A comparison of simplicial and block finite elements

Sergey Korotov and Tomáš Vejchodský

Abstract In this note we discuss and compare a performance of the finite element method (FEM) on two popular types of meshes – simplicial and block ones. A special emphasis is put on the validity of discrete maximum principles and on associated (geometric) mesh generation/refinement issues in higher dimensions. As a result, we would recommend to carefully reconsider the common belief that the simplicial finite elements are very convenient to describe complicated geometries (which appear in real-life problems), and also that the block finite elements, due to their simplicity, should be used if the geometry of the solution domain allows that.

1 Introduction

Geometrically, there are two types of finite elements (FEs) which can be naturally generalized to any dimension – simplices and blocks, where by blocks we mean Cartesian products of intervals. In what follows, we shall only consider the lowest-order finite elements, i.e., linear functions on simplices and multilinear functions on blocks. In 1D, the only reasonable element is an interval which can be understood both as a simplex and a block. Therefore, we shall make comparison for the case of two and more dimensions. Namely, we concentrate on validity of discrete maximum principles and on associated geometrical issues for mesh generation and adaptivity.

Sergey Korotov
Institute of Mathematics, Tampere University of Technology, P.O. Box 553, FI–33101 Tampere, Finland, e-mail: sergey.korotov@tut.fi

Tomáš Vejchodský
Institute of Mathematics, Academy of Sciences, Žitná 25, CZ–115 67 Prague 1, Czech Republic e-mail: vejchod@math.cas.cz

2 Model problem at its finite element discretization

We consider the following test problem: Find a function u such that

$$-\Delta u + cu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded polytopic domain with Lipschitz boundary $\partial\Omega$ and $c \geq 0$. The classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (1) satisfies the maximum principle:

$$f \leq 0 \qquad \Longrightarrow \qquad \max_{x \in \overline{\Omega}} u(x) \leq \max\{0\,,\, \max_{s \in \partial\Omega} g(s)\}. \tag{2}$$

Most of FE schemes are based on the weak formulation: Find $u \in H^1(\Omega)$ such that the boundary condition $u = g$ is satisfied in the sense of traces on $\partial\Omega$ and

$$a(u,v) = \mathscr{F}(v) \quad \forall v \in H_0^1(\Omega),$$

where $a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + cuv)\,\mathrm{d}x$, $\mathscr{F}(v) = \int_\Omega fv\,\mathrm{d}x$, $c \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$.

Let $\mathscr{T}_h$ be a conforming FE mesh on $\overline{\Omega}$ with interior nodes $B_1, \dots, B_N$ lying in Ω and boundary nodes $B_{N+1}, \dots, B_{N+N^\partial}$ lying on $\partial\Omega$. Further, let V_h be a finite-dimensional subspace of $H^1(\Omega)$, associated with $\mathscr{T}_h$ and its nodes, being spanned by the basis functions $\phi_1, \phi_2, \dots, \phi_{N+N^\partial}$ with the following properties: $\phi_i \geq 0$ in $\overline{\Omega}$ (nonnegativity), $\phi_i(B_j) = \delta_{ij}$ (delta property), $i, j = 1, \dots, N+N^\partial$, and $\sum_{i=1}^{N+N^\partial} \phi_i \equiv 1$ in $\overline{\Omega}$ (partition of unity). Notice that the lowest-order finite elements on simplices and on blocks meet these requirements. We also assume that the basis functions $\phi_1, \phi_2, \dots, \phi_N$ vanish on the boundary $\partial\Omega$. Thus, they span a finite-dimensional subspace V_h^0 of $H_0^1(\Omega)$. Let, in addition, $g_h = \sum_{i=1}^{N^\partial} g_{N+i}\phi_{N+i} \in V_h$ be a suitable approximation of the function g, for example its nodal interpolant.

The FE approximation is a function $u_h = u_h^0 + g_h$ such that $u_h^0 \in V_h^0$ and

$$a(u_h, v_h) = \mathscr{F}(v_h) \quad \forall v_h \in V_h^0, \tag{3}$$

whose existence and uniqueness is also provided by the Lax-Milgram lemma.

Algorithmically, $u_h = \sum_{i=1}^{N+N^\partial} y_i \phi_i$, where y_i are the entries of the solution $\overline{\mathbf{y}} = [y_1, \dots, y_{N+N^\partial}]^\top$ of the square system of $N + N^\partial$ linear algebraic equations

$$\overline{\mathbf{A}}\,\overline{\mathbf{y}} = \overline{\mathbf{F}}, \quad \text{where} \quad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{A}^\partial \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \text{and} \quad \overline{\mathbf{F}} = \begin{bmatrix} \mathbf{F} \\ \mathbf{F}^\partial \end{bmatrix}. \tag{4}$$

In the above, $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{A}^\partial \in \mathbb{R}^{N \times N^\partial}$, $\mathbf{0}$ and $\mathbf{I}$ stand for the zero and unit matrices of appropriate sizes. The entries of $\overline{\mathbf{A}}$ are $a_{ij} = a(\phi_j, \phi_i),\ i = 1, \dots, N,\ j = 1, \dots, N + N^\partial$. The block $\mathbf{F}$ consists of entries $f_i = \mathscr{F}(\phi_i),\ i = 1, \dots, N$, and the block-vector $\mathbf{F}^\partial$ has entries $f_i^\partial = f_{N+i} = g_{N+i},\ i = 1, \dots, N^\partial$, given by the boundary data.

3 Discrete maximum principles for FEM

In this section we compare simplicial and block finite elements with respect to the so-called discrete maximum principle (DMP). For a fixed mesh $\mathscr{T}_h$, we say that the discretization (3) satisfies the DMP if

$$f \le 0 \qquad \Longrightarrow \qquad \max_{x \in \overline{\Omega}} u_h(x) \le \max\{0, \max_{s \in \partial\Omega} g_h(s)\}. \tag{5}$$

In the case of the lowest-order finite elements, it is well known [4] that the DMP is satisfied if (i) the stiffness matrix $\overline{\mathbf{A}}$ is monotone and if (ii) the row sums of $\overline{\mathbf{A}}$ are nonnegative. Condition (ii) is satisfied, because the basis functions form the partition of unity and the coefficient c is nonnegative. Sufficient conditions for (i) can be obtained from the theory of M-matrices [7]. This, in particular, requires the nonpositivity of the off-diagonal entries in the FE matrix $\overline{\mathbf{A}}$. Matrix $\overline{\mathbf{A}}$ is assembled from the local (element) FE matrices, $\overline{\mathbf{A}} = \sum_{K \in \mathscr{T}_h} \overline{\mathbf{A}}^K$, and hence it suffices to guarantee the nonpositivity of the off-diagonal entries of each $\overline{\mathbf{A}}^K$. This observation yields various geometric limitations for the finite elements which we discuss in what follows.

3.1 On entries of FE matrices for simplices

For simplicity, let us consider the Laplace operator only, i.e., $c \equiv 0$. In this case the off-diagonal entries $a_{ij}^K\,(i \neq j)$ of the local stiffness matrices $\overline{\mathbf{A}}^K$ for simplicial elements can be expressed in any dimension by the following formula [1]

$$a_{ij}^K = \int_K \nabla\phi_j \cdot \nabla\phi_i \,\mathrm{d}x = -\frac{\mathrm{meas}_{d-1}(F_i)\,\mathrm{meas}_{d-1}(F_j)}{d^2\,\mathrm{meas}_d(K)} \cos\alpha_{ij},$$

where α_{ij} stands for the dihedral angle between the facets F_i and F_j of the simplex $K \in \mathscr{T}_h$, see Fig. 1 (left).

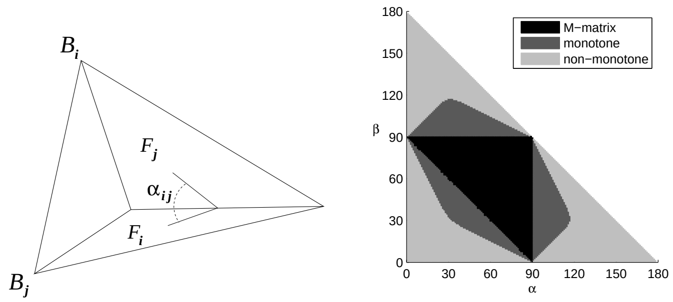

Fig. 1 The dihedral angle α_{ij} between faces F_i and F_j of a tetrahedron K (left). Results of the experiment for triangles (right).

Clearly, $a_{ij}^K \leq 0$ if and only if $\alpha_{ij} \leq \pi/2$. This nonobtuseness condition is well known for triangles and for tetrahedra, and it is crucial for the validity of DMPs [2]. For the case of general coefficients the conditions on meshes for DMP are stricter. Thus, if e.g. $c > 0$ then all dihedral angles in meshes have to be acute and, in addition, the meshes themselves have to be sufficiently fine due to the positive terms

$$\int_K \phi_j \phi_i \,\mathrm{d}x = \frac{d!}{(d+2)!} \operatorname{meas}_d(K), \quad i \neq j,$$

additionally appearing in computations, see e.g. [5, 2] for details.

Further, generalization can be obtained by requiring the stiffness matrix not to be M-matrix but to be monotone only. Theoretical handling of monotone matrices is difficult, but it can be checked numerically. Fig. 1 (right) shows results of an experiment, where we consider the Poisson problem with homogeneous Dirichlet boundary conditions. Hence, the block $\mathbf{A}$ of $\bar{\mathbf{A}}$ only is relevant. The domain Ω is a triangle. The axis in Fig. 1 (right) correspond to two angles of Ω. For each pair of angles α and β, we construct a triangulation by three steps of uniform red refinement of Ω. Then we assemble the stiffness matrix $\mathbf{A}$, and color the corresponding point according to its properties. If $\mathbf{A}$ is M-matrix (has off-diagonal entries nonpositive) then the point is black. If $\mathbf{A}$ is monotone and not M-matrix then the point is dark gray. If $\mathbf{A}$ is not monotone then the point is light gray. We clearly see that in this case the stiffness matrix is M-matrix if and only if all angles are nonobtuse (black area). Further we observe that the DMP is satisfied under favorable circumstances even for angles up to 117° (dark gray area), see also [12] for a similar 3D test.

3.2 On entries of FE matrices for blocks

The analysis of the DMP for block FE partitions can be done in the same fashion as for the simplices. The results, however, strongly depend on the dimension. For simplicity we again consider the Laplacian with homogeneous Dirichlet boundary condition. Let K be an element of a d-dimensional block mesh with edges of lengths $b_1, b_2, \ldots, b_d$. If B_i and B_j are its two vertices connected by the edge of length b_1 then the corresponding entry of the local stiffness matrix $\bar{\mathbf{A}}^K$ is

$$a_{ij}^K = \frac{b_1 b_2 \ldots b_d}{3^{d-1}} \left(\sum_{k=2}^{d} \frac{1}{2b_k^2} - \frac{1}{b_1^2} \right), \quad i \neq j. \tag{6}$$

In 2D we immediately see that $a_{ij}^K \leq 0$ if and only if $b_1/b_2 \leq \sqrt{2}$. This yields the well-known nonnarrow condition for the DMP. A rectangle K is nonnarrow if $1/\sqrt{2} \leq b_1/b_2 \leq \sqrt{2}$, where b_1 and b_2 stand for the lengths of its sides of K. It can be shown [9] that the DMP is satisfied if all rectangles in the mesh $\mathscr{T}_h$ are nonnarrow.

The nonnarrow condition guarantees that the corresponding stiffness matrix is M-matrix. A similar experiment as before reveals that this condition can be weakened

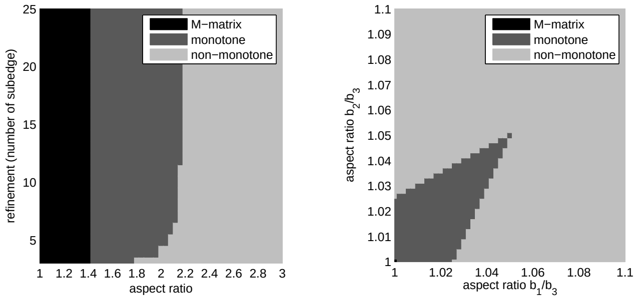

Fig. 2 The influence of the aspect ratio to the properties of the stiffness matrix **A**. Left: Ω is a rectangle $(0,b_1)\times(0,b_2)$. Right: Ω is a rectangular cuboid $(0,b_1)\times(0,b_2)\times(0,b_3)$.

if the stiffness matrix is required to be monotone only. In this experiment, we again consider $c\equiv 0$ and $g=0$. The domain is a rectangle $\Omega=(0,b_1)\times(0,b_2)$. The finite element mesh is obtained by the uniform refinement of Ω into N_{sub}^2 elements, where N_{sub} is the number of subedges induced on each edge of Ω. The axes in Fig. 2 (left) correspond to the aspect ratio b_1/b_2 of the rectangle Ω (and of all elements) and to the value N_{sub}. The results in Fig. 2 (left) indicate that the value $\sqrt{2}$ in the nonnarrow condition can be increased up to about 2.16 provided the mesh is sufficiently fine.

The 3D analysis of the trilinear elements on rectangular cuboids based on (6) gives a bit pessimistic conclusion. The stiffness matrix is M-matrix (and the DMP is satisfied) if all the elements are cubes [9]. Similar experiment as before, see Fig. 2 (right), indicates that the cubes cannot be distorted much in order to retain the stiffness matrix monotone and to satisfy the DMP. The two possible aspect ratios we have in rectangular cuboids can be at most around 1.05.

In dimensions 4 and higher, certain contributions form the local stiffness matrices are always positive. Indeed, without loss of generality we may assume that $b_1\geq b_2\geq\cdots\geq b_d$. If a_{ij}^K was nonpositive then (6) would yield

$$\frac{1}{b_1^2}\geq\sum_{k=2}^{d}\frac{1}{2b_k^2}\geq\frac{d-1}{2b_2^2}>\frac{1}{b_2^2},$$

where the last inequality holds true for $d\geq 4$. This inequality, however, contradicts the fact that $b_1\geq b_2$. Furthermore, considering the longest edge in the mesh, we see that all the contributions from all the elements surrounding this edge are positive and, hence, the corresponding off-diagonal entry in the stiffness matrix **A** is positive. Consequently, **A** is not an M-matrix. Similar experiments as before reveal that the stiffness matrix is neither monotone even on hyper-cubes. Thus, from the point of the DMP, the block finite elements are less advantageous than the simplicial elements especially for 3D and higher dimensional problems.

4 On mesh generation and adaptivity

Modern FE computations require treatment of issues like generation of a mesh with desired geometric properties and its global and local refinements preserving those properties. In the following two subsections we shall discuss these issues for both, simplices and blocks, with respect to geometric limitations imposed by the DMP.

4.1 Simplicial FE meshes (acuteness and nonobtuseness)

The practical realization of angle conditions (nonobtuseness and acuteness) is not easy. Even in 2D, an initial generation of reasonable nonobtuse and acute triangulations, especially for complicated domains, is algorithmically a hard task, see e.g. [3] for examples and literature on the subject. In 3D it is becoming even more difficult. Some results on generation and proper refinements of nonobtuse tetrahedral meshes are reported e.g. in [11] (see also [3]). But the only known positive (and very recent results) on acute meshes are the acute face-to-face tetrahedralization of the whole 3D Euclidean space [16], an infinite slab [6], some types of tetrahedra and a regular octahedron [10], and a cube [10, 17]. It is worth to mention that the last two works (the only relevant for real-life computations which are mostly done in bounded domains) are published just in summer of 2009 ! Moreover, very many acute tetrahedra are required to fill the cube by their constructions. In addition, the generated tetrahedra are very densely placed in the interior of the cube which is not so good for real computations as meshes used in practice should be dense mainly in vertices and along edges. Concerning higher dimensions, the situation with acute simplices is getting even more pessimistic. For example, it was shown in [10, 13] that the space $\mathbb{R}^d$ $(d \geq 4)$ cannot (surprisingly !) be filled face-to-face by acute simplices at all, which means that, in general, it is not possible to generate (reasonable fine) acute simplicial meshes for most of domains in higher dimensions, even for such simple as hypercubes.

In order to get more accurate FE approximations one needs to make various (global and local refinements) of the meshes preserving the desired geometric properties. For example, a triangle can be split into four similar triangles using midlines (2D red refinement) (and thus acuteness or nonobtuseness are preserved), but a tetrahedron cannot be, in general, partitioned face-to-face into several similar tetrahedrons by similar technique. After cutting four vertices of the tetrahedron off (and thus producing four similar tetrahedra), an interior octahedron remains, which can be split into four tetrahedra in three different ways. And in most of cases the resulting tetrahedra are not similar to the original one, moreover, the acuteness property cannot be preserved in any case. In addition, all further refinements should be done with a special care in order to avoid producing degenerating subtetrahedra, see [19] for details. An alternative can be to use one of bisection algorithms, see e.g. [14] and references therein, but just bisecting as such cannot obviously produce acute angles.

As far it concerns local refinements, the only results in dimension 3 and higher are known for nonobtuse simplicial partitions, see [1].

4.2 Block FE meshes (preserving the aspect ratio)

In the case of block elements global refinement is obvious. Further, one can perform local refinements with or without hanging nodes [15]. However, local refinements without hanging nodes require forced refinements far from the targeted area and, moreover, elements with high aspect ratios are actually forming. Hanging nodes are practically more demanding to use, but they overcome these difficulties. The advantage is that the resulting meshes are nested and that the aspect ratio of subelements remains unchanged. Let us remark that the sufficient geometric conditions for the DMP are the same for meshes both with and without hanging nodes.

5 Conclusions

In 2D both triangular and rectangular meshes seem to be comparable in the sense that generation and refinement of meshes yielding the DMP is well treatable in both cases. Anyway, the triangles provide more flexibility for complicated domains (e.g. for those having non-right corners). In higher dimension, block elements can be recommended if the geometry of the domain allows them and if the DMP is not an issue. In the opposite case, the simplices should be used, but then we face the above described problems with mesh generation and local refinements constrained by the dihedral angle conditions. These problems are sometimes treatable by path-simplicial meshes, which guarantee the DMP at least for the Poisson problems. In addition, the practical implementation of simplicial meshes is technically more demanding than the implementation of the blocks. This fact must be weighted as well. Let us remark that it is geometrically advantageous to use simplices and blocks together in the hybrid meshes. However, from the point of the DMP the hybrid meshes inherit the discussed disadvantages of all used types of elements. Moreover, the practical implementation of hybrid meshes is technically very demanding. For example, a 3D hybrid mesh with tetrahedra and rectangular cuboids requires also right triangular prisms and pyramids to join the elements face-to-face [18]. The DMP on prismatic meshes has been analyzed in [8]. However, up to the authors' knowledge the DMP for pyramidal elements (and therefore on hybrid 3D meshes) has not been analyzed yet.

Finally, it is interesting to mention that angle and aspect ratio conditions similar to those we discussed above also appear in the analysis of the convergence of FE approximations [5].

Acknowledgements The first author has been supported by Project no. 124619 from the Academy of Finland. The second author has been supported by Grant no. IAA100760702 of the Grant Agency of the Czech Academy of Sciences, Grant no. 102/07/0496 of the Czech Science Foundation, and by the institutional research plan no. AV0Z10190503 of the Czech Academy of Sciences.

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