simply removing from it all neighborhoods in which it is I-small.

To prove (1), let $A \subset X$ be an I-positive analytic set; I must produce its I-positive compact subset. By a result of Solecki [10], thinning out if necessary I may assume that A is in fact G_δ. Removing an open set from it, I may even assume that its intersection with any open set is either I-positive or empty. Let $A = \bigcap_n O_n$ for some inclusion-decreasing sequence of open sets $\{O_n : n \in \omega\}$. By induction on $n \in \omega$ build compact sets $K_n \subset X$ and finite collections P_n of small open subsets of X so that

- $K_0 \subset K_1 \subset \cdots \subset A$;
- P_n consists of open sets of radius $< 2^{-n}$ in some fixed compatible metric for the space X, $K_n \subset \bigcup P_n \subset O_n$ and the closure of $\bigcap P_{n+1}$ is a subset of $\bigcap P_n$;
- for every set $O \in P_n$, $K_n \cap O$ is I_n-positive.

In the end, let $K = \bigcap_n O_n$ =the closure of $\bigcup_n K_n$. This is certainly a compact set just as in the first paragraph, and so I-positive as desired.

To prove (2), it is enough to show that the set of compact I-positive sets is analytic by [12, Theorem 3.8.9]. This is easily verified: a compact set K is I-positive if it contains a nonempty compact set L such that for every open set O and every $n \in \omega$, the set $L \cap O$ is either empty or I_n-positive, and this is an analytic statement by the definability condition on the σ-ideals $I_n : n \in \omega$.

To prove (3), it will be enough to construct a winning strategy for Player I in the game $G(K, I)$ for every I-positive compact sets $K \subset X$. Player I will simply diagonalize relevant winning strategies in the games $G(L, I_n)$ for various sets L. After round n, Player I chooses finitely many balls $B_i^n : i \in k^n$ of radius $< 2^{-n}$ whose union contains the compact set K_n, whose closures are contained in the open set O_n Player II just indicated, and such that each of these balls has nonempty intersection with K. He will also choose winning strategies $\sigma_i^l : i \in k^n, l \in n$ for the games $G(K \cap \bar{B}_i^n, I_l)$ for Player I, and in all further rounds makes sure that his moves include the moves of these strategies against the sequence $O_{n+1}, O_{n+2}, \dots$ of moves of Player II. In this way, the end result of the play must be a compact set L such that for every open set O with $O \cap L \neq 0$ and every number $n \in \omega$, the set $L \cap O$ is I_n-positive, which means that $L \notin I$ as required.

□

Theorem 3.1 now easily follows. The σ-ideal H_σ is generated by the σ-ideals I_O as O ranges over all open intervals in $\mathbb{T}$ with rational endpoints. Here, the ideal I_O is generated by the set $\bigcap_{n \in b} C_n$, as b ranges over all infinite subsets of ω, and C_n denotes the closed set $\{x \in \mathbb{T} : nx \notin O\}$. The previous two theorems together yield the statement of Theorem 3.1. Thus, the quotient poset P_{H_σ} is proper, preserves Baire category, is bounding, and has the hereditary overspill.

The other forcing properties of the poset P_{H_σ} are not very orderly–it adds an independent real, collapses outer Lebesgue measure, and does not have the weak Sacks property. For the sake of brevity, I will only indicate the rather obvious term for an independent real. Let $\dot{x}_{gen}$ be the P_{H_σ}-name for the generic point in $\mathbb{T}$, and let $\dot{y} \in 2^\omega$ be the name defined by $\dot{y}(n) = 1$ iff $n\dot{x}_{gen}$ is in the upper half of the circle. It is fairly clear that $\dot{y}$ is a name for an independent real. For example, if K were a condition forcing $\forall n \in b\ \dot{y}(n) = 1$ for some ground model infinite set $b \subset \omega$, then K forces the generic $\dot{x}_{gen}$ into one of the generators of the ideal H_σ, namely the set $\{x \in \mathbb{T} : \dot{y}(n)$ is not in the lower half of the circle$\}$, which is of course impossible.

The proof of Theorem 1.1 is now easy. Suppose that the Generalized Continuum Hypothesis holds and κ is a regular cardinal. Let P be the countable support product P of κ many copies of the quotient forcing P_{H_σ}. The quotient forcing P_{H_σ} is proper and preserves Baire category, since the σ-ideal is generated by closed sets [12, Theorem 4.1.2]. The poset has just been proved to be bounding and the σ-ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Thus the countable support product of κ many copies has good behavior governed by the work of [12, Theorem 5.2.6]. A standard Δ-system argument shows that the poset P has $\aleph_2$-c.c. and so P preserves all cardinals. The preservation Theorem 2.12 shows that in the P-extension, the circle $\mathbb{T}$ is covered by ground model elements of the analytic collection U' of closed sets of uniqueness identified in Corollary 2.8. Thus, in the P-extension, the circle $\mathbb{T}$ is covered by $\aleph_1$ many closed sets of uniqueness.

The last concern is to show that in the P-extension, the circle $\mathbb{T}$ cannot be covered by fewer than κ many H-sets. This follows from the rectangular Ramsey properties of the product ideals. Suppose that $\langle \dot{C}_\alpha : \alpha \in \beta \rangle$ is a P-name for a collection of $\beta \in \kappa$ H-sets. Standard chain condition arguments show that there is an ordinal γ such that the names $\dot{C}_\alpha : \alpha \in \beta$ are supported by the product of the first γ many copies of P_{H_σ}. I claim that the generic point $\dot{x}_\gamma$ associated with the γ-th generic filter is forced not to belong to $\bigcup_{\alpha \in \beta} \dot{C}_\alpha$ as desired.

Well, assume for contradiction that $p \in P$ is a condition and $\delta \in \beta$ is an ordinal such that $p \Vdash \dot{x}_\gamma \in \dot{C}_\delta$. Let M be a countable elementary submodel of a large structure containing p etc. A standard argument will yield H_σ-positive compact sets $K_\alpha : \alpha \in \kappa \cap M$ such that the product $\Pi_\alpha K_\alpha$ consists only of M-generic sequences mmeting the condition $p \restriction \alpha$ for the product up to γ, and the function $f : \vec{x} \mapsto \dot{C}_\delta / \vec{x}$ is continuous on the product $\Pi_\alpha K_\alpha$. Note that the values of the function f depend only on $\vec{x} \restriction \gamma$ by the choice of γ. Consider the set $\{\vec{x} \in \Pi_\alpha K_\alpha : \vec{x}(\gamma) \in f(\vec{x})\}$. It cannot contain a product of H_σ-positive sets, because its sections of the γ-th coordinate are H-sets. By the rectangular Ramsey property of the product ideal, it must be the case that the complement of this set in $\Pi_\alpha K_\alpha$ contains a product $q = \Pi_\alpha K'_\alpha$ of H_σ-positive sets. A review of the definitions shows that $q \leq p$ is a condition forcing $\dot{x}_\gamma \notin \dot{C}_\delta$ as desired.

4 The loose ends

As the last point in the paper, I will prove two independence results complementary to Theorem 1.1. They show that there is a great degree of freedom in moving the covering numbers of the σ-ideals mentioned around by forcing.

Theorem 4.1. *It is consistent with ZFC that $\mathbb{T}$ is covered with $\aleph_1$ many H-sets while the continuum is very large.*

Proof. It is enough to reach for a model of ZFC in which the continuum is large while there is a P-point basis of size $\aleph_1$, such as in the product Sacks extension. For every point $x \in \mathbb{T}$ there is a set $a \subset \omega$ in the P-point ultrafilter such that the points $\{nx : n \in a\}$ converge, and therefore they avoid a certain nonempty open interval in the circle $\mathbb{T}$. This shows that the $\aleph_1$ many sets $B_{a,O} = \{x \in \mathbb{T} : \forall n \in a \; nx \notin O\}$, as a ranges over the P-point basis of size $\aleph_1$ and O ranges over all possible open intervals with rational endpoints, cover the circle $\mathbb{T}$, and they are all H-sets. □

Theorem 4.2. *It is consistent with ZFC that $\mathbb{T}$ cannot be covered by fewer than $\aleph_2$ many closed sets of uniqueness while there are dominating, nonmeager and nonnull sets of size $\aleph_1$.*

Proof. Consider the σ-ideal U_0 of sets of extended uniqueness on $\mathbb{T}$. The deep results of Debs and Saint-Raymond [2] show that this is a σ-ideal σ-generated by closed sets and it is polar. The collection of closed sets in U_0 is coanalytic, in fact $\mathbf{\Pi}^1_1$-complete by a result of Solovay and Kaufman [4, Section IV.2], and so by [12, Theorem 3.8.9] the σ-ideal U_0 is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Thus, the quotient P_{U_0} is proper, bounding, preserves Baire category, and outer Lebesgue measure by [12, Theorem 3.6.2]. Moreover, every set of uniqueness is a set of extended uniqueness, and so the poset P_{U_0} forces its generic real not to belong to any ground model coded closed sets of uniqueness. Ergo, starting with a model of the Continuum Hypothesis and iterating P_{U_0} ω_2 many times, a model of the statement of the theorem is achieved as the various preservation theorems of [12, Section 6.3] or [1, Section 6.3] show.

Note that the poset does not have the Sacks property–by the results of the previous section, it would imply a particularly strong version of overspill, and the σ-ideal U_0 does not have the overspill property. Thus, in the extension, the cofinality of the Lebesgue null ideal is $\aleph_2$. I do not know if the products of the poset P_{U_0} preserve outer Lebesgue measure, and therefore I do not know if it is possible to push the continuum beyond $\aleph_2$. □

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