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BEM for the first and second problems of the Stokes system

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Abstract: Using the integral equation method we study solutions of boundary value problems for the Stokes system in Sobolev space $H^1(G)$ in a bounded Lipschitz domain G with connected boundary. A solution of the second problem with the boundary condition $\partial \mathbf{u}/\partial \mathbf{n} - p\mathbf{n} = \mathbf{g}$ is studied both by the indirect and the direct boundary integral equation method. It is shown that we can obtain a solution of the corresponding integral equation using the successive approximation method. Nevertheless, the integral equation is not uniquely solvable. To overcome this problem we modify this integral equation. We obtain a uniquely solvable integral equation on the boundary of the domain. If the second problem for the Stokes system is solvable then the solution of the modified integral equation is a solution of the original integral equation. Moreover, the modified integral equation has a form $\mathbf{f} + S\mathbf{f} = \mathbf{g}$, where S is a contractive operator. So, the modified integral equation can be solved by the successive approximation. Then we study the first problem for the Stokes system by the direct integral equation method. We obtain an integral equation with an unknown $\mathbf{g} = \partial \mathbf{u}/\partial \mathbf{n} - p\mathbf{n}$. But this integral equation is not uniquely solvable. We construct another uniquely solvable integral equation such that the solution of the new eqution is a solution of the original integral equation provided the first problem has a solution. Moreover, the new integral equation has a form $\mathbf{g} + \tilde{S}\mathbf{g} = \mathbf{f}$, where $\tilde{S}$ is a contractive operator, and we can solve it by the successive approximation.

Keywords: Stokes system; first problem; second problem; integral equation method; successive approximation

AMS classification:35Q30; 76D10; 76D07; 65N38

1 Introduction.

The most important problems for the Stokes system are the first problem

$$\Delta \mathbf{u} = \nabla p \quad \text{in} \quad G, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad G, \tag{1}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \partial G \tag{2}$$

and the second boundary value problems. There are two relevant second boundary value problems for the Stokes system (1): One with the boundary condition

$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \mathbf{g} \quad \text{on} \quad \partial G \tag{3}$$

and the second one with the boundary condition

$$T(\mathbf{u}, p)\mathbf{n}^G = \mathbf{g} \quad \text{on} \quad \partial G. \tag{4}$$

Here $\mathbf{n} = \mathbf{n}^G$ is the outward unit normal vector of G, $\mathbf{u} = (u_1, \ldots, u_m)$ is a velocity field, p is a pressure and

$$T(\mathbf{u}, p) = 2\hat{\nabla}\mathbf{u} - pI \tag{5}$$

is the corresponding stress tensor. Here I denotes the identity matrix and

$$\hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \tag{6}$$

is the strain tensor, with $(\nabla\mathbf{u})^T$ as the matrix transposed to $\nabla\mathbf{u} = (\partial_j u_k)$, $(k, j = 1, \ldots, m)$. Remark that $\nabla \cdot \mathbf{u} = \partial_1 u_1 + \ldots + \partial_m u_m$ is the divergence of $\mathbf{u}$.

We shall suppose that G is a bounded domain with connected Lipschitz boundary in R^m. Many authors have studied the second problem with the boundary condition (4) by the integral equation method. It is a starting point for the boundary element method ([7], [16]). M. Kohr studied classical solutions of this problem on domains with smooth boundary (see [8]). B. E. J. Dahlberg, C. Kenig and G. C. Verchota studied this problem with a boundary condition $\mathbf{g} \in L^2(\partial G)$ on domains with Lipschitz boundary ([2]). D. Medková studied in [12] a weak solution in $H^1(G)$. A solution was looked for in the form of a hydrodynamical potential with an unknown density from $H^{-1/2}(\partial G)$. It was proved that a solution of the corresponding integral equation can be approximated using the successive approximation method. Then the similar result was proved also for the direct integral equation method.

In this paper we shall prove the same results for the second problem with the boundary condition (3). Then the first problem for the Stokes system (so called Stokes problem) is studied by the direct integral equation method, i.e. a solution of the Stokes problem with the boundary condition $\mathbf{f}$ is expressed in the form

$$\mathbf{u}(\mathbf{x}) = E_G\mathbf{g}(\mathbf{x}) + W_G\mathbf{f}(\mathbf{x}), \quad p(\mathbf{x}) = Q_G\mathbf{g}(\mathbf{x}) + R_G\mathbf{f}(\mathbf{x}) \qquad \mathbf{x} \in G,$$

where $\mathbf{g} = \partial\mathbf{u}/\partial\mathbf{n}^G - p\mathbf{n}^G$, $E_G\mathbf{g}$ is the hydrodynamical single layer potential with the density $\mathbf{g}$, $Q_G\mathbf{g}$ is the corresponding pressure, $W_G\mathbf{f}$ is the hydrodynamical double layer potential corresponding to the boundary condition (3) and $R_G\mathbf{f}$ is the corresponding pressure. It is shown that $\mathbf{g} = \partial\mathbf{u}/\partial\mathbf{n}^G - p\mathbf{n}^G$ is a solution of a uniquely solvable integral equation and this equation can be solved by the successive approximation method.

2 Weak solution of the second problem

If $A, B \in R^{m\times m}$ are matrices with $A = (A_{ij})$, $B = (B_{ij})$ denote

$$A : B = \sum_{i,j=1}^{3} A_{ij}B_{ij}.$$

If $X(M)$ is a vector space of real functions (or distributions) on a set M denote by $X(M,C)$ its complexification, i.e. $X(M,C) = \{v_1 + iv_2; v_1 \in X(M,R) = X(M), v_2 \in X(M)\}$. If $K = R$ or $K = C$ and $k \in N$, we denote $X(M,K^k) = \{\mathbf{u} = (u_1,\dots,u_k); u_j \in X(M,K) \text{ for } j = 1,\dots,k\}$.

In the entire article suppose that G is a bounded domain with connected Lipschitz boundary in R^m. We shall study the second boundary value problem in the Sobolev space $H^1(G;R^m)$. We denote by $H^s(G)$ the Sobolev-Slobodetski space of order s. Remark that $H^0(G) = L^2(G)$ and $H^1(G) = \{f \in L^2(G); \nabla f \in L^2(G;R^m)\}$ is equipped with the norm

$$\|f\|_{H^1(G)} = \left\{ \int_G \left[f^2 + |\nabla f|^2 \right] \, d\mathbf{x} \right\}^{1/2}.$$

If φ is a Lipschitz function on R^{m-1} and $S = \{[\mathbf{x},\varphi(\mathbf{x})]; \mathbf{x} \in R^{m-1}\}$ we say that $f \in H^s(S)$ if $f(\mathbf{x},\varphi(\mathbf{x})) \in H^s(R^{m-1})$. Since G has Lipschitz boundary there are bounded open sets $U_1,\dots,U_k$ and Lipschitz functions $\varphi_1,\dots,\varphi_k$ such that $\partial G \subset U_1 \cup \ldots \cup U_k$ and for each $j \in \{1,\dots,k\}$ there is a coordinate system such that $U_j \cap \partial G = U_j \cap S_j$ with $S_j = \{[\mathbf{x},\varphi_j(\mathbf{x})]; \mathbf{x} \in R^{m-1}\}$. Choose $\omega_j \in C^\infty(R^m)$ supported in U_j with $0 \le \omega_j \le 1$ for $j = 1,\dots,k$ such that $\omega_1 + \omega_2 + \ldots \omega_k = 1$ on a neighborhood of ∂G. We say that $f \in H^s(\partial G)$ if $\omega_j f \in H^s(S_j)$ for $j = 1,\dots,k$.

Recall that $H^{1/2}(\partial G)$ is the space of traces of $H^1(G)$ endowed with the norm

$$\|v\|_{H^{1/2}(\partial G)} = \inf\{\|u\|_{H^1(G)}; u \in H^1(G), v = u|\partial G\} \tag{7}$$

and $H^{-1/2}(\partial G)$ is the dual space of $H^{1/2}(\partial G)$.

If $(\mathbf{u},p)$ is a classical solution of the second problem for the Stokes system (1), (3) and $\mathbf{v} \in C^2(R^m,R^m)$, then the Green formula yields

$$\int_{\partial G} \mathbf{g}\cdot\mathbf{v} \, d\mathbf{y} = \int_{\partial G} \left[\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} \right] \cdot \mathbf{v} \, d\mathbf{y} = \int_G [\nabla\mathbf{u} : \nabla\mathbf{v} + \mathbf{v}\cdot\Delta\mathbf{u} - \mathbf{v}\cdot\nabla p - p(\nabla\cdot\mathbf{v})] \, d\mathbf{y}$$

$$= \int_G [\nabla\mathbf{u} : \nabla\mathbf{v} - p(\nabla\cdot\mathbf{v})] \, d\mathbf{y}.$$

We formulate a weak solution of the problem (1), (3) as follows:

Let $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$. *We say that* $\mathbf{u} \in H^1(G,R^m)$, $p \in L^2(G,R^1)$ *is a weak solution of the problem (1), (3) if* $\nabla\mathbf{u} = 0$ *in* G *and*

$$\int_G \nabla\mathbf{u} : \nabla\mathbf{v} \, d\mathbf{y} - \int_G p(\nabla\cdot\mathbf{v}) \, d\mathbf{y} = \langle \mathbf{g}, \mathbf{v} \rangle \tag{8}$$

for each $\mathbf{v} \in H^1(G,R^m)$.

It is well known that if $\mathbf{u} \in H^1(G,R^m)$, $p \in L^2(G,R^1)$, $\nabla\mathbf{u} = 0$ in G and

$$\int_G \nabla\mathbf{u} : \nabla\mathbf{v} \, d\mathbf{y} - \int_G p(\nabla\cdot\mathbf{v}) \, d\mathbf{y} = 0$$

for each $\mathbf{v} \in \mathcal{C}^\infty(G; R^m)$ with compact support in G, then $\mathbf{u} \in \mathcal{C}^\infty(G; R^m)$, $p \in \mathcal{C}^\infty(G; R^1)$ satisfy Stokes system (1).

Remark that if $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ and $\mathbf{u} \in H^1(G, R^m)$, $p \in L^2(G, R^1)$ is a weak solution of the problem (1), (3) then

$$\int_G \nabla \mathbf{u} : \nabla \mathbf{w} \; d\mathbf{y} = \langle \mathbf{g}, \mathbf{w} \rangle \quad \forall \mathbf{w} \in H^1(G, R^m), \nabla \cdot \mathbf{w} = 0. \tag{9}$$

First we study the problem of uniqueness of a solution.

Lemma 2.1. *If $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ and $\mathbf{u} \in H^1(G, R^m)$, $p \in L^2(G, R^1)$ is a weak solution of the second boundary value problem (1), (3), then $\langle \mathbf{g}, \mathbf{c} \rangle = 0$ for each constant vector function $\mathbf{c}$ (i.e. $\int \mathbf{g} = 0$). If $\mathbf{v}$, q is another weak solution of the problem (1), (3), then $p = q$ and $\mathbf{u} - \mathbf{v}$ is constant.*

Proof. If $\mathbf{c}$ is a constant function then (9) gives that $\langle \mathbf{g}, \mathbf{c} \rangle = 0$.

$\mathbf{u} - \mathbf{v}$, $p - q$ is a weak solution of the second boundary value problem for the Stokes system with zero boundary condition. Putting $\mathbf{w} = \mathbf{u} - \mathbf{v}$ we get from (9)

$$\int_G |\nabla (\mathbf{u} - \mathbf{v})|^2 \; d\mathbf{y} = 0.$$

Thus $\mathbf{u} - \mathbf{v}$ is constant. Since $\mathbf{u} - \mathbf{v}$, $p - q$ is a solution of the Stokes system then $\nabla(p - q) = \Delta(\mathbf{u} - \mathbf{v}) = 0$. Thus there exists a constant b such that $p - q = 0$. From the boundary condition $0 = \partial(\mathbf{u} - \mathbf{v})/\partial \mathbf{n} - p\mathbf{n} = -b\mathbf{n}$. Hence $b = 0$.

3 Hydrodynamical single layer potential

We shall look for a solution of the second boundary value problem for the Stokes system in the form of a hydrodynamical single layer potential. The aim of this section is to assemble some basic facts on this potential.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary $\partial\Omega$. Denote $\Omega^e := R^m \setminus \mathrm{cl}\, \Omega$ its complement with $\partial \Omega^e = \partial \Omega$. Here $\mathrm{cl}\, \Omega$ denotes the closure of Ω and $\partial\Omega$ the boundary of Ω.

Denote by ω_m the surface of the unit sphere in R^m. For $\mathbf{x} \in R^m$ and $j, k = 1, \dots, m$ define

$$E_{jk}(\mathbf{x}) = \begin{cases} \frac{1}{2\omega_m} \left[\delta_{jk} \frac{|\mathbf{x}|^{2-m}}{m-2} + \frac{x_j x_k}{|\mathbf{x}|^m} \right], & m > 2, \\ \frac{1}{4\pi} \left[\delta_{jk} \, \ln \frac{1}{|\mathbf{x}|} + \frac{x_j x_k}{|\mathbf{x}|^2} \right], & m = 2, \end{cases} \tag{10}$$

$$Q_k(\mathbf{x}) = \frac{x_k}{\omega_m |\mathbf{x}|^m}. \tag{11}$$

For $\mathbf{\Psi} = [\Psi_1, \dots, \Psi_m] \in H^{-1/2}(\partial\Omega, R^m)$ define the hydrodynamical single layer potential with density $\mathbf{\Psi}$ by

$$(E_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} E(\mathbf{x} - \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \; d\mathbf{y} \tag{12}$$

whenever it makes sense and the corresponding pressure

$$(Q_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} Q(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ d\mathbf{y}, \qquad \mathbf{x} \in R^m \setminus \partial\Omega. \tag{13}$$

Then $E_\Omega \mathbf{\Psi} \in C^\infty(R^m \setminus \partial\Omega, R^m)$, $Q_\Omega \mathbf{\Psi} \in C^\infty(R^m \setminus \partial\Omega, R^1)$, $\nabla Q_\Omega \mathbf{\Psi} - \Delta E_\Omega \mathbf{\Psi} = 0$, $\nabla \cdot E_\Omega \mathbf{\Psi} = 0$ in $R^m \setminus \partial\Omega$. We have the following decay behavior as $|\mathbf{x}| \to \infty$:

$$E_\Omega \mathbf{\Psi}(\mathbf{x}) = O(|\mathbf{x}|^{2-m}), \quad m > 2,$$

$$E_\Omega \mathbf{\Psi}(\mathbf{x}) = O(\ln |\mathbf{x}|), \quad m = 2,$$

$$Q_\Omega \mathbf{\Psi}(\mathbf{x}),\ |(\nabla E_\Omega \mathbf{\Psi})(\mathbf{x})| = O(|\mathbf{x}|^{1-m}).$$

If $m = 2$ and $\langle \mathbf{\Psi}, 1 \rangle = 0$ then

$$E_\Omega \mathbf{\Psi}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad |\nabla E_\Omega \mathbf{\Psi}(\mathbf{x})| = O(|\mathbf{x}|^{-2}).$$

If Ω is bounded then $E_\Omega : \mathbf{\Psi} \mapsto E_\Omega \mathbf{\Psi}$ represents a bounded linear operator from $H^{-1/2}(\partial\Omega, R^m)$ to $H^1(\Omega, R^m)$ and $Q_G : \mathbf{\Psi} \mapsto Q_Q \mathbf{\Psi}$ is a continuous linear operator from $H^{-1/2}(\partial\Omega, R^m)$ to $L^2(\Omega, R^1)$ (see [9], Theorem 4.4). If $\mathbf{\Psi} \in H^{-1/2}(\partial\Omega, R^m)$ then $E_\Omega \mathbf{\Psi}$ is the trace of $E_\Omega \mathbf{\Psi}$ on $\partial\Omega$. Moreover, $E_\Omega : \mathbf{\Psi} \mapsto E_\Omega \mathbf{\Psi}$ is a bounded linear operator from $H^{-1/2}(\partial\Omega; R^m)$ to $H^{1/2}(\partial\Omega; R^m)$ (see [9], Proposition 4.5).

Fix $\mathbf{y} \in \partial\Omega$ such that there is the unit outward normal $\mathbf{n}^\Omega(\mathbf{y})$ of Ω at $\mathbf{y}$. For $\mathbf{x} \in R^m \setminus \{\mathbf{y}\}$, $j, k \in \{1, \dots, m\}$ set

$$\tilde{K}^\Omega_{jk}(\mathbf{x},\mathbf{y}) = \frac{1}{2\mathcal{H}_{m-1}(\partial B(0;1))} \Bigg[\delta_{jk} \frac{(\mathbf{y}-\mathbf{x}) \cdot \mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^m}$$

$$+m \frac{(y_j - x_j)(y_k - x_k)(\mathbf{y}-\mathbf{x}) \cdot \mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^{m+2}} - \frac{(y_k - x_k) n^\Omega_j(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^m} + \frac{(y_j - x_j) n^\Omega_k(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^m} \Bigg].$$

Then $\tilde{K}^\Omega_{jk}(\mathbf{x},\mathbf{y}) = -\mathbf{n}^\Omega(\mathbf{y}) \cdot \nabla_{\mathbf{y}} E_{jk}(\mathbf{y}-\mathbf{x}) + Q_j(\mathbf{y}-\mathbf{x}) n^\Omega_k(\mathbf{y})$. For $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$, $\mathbf{x} \in \partial\Omega$ define

$$\tilde{K}'_\Omega \mathbf{\Psi}(\mathbf{x}) = \lim_{\delta \searrow 0} \int_{\partial\Omega \setminus B(x;\delta)} \tilde{K}^\Omega(\mathbf{y},\mathbf{x}) \mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathbf{y}$$

whenever this integral exists. The operator K'_Ω is a bounded linear operator on $L^2(\partial\Omega, R^m)$ (see [4], [2], [9]; compare also [1]).

If $\mathbf{x} \in \partial\Omega$, $a > 0$ denote the non-tangential approach regions of opening a at the point $\mathbf{x}$ by

$$\Gamma_a(\mathbf{x}) := \{\mathbf{y} \in \Omega; |\mathbf{x}-\mathbf{y}| < (1+a) \operatorname{dist}(\mathbf{y}, \partial\Omega)\}.$$

Denote

$$\Gamma^e_a(\mathbf{x}) := \{\mathbf{y} \in \Omega^e; |\mathbf{x}-\mathbf{y}| < (1+a) \operatorname{dist}(\mathbf{y}, \partial\Omega^e)\}$$

the non-tangential approach regions of opening a at the point $\mathbf{x}$ corresponding to $\Omega^e = R^m \setminus \mathrm{cl}\,\Omega$. We fix $a > 0$ large enough such that $\mathbf{x} \in \mathrm{cl}\,\Gamma_a(\mathbf{x}) \cap \mathrm{cl}\,\Gamma^e_a$ for every $\mathbf{x} \in \partial\Omega$. We shall write $\Gamma(\mathbf{x}) = \Gamma_a(\mathbf{x})$, $\Gamma^e(\mathbf{x}) = \Gamma^e_a(\mathbf{x})$. If now $\mathbf{v}$ is a vector function defined in Ω and $\mathbf{x} \in \partial\Omega$ then the non-tangential maximal function of $\mathbf{v}$ is defined by

$$\mathbf{v}^*(\mathbf{x}) = \sup_{\mathbf{y}\in\Gamma(\mathbf{x})} |\mathbf{v}(\mathbf{y})|$$

and

$$\mathbf{v}_+(\mathbf{x}) = \lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Gamma(\mathbf{x})}} \mathbf{v}(\mathbf{y})$$

is the non-tangential limit of $\mathbf{v}$ at $\mathbf{x}$ with respect to Ω. Similarly, if $\mathbf{v}$ is a vector function defined in Ω^e and $\mathbf{x} \in \partial\Omega$ then

$$\mathbf{v}_-(\mathbf{x}) = \lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Gamma^e(\mathbf{x})}} \mathbf{v}(\mathbf{y})$$

If Ω is bounded and $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$ then $(\nabla E_\Omega \mathbf{\Psi})^*, (Q_\Omega \mathbf{\Psi})^* \in L^2(\partial\Omega)$ and

$$[\partial E_\Omega \mathbf{\Psi}/\partial \mathbf{n} - (Q_\Omega \mathbf{\Psi})\mathbf{n}^\Omega]_+ = \frac{1}{2}\mathbf{\Psi} - \tilde{K}'_\Omega \mathbf{\Psi}, \tag{14}$$

$$[\partial E_\Omega \mathbf{\Psi}/\partial \mathbf{n} - (Q_\Omega \mathbf{\Psi})\mathbf{n}^\Omega]_- = -\frac{1}{2}\mathbf{\Psi} - \tilde{K}'_\Omega \mathbf{\Psi} \tag{15}$$

(see [4], [2], [9]; compare also [1]).

Lemma 3.1. *If $\Omega \subset R^m$ is a bounded open set with Lipschitz boundary then there is a sequence of C^∞ domains Ω_j with following properties:*

- $\mathrm{cl}\,\Omega_j \subset \Omega$.
- *There are $a > 0$ and homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, such that $\Lambda_j(\mathbf{y}) \in \Gamma_a(\mathbf{y})$ for each j and each $\mathbf{y} \in \partial\Omega$ and $\sup\{|\mathbf{y} - \Lambda_j(\mathbf{y})|; \mathbf{y} \in \partial\Omega\} \to 0$ as $j \to \infty$.*
- *There are positive functions ω_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial\Omega$,*

  $$\int_E \omega_j \, d\mathbf{y} = \int_{\Lambda_j(E)} 1 \, d\mathbf{y},$$

  and so that $\omega_j \to 1$ pointwise a.e. and in $L^2(\partial\Omega, R^1)$.
- *The normal vectors to Ω_j, $\mathbf{n}(\Lambda_j(\mathbf{y}))$, converge pointwise a.e. and in $L^2(\partial\Omega, R^m)$ to $\mathbf{n}(\mathbf{y})$.*

(For the proof see [17], Theorem 1.12.)

Proposition 3.2. *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary. Then $\tilde{K}'_\Omega$ can be extended as a bounded linear operator on $H^{-1/2}(\partial\Omega, R^m)$. If $\mathbf{\Psi} \in H^{-1/2}(\partial\Omega, R^m)$ then $[\partial E_\Omega \mathbf{\Psi}/\partial \mathbf{n} - (Q_\Omega \mathbf{\Psi})\mathbf{n}^\Omega]_+ = \frac{1}{2}\mathbf{\Psi} - \tilde{K}'_\Omega \mathbf{\Psi}$, i.e. $\mathbf{u} = E_\Omega \mathbf{\Psi}$, $p = Q_\Omega \mathbf{\Psi}$ is a weak solution of the problem (1), (3) if and only if $\frac{1}{2}\mathbf{\Psi} - \tilde{K}'_\Omega \mathbf{\Psi} = \mathbf{g}$.*

Proof. Since E_Ω is a bounded linear operator from $H^{-1/2}(\partial\Omega, R^m)$ into $H^1(\Omega, R^m)$ and Q_Ω is a bounded linear operator from $H^{-1/2}(\partial\Omega, R^m)$ into $L^2(\Omega)$, we infer that $\mathbf{\Psi} \mapsto \partial E_\Omega \mathbf{\Psi}/\partial \mathbf{n} - Q_\Omega \mathbf{\Psi} \mathbf{n}$ is a bounded linear operator on $H^{-1/2}(\partial\Omega, R^m)$. If $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$ and $\mathbf{v} \in \mathcal{C}^\infty(R^m, R^m)$ then Lemma 3.1 and Green's formula yield

$$\langle [\partial E_\Omega \mathbf{\Psi}/\partial \mathbf{n} - (Q_\Omega \mathbf{\Psi})\mathbf{n}^\Omega]_+, \mathbf{v}\rangle = \langle \frac{1}{2}\mathbf{\Psi} - \tilde{K}'_\Omega \mathbf{\Psi}, \mathbf{v}\rangle = \int_\Omega [\nabla E_\Omega \mathbf{\Psi} : \nabla \mathbf{v} - Q_\Omega \mathbf{\Psi}(\nabla \cdot \mathbf{v})].$$

Since $\mathcal{C}^\infty(R^m, R^m)$ is a dense subset of $H^1(\Omega, R^m)$, this relation holds for arbitrary $\mathbf{v} \in H^1(\Omega, R^m)$. The continuity argument gives this relation for arbitrary $\mathbf{\Psi} \in H^{-1/2}(\partial\Omega, R^m)$.

Lemma 3.3. *If $\Omega \subset R^m$ is a bounded domain with Lipschitz boundary and $\mathbf{n}^\Omega$ is the unit outward normal of Ω then $E_\Omega \mathbf{n}^G = 0$, $Q_G \mathbf{n}^G = -1$ in G.*

(See [12].)

Lemma 3.4. *Let $G \subset R^m$ be a bounded domain with connected compact Lipschitz boundary, $m \geq 2$. Let $\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m)$. If $m = 2$ suppose moreover that $\int \mathbf{\Psi} = 0$. If $\langle \mathbf{\Psi}, E_G \overline{\mathbf{\Psi}}\rangle = 0$ then $E_G \mathbf{\Psi} = 0$ in R^m and there is a constant c such that $\mathbf{\Psi} = c\mathbf{n}^G$. (Here $\overline{\mathbf{\Psi}}$ denotes the complex conjugate of $\mathbf{\Psi}$).*

(See [12], Corollary 4.4.)

4 Spectral properties of the operator $\tilde{K}'_\Omega$

We shall look for a solution of the problem (1), (3) in the form of a hydrodynamical single layer potential $\mathbf{u} = E_G \mathbf{\Psi}$, $p = Q_G \mathbf{\Psi}$ with a density $\mathbf{\Psi} \in H^{-1/2}(\partial G)$. For this reason we shall study the spectrum of the operator $\frac{1}{2}I - \tilde{K}'_G$. We show that $\sigma(\frac{1}{2}I - \tilde{K}'_G) \subset \langle 0, 1\rangle$ and $\frac{1}{2}I - \tilde{K}'_G$ is a Fredholm operator of index 0. Our approach is a modification of the method used in [12] for the integral operator K'_G corresponding to the boundary value problem (1), (4).

Proposition 4.1. *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$. Let $\mathbf{\Psi}_1, \mathbf{\Psi}_2 \in H^{-1/2}(\partial\Omega, R^m)$. If $m = 2$ and Ω is unbounded suppose moreover that $\int \mathbf{\Psi}_1 = \int \mathbf{\Psi}_2 = 0$. Then*

$$\left\langle \frac{1}{2}\mathbf{\Psi}_1 - \tilde{K}'_\Omega \mathbf{\Psi}_1, E_\Omega \mathbf{\Psi}_2 \right\rangle = \int_\Omega (\nabla E_G \mathbf{\Psi}_1) : (\nabla E_\Omega \mathbf{\Psi}_2)\, d\mathbf{y}. \tag{16}$$

Put $\mathbf{\Psi} = \mathbf{\Psi}_1 + i\mathbf{\Psi}_2$ *where* i *is the imaginary unit. Denote* $\overline{\mathbf{\Psi}} = \mathbf{\Psi}_1 - i\mathbf{\Psi}_2$ *the conjugate of* $\mathbf{\Psi}$*. Then*

$$\left\langle \frac{1}{2}\mathbf{\Psi} - \tilde{K}'_\Omega \mathbf{\Psi}, E_\Omega \overline{\mathbf{\Psi}} \right\rangle = \int_\Omega (|\nabla E_\Omega \mathbf{\Psi}_1|^2 + |\nabla E_\Omega \mathbf{\Psi}_2|^2)\ d\mathbf{y} \geq 0. \tag{17}$$

Proof. We show (16). Suppose first that Ω is bounded. Proposition 2.2 gives that $\mathbf{u} = E_\Omega \mathbf{\Psi}_1$, $p = Q_\Omega \mathbf{\Psi}_1$ is a weak solution of the problem (1), (3) with $\mathbf{g} = \frac{1}{2}\mathbf{\Psi}_1 - \tilde{K}'_\Omega \mathbf{\Psi}_1$. Since $\mathbf{v} = E_\Omega \mathbf{\Psi}_2 \in H^1(\Omega; R^m)$ and $\nabla \cdot E_\Omega \mathbf{\Psi}_2 = 0$ in Ω, we obtain (16) from (8).

Let now Ω be unbounded. Fix $R > 0$ such that $\partial\Omega \subset B(0;R)$ and denote $\Omega(R) = \Omega \cap B(0;R)$. Put $\mathbf{\Psi}_1 = 0 = \mathbf{\Psi}_2$ on $\partial B(0;R)$. Then

$$\int_{\Omega(R)} (\nabla E_\Omega \mathbf{\Psi}_1) : (\nabla E_\Omega \mathbf{\Psi}_2)\ d\mathbf{y} = \left\langle \frac{1}{2}\mathbf{\Psi}_1 - \tilde{K}'_{\Omega(R)} \mathbf{\Psi}_1, E_\Omega \mathbf{\Psi}_2 \right\rangle$$

$$= \left\langle \frac{1}{2}\mathbf{\Psi}_1 - \tilde{K}'_\Omega \mathbf{\Psi}_1, E_\Omega \mathbf{\Psi}_2 \right\rangle + \int_{\partial B(0;R)} \left[\frac{\partial E_\Omega \mathbf{\Psi}_1}{\partial \mathbf{n}} + (Q_\Omega \mathbf{\Psi}_1)\mathbf{n} \right] \cdot E_\Omega \mathbf{\Psi}_2\ d\mathbf{y}.$$

If $R \to \infty$ then the decay properties of hydrodynamical potentials give (16).

Using (16) we get

$$\left\langle \frac{1}{2}\mathbf{\Psi} - \tilde{K}'_\Omega \mathbf{\Psi}, E_\Omega \overline{\mathbf{\Psi}} \right\rangle = \left\langle \frac{1}{2}\mathbf{\Psi}_1 - \tilde{K}'_\Omega \mathbf{\Psi}_1, E_\Omega \mathbf{\Psi}_1 \right\rangle + \left\langle \frac{1}{2}\mathbf{\Psi}_2 - \tilde{K}'_\Omega \mathbf{\Psi}_2, E_\Omega \mathbf{\Psi}_2 \right\rangle$$

$$-i\left\langle \frac{1}{2}\mathbf{\Psi}_1 - \tilde{K}'_\Omega \mathbf{\Psi}_1, E_\Omega \mathbf{\Psi}_2 \right\rangle + i\left\langle \frac{1}{2}\mathbf{\Psi}_2 - \tilde{K}'_\Omega \mathbf{\Psi}_2, E_\Omega \mathbf{\Psi}_1 \right\rangle = \int_\Omega |\nabla E_\Omega \mathbf{\Psi}_1|^2\ d\mathbf{y}$$

$$+ \int_\Omega |\nabla E_\Omega \mathbf{\Psi}_2|^2\ d\mathbf{y} - i\int_\Omega (\nabla E_\Omega \mathbf{\Psi}_1) : (\nabla E_\Omega \mathbf{\Psi}_2)\ d\mathbf{y}$$

$$+i\int_\Omega (\nabla E_\Omega \mathbf{\Psi}_1) : (\nabla E_\Omega \mathbf{\Psi}_2)\ d\mathbf{y} = \int_\Omega [|\nabla E_\Omega \mathbf{\Psi}_1|^2 + |\nabla E_\Omega \mathbf{\Psi}_2|^2]\ d\mathbf{y} \geq 0.$$

Corollary 4.2. *Let* $\Omega \subset R^m$ *be an open set with compact Lipschitz boundary,* $m \geq 2$*. Let* $\mathbf{\Psi} \in H^{-1/2}(\partial\Omega, C^m)$*. If* $m = 2$ *suppose moreover that* $\int \mathbf{\Psi} = 0$*. Then*

$$\langle \mathbf{\Psi}, E_\Omega \overline{\mathbf{\Psi}} \rangle = \int_{R^m \setminus \partial\Omega} |\nabla E_\Omega \mathbf{\Psi}|^2\ d\mathbf{y} \geq 0. \tag{18}$$

Proof. Put $C = R^m \setminus \operatorname{cl}\Omega$. Since $\tilde{K}'_\Omega = -\tilde{K}'_C$ we get using Proposition 4.1

$$\langle \mathbf{\Psi}, E_\Omega \overline{\mathbf{\Psi}} \rangle = \left\langle \frac{1}{2}\mathbf{\Psi} - \tilde{K}'_\Omega \mathbf{\Psi}, E_\Omega \overline{\mathbf{\Psi}} \right\rangle + \left\langle \frac{1}{2}\mathbf{\Psi} - K'_C \mathbf{\Psi}, E_\Omega \overline{\mathbf{\Psi}} \right\rangle = \int_{R^m \setminus \partial\Omega} |\nabla E_\Omega \mathbf{\Psi}|^2.$$

Definition 4.3. *Lex X, Y be Banach spaces. Denote by I the identity operator on X. If M is a subspace of X denote by $\dim M$ the dimension of M. If Z is a subspace of X such that $X = M \bigoplus Z$, i.e. X is the direct sum of M and Z, denote by $\operatorname{codim} Z = \dim M$ the codimension of Z. If T is a bounded linear operator from X to Y, denote by $\operatorname{Ker} T = \{x \in X; Tx = 0\}$ the kernel of T, $\alpha(T) = \dim \operatorname{Ker} T$, $\beta(T) = \operatorname{codim} T(X)$. We say that T is upper semi-Fredholm if $T(X)$ is a closed subset of Y and $\alpha(T) < \infty$. For an upper semi-Fredholm operator T denote $i(T) = \alpha(T) - \beta(T)$ the index of T. We say that T is Fredholm if T is upper semi-Fredholm and $\beta(T) < \infty$. If X is a complex Banach space and T is a bounded linear operator on X, denote by $\sigma(T)$ the spectrum of T and by $r(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$ the spectral radius of T.*

Lemma 4.4. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then $(\frac{1}{2}I - \tilde{K}'_G)\mathbf{n}^G = \mathbf{n}^G$. Denote $H^{-1/2}(\partial G; C^m) \cap E_{-1}(C^m)$ the set of all $\mathbf{\Psi} \in H^{-1/2}(\partial G; C^m)$ such that $E_G \mathbf{\Psi}$ is constant in G. Then $H^{-1/2}(\partial G; C^m) \cap E_{-1}(C^m) = \operatorname{Ker}(\frac{1}{2}I - \tilde{K}'_G) \bigoplus \{c\mathbf{n}^G; c \in C\}$ and $\dim \operatorname{Ker}(\frac{1}{2}I - \tilde{K}'_G) \leq m$.*

Proof. $(\frac{1}{2}I - \tilde{K}'_G)\mathbf{n}^G = \mathbf{n}^G$ by Lemma 3.3 and Proposition 3.2.

If $\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - \tilde{K}'_G)$ then $\mathbf{u} = E_G \mathbf{\Psi}$, $p = Q_G \mathbf{\Psi}$ is a weak solution of the Neumann problem for the Stokes system (1), (3) with the boundary condition $\mathbf{g} = 0$ (see Proposition 3.2). Lemma 2.1 gives that $\mathbf{u} = E_G \mathbf{\Psi}$ is constant. Let now $\mathbf{\Phi} \in H^{-1/2}(\partial G; C^m) \cap E_{-1}(C^m)$. Then $\nabla Q_G \mathbf{\Phi} = \Delta E_G \mathbf{\Phi} = 0$ in G. So, there is a constant c such that $Q_G \mathbf{\Phi} = c$ in G. Put $\mathbf{\Psi} = \mathbf{\Phi} + c\mathbf{n}^G$. Lemma 3.3 gives that $E_G \mathbf{\Psi} = E_G \mathbf{\Phi}$ is constant and $Q_G \mathbf{\Psi} = 0$ in G. Thus $\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - \tilde{K}'_G)$ by Proposition 3.2. Since $\mathbf{n}^G \notin \operatorname{Ker}(\frac{1}{2}I - \tilde{K}'_G)$, we infer that $\operatorname{Ker}(\frac{1}{2}I - \tilde{K}'_G) \bigoplus \{c\mathbf{n}^G; c \in C\} = H^{-1/2}(\partial G; C^m) \cap E_{-1}(C^m)$.

Suppose that $\mathbf{\Psi} \in \operatorname{Ker}(\frac{1}{2}I - \tilde{K}'_G)$, $\langle \mathbf{\Psi}, \mathbf{c} \rangle = 0$ for each $\mathbf{c} \in R^m$. Then there exists $\mathbf{b} \in C^m$ such that $E_G \mathbf{\Psi} = \mathbf{b}$ on the closure of G. Thus $0 = \langle \mathbf{\Psi}, \overline{\mathbf{b}} \rangle = \langle \mathbf{\Psi}, E_G \overline{\mathbf{\Psi}} \rangle$ and Lemma 3.4 gives that $\mathbf{\Psi} = d\mathbf{n}^G$ for some $d \in C$. Since $\mathbf{n}^G \notin \operatorname{Ker}(\frac{1}{2}I - \tilde{K}'_G)$, we infer that $\mathbf{\Psi} = 0$. This gives $\dim \operatorname{Ker}(\frac{1}{2}I - \tilde{K}'_G) \leq m$.

Proposition 4.5. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary. If $\lambda \in C$ is an eigenvalue of $\frac{1}{2}I - \tilde{K}'_G$ in $H^{-1/2}(\partial G, C^m)$ then $0 \leq \lambda \leq 1$.*

Proof. Let $\mathbf{\Psi}$ be an eigenfunction corresponding to an eigenvalue λ. We can suppose that $\lambda \neq 0$. Then $\mathbf{\Psi} = (\frac{1}{2}I - \tilde{K}'_G)\frac{1}{\lambda}\mathbf{\Psi}$. Proposition 3.2 and Lemma 2.1 give that $\int \mathbf{\Psi} = 0$. If $\mathbf{\Psi} = b\mathbf{n}^G$, $b \in C$, then $\lambda = 1$ by Lemma 4.4. Suppose now that $\mathbf{\Psi} \neq b\mathbf{n}^G$. Then

$$\langle \mathbf{\Psi}, E_G \overline{\mathbf{\Psi}} \rangle = \int\limits_{R^m \setminus \partial G} |\nabla E_G \mathbf{\Psi}|^2 \, d\mathbf{x} > 0$$

by Lemma 3.4 and Corollary 4.2. According to Proposition 4.1 and Corollary 4.2

$$\int\limits_G |\nabla E_G \mathbf{\Psi}(x)|^2 = \left\langle \frac{1}{2}\mathbf{\Psi} - \tilde{K}'_G \mathbf{\Psi}, E_G \overline{\mathbf{\Psi}} \right\rangle = \langle \lambda \mathbf{\Psi}, E_G \overline{\mathbf{\Psi}} \rangle = \lambda \int\limits_{R^m \setminus \partial G} |\nabla E_G \mathbf{\Psi}|^2.$$

Therefore

$$0 \le \lambda = \frac{\int\limits_G |\hat{\nabla} E_G \mathbf{\Psi}|^2 \; d\mathbf{x}}{\int\limits_{R^m \setminus \partial G} |\hat{\nabla} E_G \mathbf{\Psi}|^2 \; d\mathbf{x}} \le 1.$$

Proposition 4.6. *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \ge 2$. Then there is a closed subspace Y of $H^{-1/2}(\partial\Omega, R^m)$ with finite codimension such that $\sqrt{\langle \mathbf{\Psi}, E_\Omega \mathbf{\Psi}\rangle}$, $\|E_\Omega \mathbf{\Psi}\|_{H^{1/2}(\partial\Omega)}$, $\sqrt{\langle [(1/2)I - \tilde{K}'_\Omega]\mathbf{\Psi}, E_\Omega \mathbf{\Psi}\rangle}$ are three norms on Y which are equivalent to the original norm.*

Proof. First we show that there exist a closed subspace X of $H^{-1/2}(\partial\Omega, R^m)$ with finite codimension and a constant C_1 such that

$$\|\mathbf{\Psi}\|_{H^{-1/2}(\partial\Omega)} \le C_1 \|E_\Omega \mathbf{\Psi}\|_{H^{1/2}(\partial\Omega)} \quad \forall \mathbf{\Psi} \in X. \tag{19}$$

For $\partial\Omega$ connected see [12], Proposition 4.11. Denote $S_1, \dots, S_k$ all components of $\partial\Omega$. Then E_Ω is an upper semi-Fredholm operator from $H^{-1/2}(S_j, R^m)$ to $H^{1/2}(S_j, R^m))$ for each j (see [13], §16, Theorem 8). If $j \ne l$ then E_Ω is a compact linear operator from $H^{-1/2}(S_j, R^m)$ to $H^{1/2}(S_l, R^m))$. Thus E_Ω is an upper semi-Fredholm operator from $H^{-1/2}(\partial\Omega, R^m)$ to $H^{1/2}(\partial\Omega, R^m))$ (see [13], §16, Theorem 16). According to [15], Lemma 5.1 there exists a closed subspace X of $H^{-1/2}(\partial\Omega, R^m)$ such that $H^{-1/2}(\partial\Omega, R^m) = X \bigoplus \operatorname{Ker} E_\Omega$. The relation (19) is a consequence of [13], §16, Theorem 10 and [3], Theorem 1.42.

Denote

$$r(\mathbf{v}) = \|\nabla \mathbf{v}\|_{L^2(\Omega)} + \left| \int\limits_\Omega \mathbf{v} \; \mathrm{d}\mathcal{H}_m \right|.$$

Then $r(\mathbf{v})$ is an equivalent norm on $H^1(\Omega; R^m)$ (see [10], Chapter 1, §1.5.4). Set

$$V = \left\{ \mathbf{v} \in H^1(\Omega, R^m); \int\limits_{\partial\Omega} \mathbf{v} \; d\mathbf{y} = 0 \right\}.$$

Then there is a positive constant C_2 such that $\|\mathbf{v}\|_{H^1(\Omega)} \le C_2 \|\nabla \mathbf{v}\|_{L^2(\Omega)}$ for all $\mathbf{v} \in V$. Denote $Y = \{\mathbf{\Psi} \in X; E_\Omega \mathbf{\Psi} \in V, \int \mathbf{\Psi} = 0\}$. Since E_Ω is a continuously invertible operator X onto $E_\Omega(X) \subset H^1(\Omega, R^m)$ and V is a closed subspace of $H^1(\Omega, R^m)$ with finite codimension, Y is a closed subspace of $H^{-1,2}(\Omega, R^m)$ with finite codimension. Fix $\mathbf{\Psi} \in Y$. Since $E_\Omega \mathbf{\Psi}$ is the trace of $E_\Omega \mathbf{\Psi}$ on $\partial\Omega$ we obtain using (19), (7), Proposition 4.1 and Corollary 4.2

$$\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial\Omega, R^m)} \le C_1 \|E_\Omega \mathbf{\Psi}\|^2_{H^{1/2}(\partial\Omega, R^m)} \le C_1 \|E_\Omega \mathbf{\Psi}\|^2_{H^1(\Omega, R^m)}$$

$$\le C_1 C_2 \int\limits_\Omega |\nabla E_\Omega \mathbf{\Psi}|^2 \; d\mathbf{y} = C_1 C_2 \langle [(1/2)I - \tilde{K}'_\Omega]\mathbf{\Psi}, E_\Omega \mathbf{\Psi}\rangle$$

$$\le C_1 C_2 \int\limits_{R^m \setminus \partial\Omega} |\nabla E_\Omega \mathbf{\Psi}|^2 \; d\mathbf{y} = C_1 C_2 \langle \mathbf{\Psi}, E_\Omega \mathbf{\Psi}\rangle.$$

Theorem 4.7. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then $\sigma(\frac{1}{2}I - \tilde{K}'_G) \subset \langle 0,1\rangle$ in $H^{-1/2}(\partial G, C^m)$ and $\frac{1}{2}I - \tilde{K}'_G$ is a Fredholm operator with index* 0.

Proof. Proposition 4.6 yields that there is a closed subspace Y of finite codimension and a positive constant L such that $\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)} \leq L\langle(\mathbf{\Psi}, E_G\mathbf{\Psi}\rangle$, $\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)} \leq L\langle(\frac{1}{2}I - \tilde{K}'_G)\mathbf{\Psi}, E_G\mathbf{\Psi}\rangle$ for each $\mathbf{\Psi} \in Y$. Put $Z = \{\mathbf{\Psi} = \mathbf{\Psi}_1 + i\mathbf{\Psi}_2; \mathbf{\Psi}_1, \mathbf{\Psi}_2 \in Y$ and $\int \mathbf{\Psi} = 0\}$. Then Z is a closed subspace of $H^{-1/2}(\partial G, C^m)$ with finite codimension. Proposition 4.1 and Corollary 4.2 give

$$\|\mathbf{\Psi}\|^2_{H^{-1/2}} \leq L\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle, \quad \|\mathbf{\Psi}\|^2_{H^{-1/2}} \leq L\langle [(1/2)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \tag{20}$$

for each $\mathbf{\Psi} \in Z$.

If $\lambda \in R$ then (20) gives

$$\langle [(1/2-\lambda)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = \langle [(1/2)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle - \lambda\langle(\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \in R. \tag{21}$$

If $\lambda < 0$ then (20) and (21) give

$$L\langle [(1/2-\lambda)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \geq L\langle [(1/2)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \geq \|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)}.$$

If $\lambda > 1$ then Corollary 4.2, Proposition 4.1 and (20) give

$$L|\langle [(1/2-\lambda)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle| \geq L\{\lambda\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle - \langle [(1/2)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle\}$$

$$= L\lambda \int\limits_{R^m\setminus\partial G} |\nabla E_G\mathbf{\Psi}|^2 \, d\mathbf{y} - L\int\limits_G |\nabla E_G\mathbf{\Psi}|^2 \, d\mathbf{y} \geq L(\lambda-1) \int\limits_{R^m\setminus\partial G} |\nabla E_G\mathbf{\Psi}|^2 \, d\mathbf{y}$$

$$= L(\lambda-1)\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \geq (\lambda-1)\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)}.$$

If $\lambda = \lambda_1 + i\lambda_2 \in C$, $\lambda_2 \neq 0$ and $\mathbf{\Psi} \in Z$ then (20) and (21) give

$$|\langle [(1/2-\lambda)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle| = |\langle [(1/2-\lambda_1)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle - i\lambda_2\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle|$$

$$\geq |\lambda_2||\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle| \geq |\lambda_2|L^{-1}\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)}.$$

Fix $\lambda \in C \setminus \langle 0,1\rangle$. We have proved that there is a positive constant M such that

$$\|\mathbf{\Psi}\|^2_{H^{-1/2}(\partial G)} \leq M\langle [(1/2-\lambda)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle.$$

for each $\mathbf{\Psi} \in Z$. If $\mathbf{\Psi} \in Z \setminus \{0\}$ then

$$\|\mathbf{\Psi}\|_{H^{-1/2}(\partial G)} \leq M\langle [(1/2-\lambda)I - \tilde{K}'_G]\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle / \|\mathbf{\Psi}\|_{H^{-1/2}(\partial G)}$$

$$\leq M\|E_G\|_{H^{-1/2}(\partial G)\to H^{-1/2}(\partial G)}\|[(1/2-\lambda)I - \tilde{K}'_G]\mathbf{\Psi}\|_{H^{-1/2}(\partial G)}.$$

So, the operator $\frac{1}{2}I - \tilde{K}'_G - \lambda I$ is upper semi-Fredholm by [13], § 16, Theorem 8. Since the index $i(\frac{1}{2}I - \tilde{K}'_G - \mu I)$ is constant on $C \setminus \langle 0,1\rangle$ (see [13],§ 18, Corollary 3) and $\frac{1}{2}I - \tilde{K}'_G - \mu I$ is invertible for $|\mu| > \|\frac{1}{2}I - \tilde{K}'_G\|$ (see [15], Lemma 6.5), we infer that $i(\frac{1}{2}I - \tilde{K}'_G - \lambda I) = 0$. Thus $\frac{1}{2}I - \tilde{K}'_G - \lambda I$ is a Fredholm operator with index 0. If $\lambda \neq 0$ then $\alpha(\frac{1}{2}I - \tilde{K}'_G - \lambda I) = 0$ by Proposition 4.5 and $i(\frac{1}{2}I - \tilde{K}'_G - \lambda I) = 0$ forces that the operator $\frac{1}{2}I - \tilde{K}'_G - \lambda I$ is onto. Therefore $\frac{1}{2}I - \tilde{K}'_G - \lambda I$ is a continuously invertible operator (see [3], Theorem 1.42).

5 Indirect BEM

In this section we shall study the problem (1), (3) for a bounded domain G with connected Lipschitz boundary using the indirect boundary integral equation method. We shall look for a solution in the form of a hydrodynamical single layer potential $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ with a density $\mathbf{\Psi} \in H^{-1/2}(\partial G)$. We have proved that $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ is a solution of the problem if and only if $\frac{1}{2}\mathbf{\Psi} - \tilde{K}'_G\mathbf{\Psi} = \mathbf{g}$ (see Proposition 3.2). We determine the necessary and sufficient condition for the solvability of the problem. Moreover, we prove that the integral equation $\frac{1}{2}\mathbf{\Psi} - \tilde{K}'_G\mathbf{\Psi} = \mathbf{g}$ can be solved by the successive approximation.

In the numerical practice we approximate $\mathbf{g}$, so we solve the equation $\frac{1}{2}\tilde{\mathbf{\Psi}} - \tilde{K}'_G\tilde{\mathbf{\Psi}} = \tilde{\mathbf{g}}$ where $\tilde{\mathbf{g}}$ is close to $\mathbf{g}$. Since the operator $\frac{1}{2}I - \tilde{K}'_G$ is not invertible this equation might not be solvable. To overcome this difficulty we define a modified operator

$$M'\mathbf{\Psi} = \tilde{K}'_G\psi - \frac{1}{c}\int\limits_{\partial G}\mathbf{\Psi}\ \mathrm{d}\mathbf{y}, \qquad c = \int\limits_{\partial G} 1\ \mathrm{d}\mathbf{y}. \tag{22}$$

We show that the integral equation $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ is uniquely solvable and if the problem (1), (3) is solvable and $\mathbf{\Psi}$ is a solution of the equation $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ then $\frac{1}{2}\mathbf{\Psi} - \tilde{K}'_G\mathbf{\Psi} = \mathbf{g}$. We show that the modified equation $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ can be solved by the successive approximation.

Proposition 5.1. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary. Then $H^{-1/2}(\partial G, C^m)$ is the direct sum of $\mathrm{Ker}(\frac{1}{2}I - \tilde{K}'_G)$ and $(\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m)) = \{\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m);\ \int \mathbf{\Psi} = 0\}$. If we denote by L'_G the restriction of $\tilde{K}'_G$ onto $(\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m))$, then $\sigma(\frac{1}{2}I - L'_G) \subset (0, 1\rangle$.*

Proof. If $\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m)$, then $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ is a solution of the problem (1), (3) with the boundary condition $\mathbf{g} = (\frac{1}{2}I - \tilde{K}'_G)\mathbf{\Psi}$ by Proposition 3.2. Lemma 2.1 gives that $\int \mathbf{\Psi} = 0$. Thus $(\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m)) \subset \{\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m); \int \mathbf{\Psi} = 0\}$ and $\mathrm{codim}(\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m)) \geq m$. Lemma 4.4 gives $\dim \mathrm{Ker}(\frac{1}{2}I - \tilde{K}'_G) \leq m$. So $\mathrm{codim}(\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m)) = \dim \mathrm{Ker}(\frac{1}{2}I - \tilde{K}'_G) = m$ by Theorem 4.7. Hence $(\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m)) = \{\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m);\ \int \mathbf{\Psi} = 0\}$.

Let now $\mathbf{\Psi} \in \mathrm{Ker}(\frac{1}{2}I - \tilde{K}'_G) \cap (\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m))$. Then $\int \mathbf{\Psi} = 0$. Since $E_G\mathbf{\Psi}$ is constant on $\overline{G}$ by Lemma 4.4, we obtain $\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = 0$. Since $\int \mathbf{\Psi} = 0$, Lemma 3.4 gives that $\mathbf{\Psi} = b\mathbf{n}^G$ for some $b \in C$. Since $\mathbf{n}^G \notin \mathrm{Ker}(\frac{1}{2}I - \tilde{K}'_G)$ by Lemma 4.4, we infer that $b = 0$. Since $\mathrm{Ker}(\frac{1}{2}I - \tilde{K}'_G) \cap (\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m)) = \{0\}$ and $\mathrm{codim}(\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m)) = \dim \mathrm{Ker}(\frac{1}{2}I - \tilde{K}'_G)$, we deduce that $H^{-1/2}(\partial G, C^m) = \mathrm{Ker}(\frac{1}{2}I - \tilde{K}'_G) \bigoplus (\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m))$.

Since $H^{-1/2}(\partial G, C^m) = \mathrm{Ker}(\frac{1}{2}I - \tilde{K}'_G) \bigoplus (\frac{1}{2}I - \tilde{K}'_G)(H^{-1/2}(\partial G, C^m))$, we have $\sigma(\frac{1}{2}I - L'_G) \subset \sigma(\frac{1}{2}I - \tilde{K}'_G) \subset \langle 0, 1\rangle$. Moreover, the operator $(\frac{1}{2}I - L'_G)$ is one-to-one and onto. Thus $0 \notin \sigma(\frac{1}{2}I - L'_G)$ (see [3], Theorem 1.42.)

Proposition 5.2. *Let X be a Banach space, T be a bounded linear operator on X. Suppose that X is the direct sum of* $\mathrm{Ker}(I-T)$ *and* $(I-T)(X)$. *Denote by $\tilde{T}$ the restriction of T onto $(I-T)(X)$. Suppose that*

$$\lim_{j\to\infty} \|\tilde{T}^j\|^{1/j} < 1. \tag{23}$$

Fix now $y \in (I-T)(X)$, $x_0 \in X$. Put

$$x_{j+1} = Tx_j + y \tag{24}$$

for a nonnegative integer j. Then there exists

$$x = \lim_{j\to\infty} x_j$$

and

$$\|x - x_j\| \le Cq^j(\|y\| + \|x_0\|) \tag{25}$$

for arbitrary j, where $C > 0$, $0 < q < 1$ are constants depending only on T.

(For the proof see ([11]), Proposition 3.)

Theorem 5.3. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \ge 2$. Fix $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$. Then there is a weak solution of the problem (1), (3) if and only if $\int \mathbf{g} = 0$. Suppose now that $\int \mathbf{g} = 0$ and $\mathbf{\Psi}_0 \in H^{-1/2}(\partial G, R^m)$. For a nonnegative integer k put*

$$\mathbf{\Psi}_{k+1} = [(1/2)I + \tilde{K}'_G]\mathbf{\Psi}_k + \mathbf{g}. \tag{26}$$

Then there is $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ such that $\mathbf{\Psi}_k \to \mathbf{\Psi}$ in $H^{-1/2}(\partial G, R^m)$ as $k \to \infty$. Moreover, there are constants $0 < q < 1$, $C > 0$ depending only on G such that

$$\|\mathbf{\Psi}_k - \mathbf{\Psi}\|_{H^{-1/2}(\partial G, R^m)} \le Cq^k \left(\|\mathbf{g}\|_{H^{-1/2}(\partial G, R^m)} + \|\mathbf{\Psi}_0\|_{H^{-1/2}(\partial G, R^m)} \right). \tag{27}$$

If we put $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$ then $\mathbf{u}$, p is a weak solution of the problem (1), (3).

Proof. Suppose first that there is a weak solution of the problem (1), (3). Lemma 2.1 gives that $\int \mathbf{g} = 0$.

Suppose now that $\int \mathbf{g} = 0$. Set $T = (1/2)I + \tilde{K}'_G$, $\tilde{T}$ the restriction of T onto $[(1/2)I - \tilde{K}'_G](H^{-1/2}(\partial G, C^m))$. Proposition 5.1 gives that $H^{-1/2}(\partial G, R^m) = \mathrm{Ker}(I-T) \bigoplus (I-T)(H^{-1/2}(\partial G, R^m))$ and $\sigma(I - \tilde{T}) \subset (-1, 1)$. Since $r(\tilde{T}) < 1$, [18], Chapter VIII, §2 gives (23). According to Proposition 5.2 there exists $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ such that $\mathbf{\Psi}_k \to \mathbf{\Psi}$ as $k \to \infty$ in $H^{-1/2}(\partial G, R^m)$ and (27) holds with constants $0 < q < 1$, $C > 0$ depending only on G.

Put $\mathbf{u} = E_G\mathbf{\Psi}$, $p = Q_G\mathbf{\Psi}$. Letting $k \to \infty$ in (26) we get $\mathbf{\Psi} = [(1/2)I + \tilde{K}'_G]\mathbf{\Psi} + \mathbf{g}$. Proposition 3.2 forces that $\mathbf{u}$, p is a weak solution of the problem (1), (3).

Proposition 5.4. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then the operator $\frac{1}{2}I - M'$ is continuously invertible in the space $H^{-1/2}(\partial G, R^m)$. If $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$, $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ and $\int \mathbf{g} = 0$, then $\int \mathbf{\Psi} = 0$ and $\frac{1}{2}\mathbf{\Psi} - \tilde{K}'_G\mathbf{\Psi} = \mathbf{g}$.*

Proof. Suppose first that $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ and $\int \mathbf{g} = 0$. By virtue of Proposition 5.1

$$0 = \int\limits_{\partial G} \mathbf{g}\ \mathrm{d}\mathbf{y} = \int\limits_{\partial G} \left(\frac{1}{2}I - \tilde{K}'_G\right)\mathbf{\Psi}\ \mathrm{d}\mathbf{y} + \frac{1}{c}\int\limits_{\partial G} \mathbf{\Psi}\ \mathrm{d}\mathbf{y} = \frac{1}{c}\int\limits_{\partial G} \mathbf{\Psi}\ \mathrm{d}\mathbf{y}.$$

Therefore $\frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi} = \frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$.

Now we prove that $\frac{1}{2}I - M'$ is one-to-one. Suppose $(\frac{1}{2}I - M')\mathbf{\Psi} = 0$. Then $\int \mathbf{\Psi} = 0$ and $\frac{1}{2}\mathbf{\Psi} - \tilde{K}'_G\mathbf{\Psi} = \frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = 0$. Since $\frac{1}{2}I - \tilde{K}'_G$ is injective on $\{\mathbf{f} \in H^{-1/2}(\partial G, R^m); \int \mathbf{f} = 0\}$ by Proposition 5.1, we infer that $\mathbf{\Psi} = 0$.

The operator $M' - \tilde{K}'_G$ is a finite rank operator and therefore compact (see [15], p. 88). Since $\frac{1}{2}I - \tilde{K}'_G$ is a Fredholm operator with index 0 by Theorem 4.7, the operator $\frac{1}{2}I - M'$ is a Fredholm operator with index 0, too (see [13], § 16, Theorem 16). Since $\frac{1}{2}I - M'$ is one-to-one, it is also onto and therefore continuously invertible (see [3], Theorem 1.42).

Proposition 5.5. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then there is an equivalent norm on $H^{-1/2}(\partial G, C^m)$ such that $\|\frac{1}{2}I + M'\| \leq q < 1$. Let now $\mathbf{g} \in H^{-1/2}(\partial G, C^m)$, $\int \mathbf{g} = 0$. Fix $\mathbf{\Psi}_0 \in H^{-1/2}(\partial G, C^m)$. For a nonnegative integer k put*

$$\mathbf{\Psi}_{k+1} = \left(\frac{1}{2}I + M'\right)\mathbf{\Psi}_k + \mathbf{g}.$$

Then $\mathbf{\Psi}_k \to \mathbf{\Psi}$ in $H^{-1/2}(\partial G, C^m)$, $\frac{1}{2}\mathbf{\Psi} - M'\mathbf{\Psi} = \mathbf{g}$ and $\|\mathbf{\Psi} - \mathbf{\Psi}_j\| \leq q^j[\|\mathbf{g}\| + \|\mathbf{\Psi}_0\|]$ for arbitrary j.

Proof. Let λ be an eigenvalue of $\frac{1}{2}I - M'$ and $\mathbf{\Psi}$ be a corresponding eigenvector. Then $\mathbf{\Psi} = \mathbf{f} + \mathbf{g}$, where $\mathbf{g}$ is constant and $\int \mathbf{f} = 0$. We have

$$\lambda\mathbf{f} + \lambda\mathbf{g} = \left(\frac{1}{2}I - M'\right)\mathbf{\Psi} = \left(\frac{1}{2}I - K'_G\right)\mathbf{\Psi} + \mathbf{g}.$$

By virtue of Proposition 5.1

$$\lambda\int \mathbf{g} = \int \lambda\mathbf{f} + \int \lambda\mathbf{g} = \int\left(\frac{1}{2}I - K'_G\right)\mathbf{\Psi} + \int \mathbf{g} = \int \mathbf{g}.$$

If $\mathbf{g} \neq 0$ then $\lambda = 1$. If $\mathbf{g} = 0$ then $\mathbf{\Psi} = \mathbf{f} \in [(1/2)I - \tilde{K}'_G](H^{-1/2}(\partial G, C^m))$ by Proposition 5.1. Since λ is an eigenvalue of $[(1/2)I - L'_G]$, Proposition 5.1 gives that $0 < \lambda \leq 1$.

Fix $\lambda \in C \setminus \langle 0, 1\rangle$. The operator $\frac{1}{2}I - \tilde{K}'_G - \lambda I$ is a Fredholm operator with index 0 by Theorem 4.7. Since $M' - \tilde{K}'_G$ is a finite rank operator and so compact (see [15], p. 88), the operator $\frac{1}{2}I - M' - \lambda I$ is a Fredholm operator with index 0 (see [13], §16, Theorem 16). If $\lambda \in \sigma(\frac{1}{2}I - M')$ then λ is an eigenvalue of $\frac{1}{2}I - M'$. We have proved that λ is not an eigenvalue of $\frac{1}{2}I - M'$. Thus $\sigma(\frac{1}{2}I - M') \subset (0, 1\rangle$. Since $\sigma(\frac{1}{2}I + M') \subset \langle 0, 1)$ we have $r(\frac{1}{2}I + M') < 1$. If we fix $r(\frac{1}{2}I + M') < q < 1$ then there exists an equivalent norm $\| \cdot \|$ on $H^{-1/2}(\partial G, C^m)$ such that $\|\frac{1}{2}I + M'\| \le q$ (see [6]). The rest is a consequence of Proposition 5.2.

6 Double layer potentials

Now we define a hydrodynamical double layer potential corresponding to the boundary condition (3). Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary. Fix $\mathbf{y} \in \partial\Omega$ such that there is the unit outward normal $\mathbf{n}^\Omega(\mathbf{y})$ of Ω at $\mathbf{y}$. For $\mathbf{x} \in R^m \setminus \{\mathbf{y}\}$, $j, k \in \{1, \ldots, m\}$ set

$$R_k^\Omega(\mathbf{x}, \mathbf{y}) = \frac{1}{\mathcal{H}_{m-1}(\partial B(0;1))} \left[\frac{n_k^\Omega(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^m} - \frac{m(y_k - x_k)(\mathbf{y}-\mathbf{x}) \cdot \mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{m+2}} \right].$$

Then $R_k^\Omega(\mathbf{x}, \mathbf{y}) = \mathbf{n}^\Omega(\mathbf{y}) \cdot \nabla_\mathbf{y} Q_k(\mathbf{y} - \mathbf{x})$.

For $\mathbf{\Psi} = [\Psi_1, \ldots, \Psi_m] \in L^2(\partial\Omega, R^m)$ define the corresponding hydrodynamical double layer potential with density $\mathbf{\Psi}$ by

$$(W_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} \tilde{K}^\Omega(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}) \tag{28}$$

and the corresponding pressure

$$(R_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} R^\Omega(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y})$$

in $R^m \setminus \partial\Omega$. Then $(W_\Omega \mathbf{\Psi}, R_\Omega \mathbf{\Psi}) \in C^\infty(R^m \setminus \partial\Omega, R^{m+1})$ solve the Stokes system

$$\nabla R_\Omega \mathbf{\Psi} - \Delta W_\Omega \mathbf{\Psi} = 0, \quad \nabla \cdot W_\Omega \mathbf{\Psi} = 0 \quad \text{in } R^m \setminus \partial\Omega.$$

We have the following decay behavior as $|\mathbf{x}| \to \infty$:

$$(W_\Omega \mathbf{\Psi})(\mathbf{x}) = O(|\mathbf{x}|^{1-m}),$$

$$|(\nabla W_\Omega \mathbf{\Psi})(\mathbf{x})|, \; R_\Omega \mathbf{\Psi}(\mathbf{x}) = O(|\mathbf{x}|^{-m}).$$

Since $\tilde{K}_{jk}^\Omega(\mathbf{x}, \mathbf{y}) = -\mathbf{n}^\Omega(\mathbf{y}) \cdot \nabla_\mathbf{y} E_{jk}(\mathbf{y} - \mathbf{x}) + Q_j(\mathbf{y} - \mathbf{x}) n_k^\Omega(\mathbf{y})$, we have $(W_\Omega f)^* \le (|\nabla E_\Omega f|)^* + (Q_\Omega f)^* \in L^2(\partial\Omega)$.

If $\mathbf{x} \in \partial\Omega$ define

$$\tilde{K}_\Omega \mathbf{\Psi}(\mathbf{x}) = \lim_{\epsilon \searrow 0} \int_{\partial\Omega \setminus B(\mathbf{x};\epsilon)} \tilde{K}^\Omega(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, d\mathbf{y}.$$

whenever this limit exists. Clearly, $\tilde{K}_\Omega$ is the adjoint operator of $\tilde{K}'_\Omega$. Thus $\tilde{K}_\Omega$ is a bounded linear operator on $L^2(\partial\Omega, R^m)$ and on $H^{1/2}(\partial\Omega, R^m)$. Moreover,

$$[W_\Omega \mathbf{\Psi}]_+(\mathbf{x}) = \frac{1}{2}\mathbf{\Psi}(\mathbf{z}) + \tilde{K}_\Omega \mathbf{\Psi}(\mathbf{z}), \quad [W_\Omega \mathbf{\Psi}]_-(\mathbf{x}) = -\frac{1}{2}\mathbf{\Psi}(\mathbf{z}) + \tilde{K}_\Omega \mathbf{\Psi}(\mathbf{z}) \tag{29}$$

for almost all $\mathbf{x} \in \partial\Omega$ (see [9], Proposition 3.2). If $\mathbf{\Psi} \in H^{1/2}(\partial\Omega)$ then $W_\Omega \mathbf{\Psi} \in H^1(\Omega, R^m)$ (see [9], Theorem 4.4) and $\frac{1}{2}\mathbf{\Psi} + \tilde{K}_\Omega \mathbf{\Psi}$ is the trace of $W_\Omega \mathbf{\Psi}$.

Proposition 6.1. *Let $G \subset R^m$ be a bounded open set with connected Lipschitz boundary, $m \geq 2$. Let $\mathbf{u}$, p be a weak solution of the problem (1), (3) with $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$. Then*

$$\mathbf{u}(\mathbf{x}) = E_G \mathbf{g}(\mathbf{x}) + W_G \mathbf{u}(\mathbf{x}), \quad p(\mathbf{x}) = Q_G \mathbf{g}(\mathbf{x}) + R_G \mathbf{u}(\mathbf{x}) \qquad \mathbf{x} \in G, \tag{30}$$

$$E_G \mathbf{g}(\mathbf{x}) + W_G \mathbf{u}(\mathbf{x}) = 0, \quad Q_G \mathbf{g}(\mathbf{x}) + R_G \mathbf{u}(\mathbf{x}) = 0 \qquad \mathbf{x} \notin \operatorname{cl} G. \tag{31}$$

Proof. If $\mathbf{u} \in \mathcal{C}^2(\overline{G}, R^m)$, $p \in \mathcal{C}^1(\overline{G})$ then this result is well-known (see [14], p. 29).

Suppose now that $\mathbf{g} \in L^2(\partial G, R^m)$. Then $\mathbf{u}^* + p^* \in L^2(\partial G)$ and $\mathbf{g}$ is the nontangential limit $\partial\mathbf{u}/\partial\mathbf{n} - p\mathbf{n}$ at almost all points of ∂G (see [4], Theorem 2.9). Let Ω_j be domains from Lemma 3.1. Then (30), (31) hold true for Ω_n. Using Lebesgue lemma we obtain these equalities for G.

Let now $\mathbf{g}$ be general. According to Theorem 5.3 and Lemma 2.1 there exists $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ and $\mathbf{c} \in R^m$ such that $\mathbf{u} = E_G \mathbf{\Psi} + \mathbf{c}$, $p = Q_G \mathbf{\Psi}$. Choose $\mathbf{\Psi}_k \in L^2(\partial G, R^m)$ such that $\mathbf{\Psi}_k \to \mathbf{\Psi}$ in $H^{-1/2}(\partial G, R^m)$. Put $\mathbf{u_k} = E_G \mathbf{\Psi}_k + \mathbf{c}$, $p_k = Q_G \mathbf{\Psi}_k$, $\mathbf{g}_k = \partial\mathbf{u}_k/\partial\mathbf{n} - p_k\mathbf{n}$. Then $\mathbf{g}_k \in L^2(\partial G, R^m)$ by [4]. So, (30), (31) hold for $\mathbf{u}_k$, p_k and $\mathbf{g}_k$. If $k \to \infty$ we get (30), (31).

Corollary 6.2. *Let $G \subset R^m$ be a bounded open set with connected Lipschitz boundary, $m \geq 2$. Then $E_G \mathbf{n}^G \equiv 0$, $Q_G \mathbf{n}^G = 0$ in $R^m \setminus \operatorname{cl} G$.*

Proof. We use Proposition 6.1 for $\mathbf{u} \equiv 0$, $p = 1$.

7 Direct BEM

Let now $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$, $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$ be such that $\int \mathbf{g} = 0$. According to Theorem 5.3 there is a weak solution $\mathbf{u}$, p of the problem (1), (3). Denote by $\tilde{\mathbf{u}}$ the trace of $\mathbf{u}$. Since

$$\mathbf{u}(\mathbf{x}) = E_G \mathbf{g}(\mathbf{x}) + W_G \tilde{\mathbf{u}}(\mathbf{x}), \tag{32}$$

$$p(\mathbf{x}) = Q_G \mathbf{g}(\mathbf{x}) + R_G \tilde{\mathbf{u}}(\mathbf{x}) \tag{33}$$

in G it is enough to determine $\tilde{\mathbf{u}}$. Using boundary behavior of hydrodynamical potentials we get

$$\frac{1}{2}\tilde{\mathbf{u}} - \tilde{K}_G \tilde{\mathbf{u}} = E_G \mathbf{g} \quad \text{on } \partial G. \tag{34}$$

Proposition 7.1. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \ge 2$. Then $\frac{1}{2}I - \tilde K_G$ is a Fredholm operator with index 0 in $H^{1/2}(\partial G, C^m)$, $H^{1/2}(\partial G, C^m) = \mathrm{Ker}(\frac{1}{2}I - \tilde K_G) \bigoplus (\frac{1}{2}I - \tilde K_G)(H^{1/2}(\partial G, C^m))$ and $\mathrm{Ker}(\frac{1}{2}I - \tilde K_G) = C^m$. If we denote by L_G the restriction of $\tilde K_G$ onto $(\frac{1}{2}I - \tilde K_G)(H^{1/2}(\partial G, C^m))$ then $\sigma(\frac{1}{2}I - L_G) \subset (0, 1\rangle$.*

Proof. Since $\frac{1}{2}I - \tilde K_G$ and $\frac{1}{2}I - \tilde K_G'$ are adjoint operators, $\frac{1}{2}I - \tilde K_G$ is a Fredholm operator with index 0 and $\sigma(\frac{1}{2}I - \tilde K_G) \subset \langle 0, 1\rangle$ by Theorem 4.7, [15], Theorem 5.15, and [15], Theorem 6.24. According to Proposition 5.1 and [15], Chapter 3, §3.3, we have $(\frac{1}{2}I - \tilde K_G)(H^{1/2}(\partial G, C^m)) = \{\mathbf{w} \in H^{1/2}(\partial G, C^m); \langle \mathbf{\Psi}, \mathbf{w}\rangle = 0\ \forall \mathbf{\Psi} \in \mathrm{Ker}(\frac{1}{2}I - \tilde K_G')\}$ and $\mathrm{Ker}(\frac{1}{2}I - \tilde K_G) = \{\mathbf{w} \in H^{1/2}(\partial G, C^m); \langle \mathbf{\Psi}, \mathbf{w}\rangle = 0\ \forall \mathbf{\Psi} \in (\frac{1}{2}I - \tilde K_G')(H^{-1/2}(\partial G, C^m))\} = C^m$. Since $H^{-1/2}(\partial G, C^m)$ is the direct sum of $\mathrm{Ker}(\frac{1}{2}I - \tilde K_G')$ and $(\frac{1}{2}I - \tilde K_G')(H^{-1/2}(\partial G, C^m))$ we deduce $H^{1/2}(\partial G, C^m) = \mathrm{Ker}(\frac{1}{2}I - \tilde K_G) \bigoplus (\frac{1}{2}I - \tilde K_G)(H^{1/2}(\partial G, C^m))$. This forces $\sigma(\frac{1}{2}I - L_G) \subset (0, 1\rangle$.

Theorem 7.2. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \ge 2$, $\mathbf{g} \in H^{-1/2}(\partial G, R^m)$, $\int \mathbf{g} = 0$. Fix $\tilde{\mathbf{u}}_0 \in H^{1/2}(\partial G, R^m)$. For a nonnegative integer k put*

$$\tilde{\mathbf{u}}_{k+1} = [(1/2)I + \tilde K_G]\tilde{\mathbf{u}}_k + E_G \mathbf{g}. \tag{35}$$

Then there is $\tilde{\mathbf{u}} \in H^{1/2}(\partial G, R^m)$ such that $\tilde{\mathbf{u}}_k \to \tilde{\mathbf{u}}$ in $H^{1/2}(\partial G, R^m)$ as $k \to \infty$. Moreover, there are constants $0 < q < 1$, $C > 0$ depending only on G such that

$$\|\tilde{\mathbf{u}}_k - \tilde{\mathbf{u}}\|_{H^{1/2}(\partial G, R^m)} \le C q^k \left(\|\mathbf{g}\|_{H^{-1/2}(\partial G, R^m)} + \|\tilde{\mathbf{u}}_0\|_{H^{1/2}(\partial G, R^m)} \right). \tag{36}$$

The function $\tilde{\mathbf{u}}$ is a solution of the equation (34). If $\mathbf{u}$, p are given by (32), (33) in G, then $\mathbf{u}$, p is a weak solution of the problem (1), (3) and $\tilde{\mathbf{u}}$ is the trace of $\mathbf{u}$ on ∂G.

Proof. Put $T = (1/2)I + \tilde K_G$ and denote by $\tilde T$ the restriction of T onto $[(1/2)I - \tilde K_G](H^{1/2}(\partial G, C^m))$. Proposition 7.1 gives that $H^{1/2}(\partial G, R^m) = \mathrm{Ker}(I - T) \bigoplus (I - T)(H^{1/2}(\partial G, R^m))$ and $\sigma(I - \tilde T) \subset (-1, 1)$. Since $r(\tilde T) < 1$, [18], Chapter VIII, §2 gives (23). According to Theorem 5.3 there is a weak solution $\mathbf{v}$, q of the problem (1), (3). By virtue of (32), (33) and (34) we receive that $E_G \mathbf{g} \in (I - T)(H^{1/2}(\partial G, R^m))$. Proposition 5.2 gives that there is $\tilde{\mathbf{u}} \in H^{1/2}(\partial G, R^m)$ such that $\tilde{\mathbf{u}}_k \to \tilde{\mathbf{u}}$ as $k \to \infty$ in $H^{1/2}(\partial G, R^m)$ and

$$\|\tilde{\mathbf{u}}_k - \tilde{\mathbf{u}}\|_{H^{1/2}(\partial G, R^m)} \le \tilde C q^k \left(\|E_G \mathbf{g}\|_{H^{1/2}(\partial G, R^m)} + \|\tilde{\mathbf{u}}_0\|_{H^{1/2}(\partial G, R^m)} \right).$$

holds with constants $0 < q < 1$, $\tilde C > 0$ depending only on G. So, (36) holds with $C = \tilde C(1 + \|E_G\|)$.

Letting $k \to \infty$ we get that $\tilde{\mathbf{u}}$ is a solution of the equation (34). Since $\mathbf{v}$ is also a solution of the equation (34), Proposition 2.1 forces that $\mathbf{w} = \tilde{\mathbf{u}} - \mathbf{v} \in R^m$. Since $\mathbf{v}$, q is a solution of the problem (1), (3), we have $\mathbf{v} = E_G \mathbf{g} + W_G \mathbf{v}$, $q = Q_G \mathbf{g} + R_G \mathbf{v}$ in G. Since $\mathbf{v} + \mathbf{w}$, q is a solution of the problem (1), (3), we have also $\mathbf{v}+\mathbf{w} = E_G\mathbf{g} + W_G(\mathbf{v}+\mathbf{w}) = \mathbf{u}$, $q = Q_G\mathbf{g} + R_G(\mathbf{v}+\mathbf{w}) = p$ in G. Thus $\tilde{\mathbf{u}} = \mathbf{v}+\mathbf{w}$ is the trace of $\mathbf{u} = \mathbf{v}+\mathbf{w}$ on ∂G.

Proposition 7.3. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Put*

$$M\mathbf{\Psi} = \tilde{K}_G\psi - \frac{1}{c}\int\limits_{\partial G} \mathbf{\Psi}\ \mathrm{d}\mathbf{y}, \qquad c = \int\limits_{\partial G} 1\ \mathrm{d}\mathbf{y}.$$

The operator $\frac{1}{2}I - M$ is continuously invertible in $H^{1/2}(\partial G, C^m)$. If $\mathbf{f} \in H^{1/2}(\partial G, C^m)$, $\mathbf{h} \in (\frac{1}{2}I - \tilde{K}_G)(H^{1/2}(\partial G, C^m))$ and $\frac{1}{2}\mathbf{f} - M\mathbf{f} = \mathbf{h}$, then $\frac{1}{2}\mathbf{f} - \tilde{K}_G\mathbf{f} = \mathbf{h}$.

Proof. Since $\frac{1}{2}I - M'$ is continuously invertible by Proposition 5.4, its adjoint operator $\frac{1}{2}I - M$ is also continuously invertible (see [15], Theorem 6.24). We have $H^{1/2}(\partial G, C^m) = C^m \bigoplus (\frac{1}{2}I - \tilde{K}_G)(H^{1/2}(\partial G, C^m))$ by Proposition 5.1. Since $\frac{1}{2}\mathbf{f} - \tilde{K}_G\mathbf{f} \in (\frac{1}{2}I - \tilde{K}_G)(H^{1/2}(\partial G, C^m))$, $(\tilde{K}_G - M)\mathbf{f} \in C^m$ and $\mathbf{h} = [\frac{1}{2}\mathbf{f} - \tilde{K}_G\mathbf{f}] + (\tilde{K}_G - M)\mathbf{f} \in (\frac{1}{2}I - \tilde{K}_G)(H^{1/2}(\partial G, C^m))$, we infer that $(\tilde{K}_G - M)\mathbf{f} = 0$.

Theorem 7.4. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then there is an equivalent norm on $H^{1/2}(\partial G, R^m)$ such that $\|\frac{1}{2}I + M\| \leq q < 1$. Let now $\mathbf{h} \in H^{1/2}(\partial G, R^m)$. Fix $\mathbf{f}_0 \in H^{1/2}(\partial G, R^m)$. For a nonnegative integer k put*

$$\mathbf{f}_{k+1} = \left(\frac{1}{2}I + M\right)\mathbf{f}_k + \mathbf{h}.$$

Then $\mathbf{f}_k \to \mathbf{f}$ in $H^{1/2}(\partial G, R^m)$, $\frac{1}{2}\mathbf{f} - M\mathbf{f} = \mathbf{h}$ and $\|\mathbf{f} - \mathbf{f}_j\| \leq q^j[\|\mathbf{h}\| + \|\mathbf{f}_0\|]$ for arbitrary j.

Proof. Since there is an equivalent norm on $H^{-1/2}(\partial G, C^m)$ such that $\|\frac{1}{2}I + M'\| \leq q < 1$ (see Proposition 5.5), we have $\|\frac{1}{2}I + M\| = \|\frac{1}{2}I + M'\| \leq q < 1$ (see [15], Theorem 3.3). The rest is a consequence of Proposition 5.2.

8 BEM for the Stokes problem

Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary. We would like to construct a solution of the Stokes problem, i.e. $\mathbf{u} \in H^1(G, R^m)$, $p \in L^2(G)$ such that (1) holds and $\mathbf{u} = \mathbf{f}$ on ∂G, where $\mathbf{f} \in H^{1/2}(\partial G, R^m)$ be given. It is well-known that this problem is solvable if and only if

$$\int\limits_{\partial G} \mathbf{f} \cdot \mathbf{n}^G\ \mathrm{d}\mathbf{y} = 0, \tag{37}$$

a velocity $\mathbf{u}$ is unique and a pressure p is unique up to an additive constant (see [5]). Denote

$$\mathbf{g} = \partial\mathbf{u}/\partial\mathbf{n}^G - p\mathbf{n}^G. \tag{38}$$

Then

$$\mathbf{u}(\mathbf{x}) = E_G\mathbf{g}(\mathbf{x}) + W_G\mathbf{f}(\mathbf{x}), \quad p(\mathbf{x}) = Q_G\mathbf{g}(\mathbf{x}) + R_G\mathbf{f}(\mathbf{x}) \qquad \mathbf{x} \in G \tag{39}$$

by Proposition 6.1. In this section we calculate $\mathbf{g}$ by the successive approximation. Set

$$\mathbf{h} = \partial(W_G\mathbf{f})/\partial\mathbf{n}^G - (R_G\mathbf{f})\mathbf{n}^G. \tag{40}$$

(39) and Proposition 3.2 give $\mathbf{g} = \mathbf{g}/2 - \tilde{K}'_G\mathbf{g} + \mathbf{h}$, i.e

$$\left(\frac{1}{2}I + K'_G\right)\mathbf{g} = \mathbf{h}. \tag{41}$$

Proposition 8.1. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary. Then $\frac{1}{2}I + \tilde{K}'_G$ is a Fredholm operator with index 0 in the space $H^{-1/2}(\partial G, C^m)$.*

Proof. Fix $R > 0$ such that $\overline{G} \subset B(0; R)$ and put $\Omega = B(0; R)\setminus\overline{G}$. According to Proposition 4.6 there exists a closed subspace Y of $H^{-1/2}(\partial\Omega, C^m)$ with finite codimension such that $\frac{1}{2}I - \tilde{K}'_\Omega$ is a continuously invertible operator from Y onto a Banach space $(\frac{1}{2}I - \tilde{K}'_\Omega)(Y)$. Thus $\frac{1}{2}I - \tilde{K}'_\Omega$ is an upper semi-Fredholm operator on $H^{-1/2}(\partial\Omega, C^m)$ by [13], §16, Theorem 8. Moreover, $\frac{1}{2}I - \tilde{K}'_\Omega$ is an upper semi-Fredholm operator from $H^{-1/2}(\partial G, C^m)$ to $H^{-1/2}(\partial\Omega, C^m)$ by [13], §16, Theorem 10. If $\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m)$, then $(\frac{1}{2}I - \tilde{K}'_\Omega)\mathbf{\Psi} - (\frac{1}{2}I + \tilde{K}'_G)\mathbf{\Psi} = [\partial(E_G\mathbf{\Psi})/\partial\mathbf{n} - (R_G\mathbf{\Psi})\mathbf{n}]|\partial B(0; R)$ and thus $(\frac{1}{2}I - \tilde{K}'_\Omega) - (\frac{1}{2}I + \tilde{K}'_G)$ is a compact linear operator from $H^{-1/2}(\partial G, C^m)$ to $H^{-1/2}(\partial\Omega, C^m)$. So, $\frac{1}{2}I + \tilde{K}'_G$ is an upper semi-Fredholm operator from $H^{-1/2}(\partial G, C^m)$ to $H^{-1/2}(\partial\Omega, C^m)$ by [13], §16, Theorem 16. Clearly, $\frac{1}{2}I + \tilde{K}'_G$ is an upper semi-Fredholm operator on $H^{-1/2}(\partial G, C^m)$. If $\lambda > 1/2$ then $\lambda I + \tilde{K}'_G$ is a Fredholm operator with index 0 on $H^{-1/2}(\partial G, C^m)$ by Proposition 4.7. Since the index is constant on each component of semi-Fredholmness by [13], §18, Corollary 3, we infer that $\frac{1}{2}I + \tilde{K}'_G$ is a Fredholm operator with index 0 on $H^{-1/2}(\partial G, C^m)$.

Proposition 8.2. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary. Then* $\mathrm{Ker}[(1/2)I + \tilde{K}'_G] = \{c\mathbf{n}^G; c \in R\}$ *and* $H^{-1/2}(\partial G, R^m) = [(1/2)I + \tilde{K}'_G](H^{-1/2}(\partial G, R^m)) \bigoplus \mathrm{Ker}[(1/2)I + \tilde{K}'_G]$.

Proof. $\mathbf{n}^G \in \mathrm{Ker}[(1/2)I + \tilde{K}'_G]$ by Lemma 4.4..

Suppose that $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$, $[(1/2)I + \tilde{K}'_G]\mathbf{\Psi} = c\mathbf{n}^G$, $c \in R$. Then $\int \mathbf{\Psi} = \int\{c\mathbf{n}^G + [(1/2)I - \tilde{K}'_G]\mathbf{\Psi}\} = 0$ by Proposition 5.1. Put $\Omega = R^m \setminus \mathrm{cl}\, G$. According to (37) and Proposition 4.1

$$0 = \int_{\partial G} c\mathbf{n}^G \cdot E_G\mathbf{\Psi}\, \mathrm{d}\mathbf{y} = \int_{\partial G} \{[(1/2)I - \tilde{K}'_\Omega]\mathbf{\Psi}\} \cdot E_G\mathbf{\Psi}\, \mathrm{d}\mathbf{y} = \int_{\Omega} |\nabla E_G\mathbf{\Psi}|^2\, \mathrm{d}\mathbf{y}.$$

So, $E_G\mathbf{\Psi}$ is constant in Ω. Since $E_G\mathbf{\Psi}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, we infer that $E_G\mathbf{\Psi} = 0$ in Ω. Lemma 3.4 gives that there exists $\alpha \in R$ such that $\mathbf{\Psi} = \alpha\mathbf{n}^G$. Thus $c = 0$, $\mathrm{Ker}[(1/2)I + \tilde{K}'_G] = \{\beta\mathbf{n}^G; \beta \in R\}$ and $\{\beta\mathbf{n}^G; \beta \in R\} \cap [(1/2)I + \tilde{K}'_G](H^{-1/2}(\partial G, R^m)) = \{0\}$. Since $\dim \mathrm{Ker}[(1/2)I + \tilde{K}'_G] = 1$ and

$(1/2)I+\tilde{K}'_G$ is a Fredholm operator with index 0 by Proposition 8.1, we deduce that $H^{-1/2}(\partial G, R^m) = [(1/2)I + \tilde{K}'_G](H^{-1/2}(\partial G, R^m)) \bigoplus \mathrm{Ker}[(1/2)I + \tilde{K}'_G]$.

Proposition 8.3. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary. Define*

$$\hat{M}\mathbf{\Psi} = \frac{1}{2}\mathbf{\Psi} + \tilde{K}'_G\mathbf{\Psi} + \frac{1}{c}\mathbf{n}^G \int\limits_{\partial G} \mathbf{n}^G \cdot \mathbf{\Psi}, \qquad c = \int\limits_{\partial\Omega} 1 \ \mathrm{d}\mathbf{y}. \tag{42}$$

Then $\hat{M}$ is a continuously invertible operator in $H^{-1/2}(\partial G, R^m)$.

Proof. Let $\hat{M}\mathbf{\Psi} = 0$. Since $H^{-1/2}(\partial G, R^m) = [(1/2)I+\tilde{K}'_G](H^{-1/2}(\partial G, R^m)) \bigoplus \{\alpha\mathbf{n}^G; \alpha \in R\}$ by Proposition 8.2, we have $[(1/2)I + \tilde{K}'_G]\mathbf{\Psi} = 0$. According Proposition 8.2 there exists $\alpha \in R$ such that $\mathbf{\Psi} = \alpha\mathbf{n}^G$. Thus $0 = \hat{M}\mathbf{\Psi} = \alpha\mathbf{n}^G$ and $\alpha = 0$. The operator $\hat{M}$ is one to one. $\hat{M}$ is a finite dimensional perturbation of the Fredholm operator with index 0 by Proposition 8.1. So, $\hat{M}$ is a Fredholm operator with index 0 (see [13], §16, Theorem 16). Since the operator $\hat{M}$ is injective, it is continuously invertible (see [3], Theorem 1.42).

Theorem 8.4. *Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $m \geq 2$. Then there is an equivalent norm on $H^{-1/2}(\partial G, R^m)$ such that $\|I - \hat{M}\| \leq q < 1$. Let $\mathbf{f} \in H^{1/2}(\partial G; R^m)$, $\int \mathbf{f} \cdot \mathbf{n}^G \ \mathrm{d}\mathbf{y} = 0$. Let $\mathbf{h}$ be given by (40). Fix $\mathbf{g}_0 \in H^{-1/2}(\partial G, R^m)$. For a nonnegative integer k put*

$$\mathbf{g}_{k+1} = (I - \hat{M})\mathbf{g}_k + \mathbf{h}.$$

Then $\mathbf{g}_k \to \mathbf{g}$ in $H^{-1/2}(\partial G, R^m)$, $\hat{M}\mathbf{g} = [(1/2)I + \tilde{K}'_G]\mathbf{g} = \mathbf{h}$ and $\|\mathbf{g} - \mathbf{g}_j\| \leq q^j[\|\mathbf{h}\| + \|\mathbf{g}_0\|]$ for arbitrary j. If $\mathbf{u}$, p are given by (39) then $\mathbf{u}$, p is a solution of the Stokes problem with the boundary condition $\mathbf{f}$ and $\mathbf{g} = \partial\mathbf{u}/\partial\mathbf{n}^G - p\mathbf{n}^G$.

Proof. First we show that $\sigma(I - \hat{M}) \subset \langle 0, 1)$. Let $\lambda \in \sigma(I - \hat{M})$. If λ is not an eigenvalue then $\lambda \in (0, 1)$ by Theorem 4.7, Theorem 8.1 and [13], §16, Theorem 16. Let now λ be an eigenvalue with an eigenfunction $\mathbf{\Psi}$. Since $H^{-1/2}(\partial G, C^m) = [(1/2)I + \tilde{K}'_G](H^{-1/2}(\partial G, C^m)) \bigoplus \{\alpha\mathbf{n}^G; \alpha \in R\}$ by Proposition 8.2, there exist $\mathbf{\Phi} \in [(1/2)I + \tilde{K}'_G](H^{-1/2}(\partial G, C^m))$ and $\alpha \in C$ such that $\mathbf{\Psi} = \mathbf{\Phi} + \alpha\mathbf{n}^G$. Proposition 8.2 gives $[(1/2)I + \tilde{K}'_G]\mathbf{n}^G = 0$ and thus $0 = \lambda\mathbf{\Psi} - (I - \hat{M})\mathbf{\Psi} = [(1/2)I + \tilde{K}'_G]\mathbf{\Phi} + (\lambda - 1)\mathbf{\Phi} + \beta\mathbf{n}^G$ with $\beta \in C$. Hence $[(1/2)I + \tilde{K}'_G]\mathbf{\Phi} = (1 - \lambda)\mathbf{\Phi}$. If $\mathbf{\Phi} \neq 0$ then $0 \leq \lambda \leq 1$ by Proposition 4.5. If $\mathbf{\Phi} = 0$ then $\alpha \neq 0$ and $0 = \lambda\mathbf{\Psi} - (I - \hat{M})\mathbf{\Psi} = \alpha\lambda\mathbf{n}^G$ and $\lambda = 0$. This and Proposition 8.3 gives that $\sigma(I - \hat{M}) \subset \langle 0, 1)$. Thus $r(I - \hat{M}) < 1$. If we fix $r(I - \hat{M}) < q < 1$ then there exists an equivalent norm $\| \cdot \|$ on $H^{-1/2}(\partial G, C^m)$ such that $\|I - \hat{M}\| \leq q$ (see [6]).

Since $\int \mathbf{f} \cdot \mathbf{n}^G \ \mathrm{d}\mathbf{y} = 0$ there exist $\mathbf{u} \in H^1(G, R^m)$, $\tilde{p} \in L^2(G, R^m)$ solving the Stokes system in G such that $\mathbf{v} = \mathbf{f}$ on ∂G. Fix $\alpha \in R$. Put $p = \tilde{p} + \alpha$. Then $\mathbf{u}$, p is a solution of the Stokes problem with the boundary condition $\mathbf{f}$. Put $\mathbf{g} = \partial\mathbf{u}/\partial\mathbf{n} - p\mathbf{n}^G$. We can choose α in a such way that $\int \mathbf{g} \cdot \mathbf{n}^G = 0$. Then $\hat{M}\mathbf{g} = [(1/2)I + K'_G]\mathbf{g} = \mathbf{h}$. The rest is a consequence of Proposition 8.3 and Proposition 5.2.

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References

[1] Chang T.K., Pahk, D. H: Spectral properties for layer potentials associated to the Stokes equation in Lipschitz domains. Manuscripta Math. 130, 359–373 (2009)

[2] Dahlberg, B. E. J., Kenig, C., Verchota, G. C.: Boundary value problems for the systems of elastics in Lipschitz domains. Duke Math. J. 57, 795–818 (1988)

[3] Douglas, R. G.: Banach algebra techniques in operator theory. Academic Press, New York and London (1972)

[4] Fabes, E. B., Kenig, C. E., Verchota, G.C.: The Dirichlet problem for the Stokes system on Lipschitz domains. Duke Math. J. 57, 769–793 (1988)

[5] Galdi G. P.: An introduction to the Mathematical Theory of the Navier-Stokes Equations. Springer, Belin–Heidelberg–New York (1998)

[6] Gohberg, I., Marcus, A.: Some remarks on topologically equivalent norms. Izvestia Mold. Fil. Akad. Nauk SSSR 10(76), 91–95 (1960) (in Russian)

[7] Hsiao, G. C., Wendland, W.L.: Boundary Integral Equations. Springer, Heidelberg (2008)

[8] Kohr, M.: Boundary value problems for a compressible Stokes system in bounded domains in R^n. J. Comput. Appl. Math. 201, 128–145 (2007)

[9] Maz'ya, V., Mitrea, M., Shaposhnikova, T.: The inhomogeneous Dirichlet problem for the Stokes system in Lipschitz domains with unital close to VMO. Funct. Anal. Appl. 43, 217–235 (2009)

[10] Maz'ya, V.G., Poborchi, S. V.: Differentiable Functions on Bad Domains. World Scientific, Singapore (1997)

[11] Medková, D.: Integral representation of a solution of the Neumann problem for the Stokes system. Numer. Algorithms 54, 459–484 (2010)

[12] Medková, D.: Convergence of the Neumann series in BEM for the Neumann problem of the Stokes system. Acta Appl. Math. 116, 281–304 (2011)

[13] Müller, V.: Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras. Operator Theory Advances and Applications. Birkhäuser, Basel (2007)

[14] Pozrikidis, C.: Boundary integral and singularity methods for linearized viscous flow. Cambridge University Press, New York (1992)

[15] Schechter, M.: Principles of Functional Analysis. American Mathematical Society, Providence - Rhode Island (2002)

[16] Steinbach, O.: Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer, New York (2008)

[17] Verchota, G.: Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. J. Funct. Anal. 59, 572–611 (1984)

[18] Yosida, K.: Functional Analysis. Springer, Berlin (1965)

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