Preserving P-points in definable forcing*

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Abstract

I isolate a simple condition that is equivalent to preservation of P-points in definable proper forcing.

1 Introduction

Blass and Shelah [3], [2, Section 6.2] introduced the forcing property of preserving P-points. Here, a *P-point* is an ultrafilter U on ω such that every countable subset of it has a pseudo-intersection in it: $\forall a_n \in U : n \in \omega\ \exists b \in U\ |b \setminus a_n| < \aleph_0$. While the existence of P-points is unprovable in ZFC, they are plentiful under ZFC+CH. A forcing P *preserves* an ultrafilter U if every set $a \subset \omega$ in the extension either contains, or is disjoint from, a ground model element of the ultrafilter U; otherwise, P *destroys* U. The forcing P preserves P-points if it preserves all ultrafilters that happen to be P-points.

Several circumstances make this property a natural and useful tool. Every forcing adding a real number destroys some ultrafilter [2, Theorem 6.2.2]; if the forcing adds an unbounded real, then it destroys all non-P-point ultrafilters. A P-point, if preserved by a proper forcing, will again generate a P-point in the extension. Cohen and Solovay forcing both destroy all non-principal ultrafilters, and so preservation of P-points excludes the introduction of Cohen or random reals into the extension. Finally, preservation of P-points is itself preserved under the countable support iteration of proper forcing [3],[2, Theorem 6.2.6].

In the context of the theory of definable proper forcing [18], the preservation of P-points has two disadvantages: it trivializes when P-points do not exist (while the important properties of a definable forcing are typically independent of circumstances of this kind), and it refers to undefinable objects such as ultrafilters. As a result, it is not clear how difficult its verification might be, and what tools should be used for that verification. In this paper, I will resolve this situation by isolating a simple condition that is equivalent to the preservation of P-points for definable proper forcing in the theory ZFC+LC+CH. In order to state the theorem, I will need the following definitions.

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Definition 1.1. A forcing P *does not add splitting reals* if for every set $a \subset \omega$ in the extension there is an infinite ground model subset of ω which is either included in a or disjoint from it.

This is a familiar property. Some forcings do not add splitting reals (Sacks forcing, the fat tree forcing [18, Section 4.4.3], the E_0 forcing [17], or Miller forcing [13], to include a diversity of examples), others do (most notably, Cohen and random forcing, as well as all the Maharam algebras [1], and with them all definable c.c.c. forcings adding a real). Clearly, a forcing adding a splitting real preserves no nonprincipal ultrafilters. I do not think that on its own not adding splitting reals is preserved under even two-step iteration. Its conjunction with the bounding property is preserved under the countable support iteration of definable forcings by [18, Corollary 6.3.8], and it is equivalent to the preservation of Ramsey ultrafilters by [18, Section 3.4].

Definition 1.2. A forcing P has the *weak Laver property* if for every function $g \in \omega^\omega$ in the extension dominated by some ground model function there is a ground model infinite set $a \subset \omega$ and a ground model function $h : a \to \mathcal{P}(\omega)$ such that for every number $n \in a$, both $|h(n)| < 2^n$ and $g(n) \in h(n)$ hold.

The weak Laver property is less well-known, and on the surface it appears to have nothing to do with preservation of any ultrafilters. It is a weakening of the more familiar Laver [2, Definition 6.3.27] or Sacks properties. Notably, it occurs in [2, Section 7.4.D] in parallel to the proof that the Blass-Shelah forcing preserves P-points. Some more complicated variants of it, iterable in the category of arbitrary proper forcings, appeared in [16, Section 7], to guarantee the preservation of certain more complicated properties of filters on ω.

Definition 1.3. A σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subset 2^\omega \times X$ the set $\{y \in 2^\omega : A_y \in I\}$ is coanalytic.

This is a definability property of ideals studied for almost a century, considered for example by Sierpiński [9, Theorem 29.19]. It is a cornerstone of the ZFC development of the theory of definable forcing [18, Section 3.8]. A typical definable proper forcing adding a single real, adding no dominating reals, is of the form $P_I = I$-positive Borel sets ordered by inclusion, for a suitable $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal I.

Theorem 1.4. *(ZFC+LC+CH) The following are equivalent for a suitably definable proper forcing P:*

1. *P preserves P-points;*
2. *P does not add splitting real and has the weak Laver property.*

In the case that $P = P_I$ for a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal I on a Polish space the theorem is provable without the large cardinal assumptions.

The theorem can be used to swiftly argue that certain forcings preserve or do not preserve P-points. As one example, I introduced a combinatorial DPLT property of forcings in [17], and used a deep result of DiPrisco, Llopis, and Todorcevic [4] to show that forcings with this property have the Sacks property and do not add a splitting real. The posets with this property include the E_0 forcing [18, Section 4.7.1], the E_2 forcing [8], as well as certain variations of Silver and symmetric Sacks forcing [15]. Theorem 1.4 now implies that all of these forcings in fact preserve P-points; the results of [4] would be insufficient for such a conclusion. As another example, the forcings adding a bounded eventually different real never preserve P-points under CH. On the other hand, the Blass-Shelah forcing of [2, Section 7.4.D] adds an unbounded eventually different real and still preserves P-points.

The notation used in the paper follows the set theoretic standard of [6]. The shorthand LC denotes the use of suitable large cardinal assumptions. If $A \subset X \times Y$ is a set and $x \in X$ is a point, then A_x is the vertical section of the set A corresponding to x.

2 Proof of Theorem 1.4

Suppose that (2) of Theorem 1.4 fails; I will argue that (1) must fail as well. If P adds a splitting real, then P certainly destroys all nonprincipal ultrafilters. In the other case, the weak Laver property must fail for some function $f \in \omega^\omega$, and there is a condition $p \in P$ forcing that $\dot{g} < \dot{f}$ is a counterexample. Let $U_n : n \in \omega$ be pairwise disjoint sets of the respective size $f(n)$, in some way identified with $f(n)$. Let J be the ideal on the countable set $\mathrm{dom}(J) = \bigcup_n \mathcal{P}(U_n)$ generated by singletons and sets $a \subset \mathrm{dom}(J)$ such that for every number $n \in \omega$, either $a \cap \mathcal{P}(U_n) = 0$ or $|\bigcap(a \cap \mathcal{P}(U_n))| > 2^n$, or $|U_n \setminus \bigcup(a \cap \mathcal{P}(U_n))| > 2^n$.

Claim 2.1. *The ideal J is an F_σ proper ideal.*

Proof. The set F of generators is closed, and therefore compact, in the space $\mathcal{P}(\mathrm{dom}(J))$. The ideal generated by a closed set of generators is always F_σ, since the finite union map is continuous on the compact set F^n for every $n \in \omega$, its image is again a compact set, and the ideal J is the union of all of these countably many compact sets.

To see that $\mathrm{dom}(J) \notin J$, suppose that $a_i : i \in k$ are the generators of the ideal J. To show that they do not cover $\mathrm{dom}(J)$, find a number $n \in \omega$ such that $2^n > k$ and argue that there is a set $b \subset U_n$ not in any of the sets $a_i : i \in k$. First, partition k into two pieces $k = z_0 \cup z_1$ such that for $i \in z_0$, $|\bigcap(a_i \cap \mathcal{P}(U_n))| > 2^n$ holds, and for $i \in z_1$, $|U_n \setminus \bigcup(a \cap \mathcal{P}(U_n))| > 2^n$ holds. Use a counting argument to find pairwise distinct elements $u_i : i \in k$ in the set U_n so that for $i \in z_0$, $u_i \in \bigcap(a_i \cap \mathcal{P}(U_n))$ holds, and for $i \in z_1$, $u_i \notin \bigcup(a \cap \mathcal{P}(U_n))$ holds. The set $b = \{u_i : i \in z_1\}$ then belongs to none of the sets $a_i : i \in k$. □

It follows from the definition of the ideal J that the forcing P below the condition p adds a set $b \subset \mathrm{dom}(J)$ such that no ground model J-positive set can be disjoint from it, or included in it. Namely, consider the set $\dot{b} = \{c \subset U_n : \dot{g}(n) \in c, n \in \omega\}$. Suppose that $q \leq p$ is a condition, and $a \subset \text{dom}(J)$ is a J-positive set. Then, there must be infinitely many numbers $n \in \omega$ such that $a \cap \mathcal{P}(U_n) \neq 0$ and $|\bigcap(a \cap \mathcal{P}(U_n))| \leq 2^n$; since $\dot{g}$ is forced by p to be a counterexample to the weak Laver property, there must be a condition $r \leq q$ and a number $n \in \omega$ such that $r \Vdash \dot{g}(n) \notin \bigcap(\check{a} \cap \mathcal{P}(U_n))$ and therefore $r \Vdash \check{a} \not\subset \dot{b}$. Similarly, there must be infinitely many numbers $n \in \omega$ such that $a \cap \mathcal{P}(U_n) \neq 0$ and $|U_n \setminus \bigcup(a \cap \mathcal{P}(U_n))| \leq 2^n$, and by the failure of the weak Laver property, there must be a number n and a condition $r \leq q$ forcing $\dot{g}(n) \in \bigcup(a \cap \mathcal{P}(U_n))$ and so $\check{a} \cap \dot{b} \neq 0$.

It is now enough to extend the ideal J to a complement of a P-point, since then the previous paragraph shows that such a P-point cannot be preserved by the forcing P below the condition p. Such an extension exists, since the ideal J is F_σ; the construction is well-known, I am not certain to whom to attribute it, it certainly easily follows from some fairly old results.

Claim 2.2. *(CH) Whenever K is a proper F_σ ideal on a countable set, there is a P-point ultrafilter disjoint from K.*

Proof. By a result of [7], the quotient poset $\mathcal{P}(\omega)/I$ is countably saturated, in particular σ-closed. Any sufficiently generic filter over this poset will generate the desired P-point ultrafilter. Just build a modulo K descending ω_1 chain $a_\alpha : \alpha \in \omega_1$ of K-positive sets such that:

- $a_{\alpha+1}$ is either disjoint from or a subset of the α-th subset of ω in some fixed enumeration;
- a_α is modulo finite included in all sets $a_\beta : \beta \in \alpha$ for every limit ordinal α.

The first item shows that the sets $a_\alpha : \alpha \in \omega_1$ generate an ultrafilter disjoint from K, the second item is present to assure that this ultrafilter will be a P-point. The induction itself is easy. At the successor step, note that if $b \subset \omega$ is the α-th subset of ω in a given enumeration, then one of the sets $a_\alpha \cap b, a_\alpha \setminus b$ will be K-positive, and it will serve as $a_{\alpha+1}$. At the limit stage of induction, use the result of Mazur [12] to find a lower semicontinuous submeasure ϕ such that $K = \{b \subset \omega : \phi(b) < \infty\}$, enumerate $\alpha = \{\beta_n : n \in \omega\}$, and choose finite sets $b_n \subset \bigcap_{m \in n} a_{\beta_m}$ of ϕ-mass $\geq n$. The set $a_\alpha = \bigcup_n b_n$ will work. □

This completes the proof of the implication $\neg(2) \rightarrow \neg(1)$. Note that the definability of the forcing P and the large cardinal assumptions played no role here.

The implication (2)$\rightarrow$(1) is more exciting. Assume that (2) holds. There are two auxiliary claims.

Claim 2.3. *If K is an F_σ ideal on ω, $p \in P$ is a condition, and $p \Vdash \dot{b} \subset \omega$, then there are a ground model K-positive set and a condition $r \leq p$ forcing it to be either disjoint from, or a subset of, the set $\dot{b}$.*

Proof. Use the result of Mazur [12] to find a lower semicontinuous submeasure ϕ on ω such that $J = \{c \subset \omega : \phi(c) < \infty\}$. Find pairwise disjoint sets $c_n \subset \omega$ such that $\phi(c_n) > n \cdot 2^{2^n}$, this for every $n \in \omega$. Use the weak Laver property to find an infinite set $a \subset \omega$, sets $d_n \subset \mathcal{P}(c_n)$ of the respective size $\leq 2^n$, and a condition $q \leq p$ such that $q \Vdash \forall n \in \check{a}\ \dot{b} \cap \check{c}_n \in \check{d}_n$. Use the subadditivity of the submeasure ϕ to find sets $e_n \subset c_n$ of submeasure $\geq n$ such that $\forall f \in d_n\ f \cap e_n = 0 \vee e_n \subset f$, this for every $n \in a$. Thus $q \Vdash \forall n \in a\ \check{e}_n \subset \dot{b} \vee \check{e}_n \cap \dot{b} = 0$. Since P adds no splitting reals, there is a condition $r \leq q$ and an infinite subset $a' \subset a$ such that $r \Vdash \forall n \in a'\ \check{e}_n \subset \dot{b} \vee \forall n \in a'\ \check{e}_n \cap \dot{b} = 0$. In the first case, the ground model J-positive set $\bigcup_{n \in a'} e_n$ is forced to be a subset of $\dot{b}$, in the other case, this set is forced to be disjoint from $\dot{b}$ as desired. $\square$

Claim 2.4. *(ZFC+LC) If U is a P-point and J is a universally Baire ideal disjoint from U, then there is an F_σ-ideal $K \supset J$ disjoint from U. If J is analytic then no large cardinals are needed.*

The class of universally Baire sets first appeared in [5]. Its precise definition is irrelevant for the purposes of this paper. Suitable large cardinal assumptions imply that suitably definable subsets of Polish spaces are universally Baire [14], [10, Section 3.3], and analytic sets are universally Baire in ZFC. Suitable large cardinals imply that games with universally Baire payoff are determined [11] and the class of universally Baire sets is closed under projections, countable intersections, complements and other operations.

Note that claims 2.2 and 2.4 together yield a complete characterization of analytic ideals on ω that are disjoint from a P-point under CH: these are exactly those ideals that can be extended to nontrivial F_σ-ideals.

Proof. This in fact follows from the Kechris-Louveau-Woodin dichotomy [9, Theorem 21.22]. I will prove the large cardinal version with a direct determinacy argument and then use the Kechris-Louveau-Woodin dichotomy to argue for the analytic case in ZFC.

Recall the Galvin-Shelah game theoretic characterization of P-points: the ultrafilter U is a P-point if and only if Player I has no winning strategy in the P-point game where he chooses sets $a_n \in U$, Player II chooses their finite subsets $b_n \subset a_n$, and Player II wins if $\bigcup_n b_n \in U$ [2, Theorem 4.4.4]. Now consider the same game, except the winning condition for Player II is replaced with $\bigcup_n b_n \notin J$. This is certainly easier to win for Player II, and so Player I still does not have a winning strategy. Now, however, the payoff set is universally Baire and one can use the large cardinal assumptions and determinacy results [11] to argue that the game is determined and Player II must have a winning strategy σ.

Let M be a countable elementary submodel of a large enough structure containing the strategy σ. For every position $p \in M$ of the game that respects the strategy σ and ends with a move of Player II, let $u_p = \{b \in [\omega]^{<\aleph_0} : \exists a \in U\ p^\frown a^\frown b$ is a position respecting the strategy $\sigma\}$ and let $F_p = \{c \subset \omega : c$ has no subset in $u_p\}$. The sets $F_p \subset \mathcal{P}(\omega)$ are closed and disjoint from the ultrafilter U, since for every set $a \in U$ the strategy σ must answer a with its subset. Thus, the sets $F_p : p \in M$ generate an F_σ-ideal K on ω disjoint from the ultrafilter U. I must show that $J \subset K$ holds.

Suppose $c \subset \omega$ is not in the ideal K. By induction on $n \in \omega$ find sets $a_n \in U \cap M$ such that when Player I plays these sets in succession, the strategy σ always responds with a subset of c. Suppose the sets $a_n : n \in m$ have been built, and let $p \in M$ be the corresponding position of the game. Since $c \notin F_p$, there must be a set a_m such that the strategy responds to the move a_m by a subset of c. This concludes the inductive construction. In the end, the strategy σ won the infinite play against the sequence $a_n : n \in \omega$ of Player I's challenges. Thus the set $\bigcup_n b_n$ it produced was not J-positive. This set is a subset of the set c by the inductive construction, and therefore $c \notin J$ as required.

Now for the ZFC case, let J be an analytic ideal disjoint from the P-point ultrafilter U. If J can be separated from U by an F_σ set K_0, then the ideal K generated by this set is still F_σ, still disjoint from U, and it includes J as desired. If J cannot be so separated, then the Kechris-Louveau-Woodin dichotomy shows that there is a perfect set $C \subset J \cap U$ such that $C \cap U$ is countable and dense in C. I will use it to construct a winning strategy for Player I in the P-point game, yielding a contradiction and completing the proof. Let $c_n : n \in \omega$ be an enumeration of the set $C \cap U$. Player I will win by playing sets $a_n \in C \cap U$ and on the side writing down finite initial segments $b'_n \subset a_n$ which include Player II's answer b_n in such a way that

- a_n contains $\bigcup_{i \in n} b'_i$ as an initial segment;
- $a_n \neq c_n$ and c_n does not contain $\bigcup_{i \in n+1} b'_i$ as an initial segment.

This is easily possible. In the end, the set $\bigcup_{n \in \omega} b'_n \subset \omega$ is the limit of the sets $a_n \in C \cap U$, and therefore it belongs to C by the first item, and it is not equal to any of the sets in $C \cap U$ by the second item. Consequently, it must belong to the ideal J, and since the set $\bigcup_{n \in \omega} b_n$ is included in it, it means that Player I won. $\square$

The implication (2)→(1) now follows easily. Suppose P is a proper forcing, $P = P_I$ for some universally Baire σ-ideal on a Polish space X, U is a P-point, $B \in P_I$ is a condition and $B \Vdash \dot{b} \subset \omega$ is a set. I must find a condition $C \subset B$ and a set $a \in U$ such that $C \Vdash \dot{b} \cap \check{a} = 0 \vee \check{a} \subset \dot{b}$. By strengthening the condition B I may assume that there is a Borel function $f : B \to \mathcal{P}(\omega)$ such that $B \Vdash \dot{b} = \dot{f}(\dot{x}_{gen})$. Consider the set $J_0 = \{a \subset \omega : \exists C \subset B\ C \Vdash \check{a} \cap \dot{b} = 0 \vee C \Vdash \check{a} \subset \dot{b}\} = \{a \subset \omega : \{x \in B : f(x) \cap a = 0\} \notin I \vee \{x \in Ba \subset f(x)\} \notin I\}$. If it is not disjoint from the P-point U, then we are done. If $J_0 \cap U = 0$, then even the ideal J generated by J_0 is disjoint from U. The ideal J is universally Baire, and if the σ-ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then J is in fact analytic. Claim 2.4 now shows that there is an F_σ-ideal $K \supset J$ disjoint from U. Claim 2.3 shows that there is a condition $C \subset B$ and a K-positive set $a \subset \omega$ such that $C \Vdash \check{a} \cap \dot{b} = 0$ or $C \Vdash \check{a} \subset \dot{b}$. This however contradicts the definition of the set $J_o \subset K$!

3 Applications of Theorem 1.4

Theorem 1.4 can be used in two directions: to assure that certain forcings preserve P-points, and to prove that other forcings do not preserve P-points. In this brief section I will give examples of both.

In [17], I introduced the combinatorial DPLT property of σ-ideals. A σ-ideal I on a Polish space X has the DPLT property if for every Borel I-positive set $B \subset X$ there is a continuous function f from the space of increasing functions in ω^ω to B such that the images of products $\Pi_n b_n$, where b_n are increasing sequences of pairs of natural numbers, are I-positive. I proved that if the quotient forcing P_I is proper and the ideal has the DPLT property, then the quotient forcing has the Sacks property and does not add splitting reals. The following is then a direct corollary of Theorem 1.4:

Proposition 3.1. *Let I be a suitably definable σ-ideal with the DPLT property. If the forcing P_I is proper, then it preserves P-points.*

This class of forcings includes the wide Silver forcing, symmetric Sacks forcing [15], and the E_0 and E_2 forcings [18, Section 4.7] as good examples. In all of these cases, a direct proof of P-point preservation seems to be entirely out of reach.

Proposition 3.2. *(CH) If P is a forcing adding a bounded eventually different real, then P fails to preserve P-points.*

Note that every bounding forcing making the set of all ground model reals meager falls into this category essentially by [2, Theorem 2.4.7]. Thus, for example, forcing with an ideal associated with a Ramsey capacity is bounding and adds no splitting reals [18, Theorem 4.3.25], but it must destroy a P-point. On the other hand, the Blass-Shelah forcing makes the set of ground model reals meager, it is not bounding, and it preserves P-points.

Proof. It will be enough to show that P fails the weak Laver property. Suppose $\dot{g}$ and f are a P-name and a function in ω^ω respectively such that $P \Vdash \dot{g} < \check{f}$ and for every ground model function $h \in \omega^\omega$, $\dot{g} \cap \check{h}$ is finite. Let $\omega = \bigcup_n b_n$ be a partition of ω into finite sets of the respective size 2^n, let $\bar{f}(n)$ be the set $\pi_{i \in b_n} f(i)$ and let $\bar{g} \in \Pi_n \bar{f}(n)$ be the name for the function in the extension defined by $\bar{g}(n) = \dot{g} \restriction \check{b}(n)$. I claim that $\bar{f}, \bar{g}$ witness the failure of the weak Laver property.

Indeed, if $a \subset \omega$ was an infinite set, h a ground model function on a such that $h(n)$ is a subset of $\bar{f}(n)$ of size $< 2^n$ and $p \in P$ a condition forcing $\forall n \in a\ \bar{g}(n) \in \check{h}(n)$, one could find surjections $u_n : b_n \to h(n)$ for every number $n \in a$, find a function $k \in \omega^\omega$ such that for every $n \in a$ and every $i \in b_n$ it is the case that $k(i) = u_n(i)(i)$, and obtain $p \Vdash \check{k} \cap \dot{g}$ is infinite. This contradicts the assumptions on the name $\dot{g}$! $\square$

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