# Homogeneous sets of positive outer measure*

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**Abstract**

For every Polish space and a coanalytic set of its countable subsets, if there is a homogeneous set of outer measure one then there is a perfect homogeneous set. In the generic extension by a large measure algebra, if there is a homogeneous set of size continuum then there is a a perfect homogeneous set.

## 1 Introduction

In a recent paper, Tamás Mátrai showed

**Fact 1.1.** *[4] Let $X$ be a Polish space and $A \subset [X]^{\aleph_0}$ be coanalytic.*

1. *if there is a nonmeager set $C \subset X$ such that $[C]^{\aleph_0} \subset A$, then there is a perfect set $P \subset X$ such that $[P]^{\aleph_0} \subset A$;*
2. *in the iterated Sacks model and the Cohen model, if there is a set $C \subset X$ of size continuum such that $[C]^{\aleph_0} \subset A$, then there is a perfect set $P \subset X$ such that $[P]^{\aleph_0} \subset A$.*

Here, I call a set $A \subset [X]^{\aleph_0}$ coanalytic if the set $\{\vec{x} \in X^\omega : \{\vec{x}(n) : n \in \omega\} \in A\}$ is a coanalytic subset of the space $X^\omega$ equipped with the product topology. The sets $C, P$ as in the statement of the fact are called *homogeneous* for $A$.

In this note, I will adjust Mátrai's argument to treat similar questions in the context of Borel probability measures.

**Theorem 1.2.** *Let $X$ be a Polish space with a Borel probability measure and $A \subset [X]^{\aleph_0}$ be coanalytic.*

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1. *If there is a outer mass one set $C \subset X$ such that $[C]^{\aleph_0} \subset A$, then there is a perfect set $P \subset X$ such that $[P]^{\aleph_0} \subset A$;*
2. *in the random model, if there is a set $C \subset X$ of size continuum such that $[C]^{\aleph_0} \subset A$, then there is a perfect set $P \subset X$ such that $[P]^{\aleph_0} \subset A$.*

The notation in this paper follows the set theoretic standard of [2]. If $t$ is a finite binary sequence then $O_t = \{x \in 2^\omega : t \subset x\}$; similar notation is used for clopen subsets of the Baire space $\omega^\omega$. The random model is a generic extension of a model of the generalized continuum hypothesis by a measure algebra with $\kappa$ generators where $\kappa$ is a regular cardinal larger than the continuum.

# 2 The iterated null ideal

Let $X$ be a Polish space and $\mu$ a Borel probability measure on it. Let $I$ be the $\sigma$-ideal of set of $\mu$-mass zero. Consider the following ideals on the space $X^\omega$:

- the ideal $I^\omega$, the iterated Fubini power of $I$ [6, Definition 5.1.1]. Here, a set $A \subset X^\omega$ is in $I^\omega$ if Player I has a winning strategy in the game $G(A)$. In this game, Players I and II alternate for $\omega$ many rounds, in each Player I indicates a null set $C_n \subset X$ and Player II responds with a point $x_n \in X \setminus C_n$. Player II wins if $\langle x(n) : n \in \omega\rangle \in A$.
- $I^\omega_*$, the ideal of all sets $A \subset X^\omega$ for which there are sets $C_n : n \in \omega$ such that $C_n \subset X^n$ is $\mu^n$-null, where $\mu^n$ is the product measure on $X^n$, and $\forall \vec{x} \in A\ \exists n\ \vec{x} \restriction n \in C_n$.

Both $I^\omega$ and $I^\omega_*$ are easily seen to be $\sigma$-ideals. They are in general distinct; however, in a suitable context, they contain the same analytic sets:

**Proposition 2.1.** *Suppose that all $\Delta^1_2$ sets are Lebesgue measurable. If $A \subset X^\omega$ is analytic then $A \in I^\omega_* \leftrightarrow A \in I^\omega$.*

I do not know if the $\Delta^1_2$ measurability assumption is necessary. In order to prove the main theorem without this assumption, I will have to pass to a generic extension in which all $\Delta^1_2$ sets are measurable, use the proposition there, and use an absoluteness argument. Tamás Mátrai found a mistake in the original proof that did not consider this detour.

*Proof.* The left-to-right direction is easy and does not depend on the analyticity of the set $A$. It is enough to show that for every number $n \in \omega$ and every set $C \subset X^n$ of $\mu^n$-mass zero, the set $\{\vec{x} \in X^\omega : \vec{x} \restriction n \in C\}$ belongs to the ideal $I^\omega$. Indeed, assume without loss of generality that the set $C$ is Borel, and consider the strategy $\sigma$ for Player I that commands him to play the set $\{y \in X : \{\vec{z} \in X^{n \setminus m} : \langle x_0, x_1, \ldots, x_{m-2}, y\rangle^\frown \vec{z} \in C_n\}$ has positive mass$\}$ at every round $m \in n$. Use the Fubini theorem to argue by induction that the above set is of null mass, and therefore this is a legal strategy for Player II. Clearly, the strategy is winning and the left-to-right implication follows.

The right-to-left implication is more difficult. Suppose that $A \notin I^\omega_*$ and $\sigma$ is a strategy for Player I in the game $G(A)$. I must produce a counterplay against the strategy which gives a sequence in the set $A$.

Fix a continuous function $f : \omega^\omega \to X^\omega$ such that $A = \mathrm{rng}(f)$. By induction on $n \in \omega$ build points $x_n \in X$ and numbers $m_n$ so that

- the points $x_0, \dots x_{n-1}$ form a legal finite counterplay against the strategy $\sigma$;
- the set $A_n = \{\vec{z} \in X^{\omega \setminus n} : \langle x_0, \dots x_{n-1}\rangle^\frown \vec{z} \in f'' O_{\langle m_0, \dots m_{n-1}\rangle}\}$ is $I^{\omega\setminus n}_*$ positive.

Suppose that the points $x_i : i \in n$ and numbers $m_i : i \in n$ have been constructed. Given a number $m \in \omega$ consider the set $C^m_n$ of all points $y \in X$ such that the set $(A^m_n)_y \subset X^{\omega \setminus n+1}$ is $I^{\omega\setminus n}_*$ positive, where $(A^m_n)_y = \{\vec{z} \in X^{\omega\setminus n+1} : \langle x_0, \dots x_{n-1}, y\rangle^\frown \vec{z} \in f'' O_{\langle m_0, \dots m_{n-1}, m\rangle}\}$. It may not be clear how to argue at this point that these sets are measurable; I will only prove that one of them is not null.

Suppose for contradiction that these sets are all null and enclose their union in a Borel null set $C_n$. Suppose for the simplicity of the notation that the underlying space $X$ is just the Cantor space $2^\omega$. Consider the following coding of infinite sequences of $G_\delta$ null sets (any reasonable coding will do). A code is a sequence $\vec{h}$ of functions $\vec{h}_k : \omega \times (2^{<\omega})^{k\setminus n+1} \to 2 : k \in \omega$ where the set $\bigcup\{O_{\vec{t}} : \vec{h}_k(i, \vec{t}) = 1\} \subset (2^\omega)^k$ is of $\mu^{k\setminus n+1}$-mass $\leq 2^{-i}$; the $k$-th $G_\delta$ null set $C_k(\vec{h})$ coded is the intersection of all these open sets as $i$ varies over all natural numbers. The set $\{\langle y, \vec{h}\rangle : y \notin C_n$ and for every $m \in \omega$ and every sequence $\vec{z} \in (A^m_n)_y$ there is a number $k \in \omega$ such that $\vec{z} \restriction (n, k) \in \vec{C}_k(\vec{h})\}$ is coanalytic, and by Novikov-Kondo's uniformization theorem [3, Theorem 36.14] there is a coanalytic uniformization $F$ of it, with domain $X \setminus C$. For every $k > n$ define the set $C_k = \{y^\frown \vec{z} \in X^{k\setminus n} : y \notin C_n, \vec{z} \in C_k(F(y)(k))\}$. This set is $\Delta^1_2$: $y^\frown \vec{z} \in C_k$ iff $y \notin C_n$ and $\exists \vec{h}\ \vec{h} = F(y) \wedge \vec{z} \in C_k(\vec{h})$, iff $y \notin C_n$ and $\forall \vec{h}\ \vec{h} \neq F(y) \vee \vec{z} \in C_k(\vec{h})$. By the assumptions, the set $C_k$ is measurable, and the Fubini theorem shows that it is $\mu^{k\setminus n}$-null. Now the definitions imply that for every $\vec{y} \in A_n$ there is $k \geq n$ such that $\vec{z} \restriction k \setminus n \in C_n$, contradicting the second induction assumption.

Fix a number $m_n$ such that the set $C^{m_n}_n$ is not null, and choose a point $x_n$ in the set $C^{m_n}_n$ which does not belong to the null set the strategy $\sigma$ commands Player I to play at round $n$. The induction hypotheses continue to hold. In the end, the sequence $\langle x_n : n \in \omega\rangle$ belongs to the set $A$ since it is the functional value of $f$ applied to $\langle m_n : n \in \omega\rangle \in \omega^\omega$. The proposition follows.

□

# 3 The ZFC situation

Let $X$ be a Polish space and $\mu$ a Borel probability measure on it. Suppose that $A \subset [X]^{\aleph_0}$ is a coanalytic set and there is an outer mass one set $C \subset X$ such that $[C]^{\aleph_0} \subset A$. I must produce a perfect set $P \subset X$ such that $[P]^{\aleph_0} \subset A$. By the Borel isomorphism of measures theorem and the perfect set theorem it is enough to deal with the case $X = 2^\omega$ and $\mu =$the unique probability measure on $2^\omega$ invariant under coordinatewise addition. I will first show how to argue with the additional assumption of $\Delta^1_2$ measurability.

**Proposition 3.1.** *The set $B = \{\vec{x} \in X^\omega : \{\vec{x}(n) : n \in \omega\} \notin A\}$ is in the ideal $I^\omega$.*

*Proof.* Suppose that the set $B$ is $I^\omega$-positive. It is analytic, and in such a case by [6, Theorem 5.1.9] the game $G(B)$ is determined, and moreover Player II has an very simple winning strategy in the form of a Borel tree $T \subset X^\omega$ such that every node splits into $I$-positively many immediate successors, and $[T] \subset B$. However, since the set $C$ has outer mass one, it is easy to find a branch of the tree $T$ consisting solely of points in $C$. However, the definitions show that such a branch cannot be an element of the set $B$. Contradiction!

□

Since $B \in I^\omega$ and $B$ is analytic, it follows from the previous section that $B \in I^\omega_*$ and there are zero mass sets $C_n : n \in \omega$, each a subset of $X^n$ respectively, such that for every sequence $\vec{x} \in X^\omega$, if $\forall n\ \vec{x} \restriction n \notin C_n$ then $\vec{x} \notin B$. Let $M \prec H_\theta$ be a countable elementary submodel of a large enough structure containing all the sets $C_n : n \in \omega$. It will be enough to find a perfect set $P \subset X$ of points such that their finite one-to-one tuples are random generic for the model $M$, since then their infinite one-to-one sequences cannot belong to the set $B$ by the previous sentence, and therefore countably infinite subsets of the set $P$ must all belong to the set $A$. Such a perfect set of mutually random reals can be obtained from results of Mycielski [5].

To eliminate the $\Delta^1_2$ measurability assumption, let $C$ be the outer measure one homogeneous set, and pass to the random model $V[G]$. The set $C$ retains its properties there: it is still of outer measure one since the random forcing preserves outer measure, and all of its countable subsets are still subsets of $A$. To see the latter claim, if $a \subset C$ is a countable infinite set in the extension, it is covered by a countable set $b \subset C$ in the ground model. The statement $[b]^{\aleph_0} \subset A$ is coanalytic, true in the ground model, therefore true in the extension, and $a \in A$ follows.

Now, the random extension satisfies $\Delta^1_2$ measurability [1, Theorem 9.2.1]. By the work we have just done, in the random extension there must be a perfect homogeneous set $P \subset X$ such that $[P]^{\aleph_0} \subset A$. This is a $\Sigma^1_2$ statement, true in $V[G]$, and therefore, by Shoenfield absoluteness, it is true in the ground model. The theorem follows!

# 4 The random model

Suppose that $\lambda \leq \kappa$ are regular cardinals larger than the continuum. Consider the generic extension $V[G]$ obtained by measure algebra on $\kappa$ many generators.

I will show that the following holds in $V[G]$. Whenever $X$ is a Polish space and $A \subset [X]^{\aleph_0}$ is a coanalytic set such that there is a set $C \subset X$ of size $\lambda$ with $[C]^{\aleph_0} \subset A$, then there is a perfect set $P \subset X$ such that $[P]^{\aleph_0} \subset A$.

First, work in $V$. To get a particular representation of the random algebra, consider the usual product Baire measure on the space $2^\kappa$ that assigns to every set of the form $\{y \in 2^\lambda : y(\beta) = 0\} : \beta \in \kappa$ mass 1/2, and force with Baire subsets of $2^\kappa$ of positive mass, ordered by inclusion. The forcing adds a generic function $\dot{y}_{gen} \in 2^\kappa$. Let $\dot{x}_\alpha : \alpha \in \lambda$ be names for distinct elements of the set $\dot{C}$. Use the c.c.c. to find for each $\alpha \in \kappa$ a countable set $b_\alpha \subset \kappa$ and a Borel function $f : 2^{b_\alpha} \to X$ such that it is forced that $\dot{x}_\alpha = \dot{f}(\dot{y}_{gen} \restriction b_\alpha)$. Thinning out if necessary, I may assume that the sets $b_\alpha : \alpha \in \lambda$ form a $\Delta$-system with root $b$. By a standard homogeneity argument, I may assume that $b = 0$. Thinning out further, I may assume that the sets $b_\alpha$ have the same ordertype and that there is a single Borel function $g : 2^\gamma \to X$ such that $f = g \circ \pi_\alpha$ for every ordinal $\alpha \in \lambda$, where $\pi_\alpha$ denotes the transitive collapse of the set $2^{b_\alpha}$.

Now, in the generic extension $V[G]$, consider the set $C' = \{\pi_\alpha(\dot{y}_{gen} \restriction b_\alpha) : \alpha \in \lambda\} \subset 2^\gamma$. A standard argument shows that this set has outer measure one. Also, every countable subset $C'' \subset C'$ has the property that $g \restriction C''$ is one-to-one and $g''C'' \in A$. By the results of the previous section applied in the model $V[G]$, there must be a perfect set $P' \subset 2^\gamma$ with the same properties. The set $g''P'$ is uncountable and analytic, and as such contains a perfect subset $P \subset X$. It is not difficult to check that the set $P$ has the requested properties.

# References


[1] Tomek Bartoszynski and Haim Judah. *Set Theory. On the structure of the real line.* A K Peters, Wellesley, MA, 1995.

[2] Thomas Jech. *Set Theory.* Academic Press, San Diego, 1978.

[3] Alexander S. Kechris. *Classical Descriptive Set Theory.* Springer Verlag, New York, 1994.

[4] Tamás Mátrai. Infinite dimensional perfect set theorems. 2009.

[5] Jan Mycielski. Algebraic independence and measure. *Fund. Math.*, 61:165–169, 1967.

[6] Jindřich Zapletal. *Forcing Idealized.* Cambridge Tracts in Mathematics 174. Cambridge University Press, Cambridge, 2008.