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Amiran Gogatishvili
Ron Kerman

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A NEW GAUGE FUNCTIONAL CHARACTERIZING A GIVEN ORLICZ CLASS

AMIRAN GOGATISHVILI AND RON KERMAN

Abstract. We define a new gauge functional characterizing a given Orlicz class. This functional is shown to make more computable a formula for the dual of a $\mathcal{K}$-method interpolation space.

1. Introduction

Suppose A is a Young function defined by the formula

$$A(t) := \int_0^t a(s)ds, \;\; t \in \mathbb{R}_+ := (0, \infty),$$

in which $a(s)$ is an increasing function on $\mathbb{R}_+$, with $a(0+) = 0$ and $\lim_{s\to\infty} a(s) = \infty$.

Let (X, μ) be a σ-finite measure space and denote by $\mathfrak{M}(X)$ the set of μ-measurable functions on X. A function $f \in \mathfrak{M}(X)$ is said to belong to the Orlicz class $L_A(X)$ if

$$\int_X A\left(\frac{|f(x)|}{\lambda_f}\right) d\mu(x) < \infty,$$

for some $\lambda_f > 0$. The gauge norm, $\rho_A(f)$, of an $f \in L_A(X)$ is

$$\rho_A(f) := \inf\left\{\lambda > 0 : \quad \int_X A\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\}. \tag{1.1}$$

See [5, p.97] for the interesting history of the functional (1.1) that justifies the introduction of the term "gauge norm".

With the norms ρ_A in mind, we speak of the Orlicz *spaces* $L_A(X)$. These are examples of rearrangement invariant (r.i) spaces, which are defined by norms ρ whose characteristic property is that $\rho(f) = \rho(g)$ whenever $f, g \in \mathfrak{M}(X)$ are equimeasurable in the sense that $f^* = g^*$; here,

$$f^*(t) := \inf\{\lambda > 0 : \mu(\{x \in X : |f(x)| > \lambda\}) \le t\},$$

$t \in I_\mu := (0, \mu(X))$.

We are now ready to state our principal result, namely,

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Theorem 1.1. *Let $A(t) = \int_0^t a(s)ds$, $t \in \mathbb{R}_+$, be a Young function with $a(s)$ absolutely continuous. Define*

$$c(t) := t\frac{d}{dt}\left(\frac{A(t)}{t}\right) = a(t) - \frac{A(t)}{t} = \frac{1}{t}\int_0^t sa'(s)ds$$

and set

$$C(t) := \int_0^t c(s)ds, \quad t \in \mathbb{R}_+.$$

Let (X, μ) be a σ-finite measure space and suppose the (increasing) function C satisfies

$$\int_{\mathbb{R}_+} C\left(\frac{k}{1+t}\right) dt < \infty, \tag{1.2}$$

for some $k > 0$. Then,

$$\frac{1}{2}\rho_{\Gamma_C}(f) \le \rho_A(f) \le \rho_{\Gamma_C}(f), \ \ f \in \mathfrak{M}(X), \tag{1.3}$$

in which

$$\rho_{\Gamma_C}(f) := \inf\left\{\lambda > 0 : \int_{\mathbb{R}_+} C\left(t^{-1}\int_0^t f^*(s)ds\right) dt \le 1\right\}.$$

Remarks 1.2

1. We observe that $A(t) = \int_0^t a(s)ds$ and $\mathbf{A}(t) = \int_0^t \frac{A(s)}{s}ds$ give rise to the same Orlicz class, so there is no essential loss of generality in the assumption of Theorem 1.1 that $a(s)$ is absolutely continuous.
2. When $\mu(X) < \infty$, we may take $a(s)$, and hence $c(s)$, equal to 0 on $(0, 1)$. In this case (1.2) is automatically true, so (1.3) holds with no essential restrictions.

The result of applying (1.3) to the representation of norms dual to the $\mathcal{K}$-method interpolation norms requires some background to even state, so we postpone it to (the last) section 4.

In Section 2 we consider r.i. spaces with special emphasis on the Orlicz case.

Section 3 contains the proof of Theorem 1.1 along with a remark and an example.

2. Rearrangement invariant spaces

Let (X, μ) be a σ-finite measure space. Denote by $\mathfrak{M}(X)$ the set of μ-measurable real-valued functions on X and by $\mathfrak{M}_+(x)$ the nonnegative functions in $\mathfrak{M}(X)$. A Banach function norm is a functional $\rho : \mathfrak{M}_+(X) \to \mathbb{R}_+$ satisfying

(A1) $\rho(f) = 0$ if and only if $f = 0$ μ- a.e.,
(A2) $\rho(cf) = c\rho(f)$, $c \ge 0$,
(A3) $\rho(f + g) \le= \rho(f) + \rho(g)$,
(A4) $0 \le f_n \uparrow f$ implies $\rho(f_n) \uparrow \rho(f)$,
(A5) $|E| < \infty$ implies $\rho(\chi_E) < \infty$,
(A6) $|E| < \infty$ implies $\int_E f d\mu \le c_E(\rho)\rho(f)$, for some constant $c_E(\rho)$ depending on E and ρ but not on $f \in \mathfrak{M}_+(X)$.

Furthermore, as mentioned in the introduction, a Banach function norm is said to be rearrangement invariant if $\rho(f)=\rho(g)$ whenever $f,g\in\mathfrak{M}_+(X)$ are equimeasurable in the sense that $f^*=g^*$; the nonincreasing rearrangement, f^*, of $f\in\mathfrak{M}(X)$ on $\mathbb{R}_+$ is defined as

$$f^*(t):=\inf\{\lambda>0:\mu(\{x\in X:|f(x)|>\lambda\})\le t\},$$

$t\in I_\mu:=(0,\mu(X))$.

It satisfies the property

$$|\{t\in\mathbb{R}_+:f^*(t)>\tau\})|=\mu(\{x\in X:|f(x)|>\tau\}),\ f\in\mathfrak{M}(X),\ \tau\in I_\mu.$$

Now, although the mapping $f\mapsto f^*$ is not subadditive, the mapping $f\mapsto t^{-1}\int_0^t f^*(s)ds$ is, namely

$$t^{-1}\int_0^t (f+g)^*(s)ds\le t^{-1}\int_0^t f^*(s)ds+t^{-1}\int_0^t g^*(s)ds, \tag{2.1}$$

for all $f,g\in\mathfrak{M}(X)$, $t\in\mathbb{R}_+$. The Kothe dual of a Banach function norm ρ is another such norm, ρ', with

$$\rho'(g):=\sup_{\rho(f)\le 1}\int_X fg\mu,\quad f,g\in\mathfrak{M}_+(X).$$

It is obeys the Principle of Duality; that is,

$$\rho'':=(\rho')'=\rho.$$

The space $L_\rho(X)$ is the vector space

$$\{f\in\mathfrak{M}(X):\rho(|f|)<\infty\},$$

together with the norm

$$\|f\|_{L_\rho}:=\rho(|f|).$$

This Banach space is said to be an r.i. space provided ρ is an r.i. function norm.

The gauge norm, ρ_A, defined in (1.2) in terms of the Young function $A(t)=\int_0^t a(s)ds$, $t\in\mathbb{R}_+$, is an r.i. norm; indeed,

$$\rho_A(f)=\inf\{\lambda>0:\int_{I_\mu}A\left(\frac{f^*(t)}{\lambda}\right)dt\le 1\},\quad f\in\mathfrak{M}(X).$$

Its Kothe dual, ρ_A', satisfies

$$\rho_{\widetilde{A}}(g)\le\rho_A'(g)\le 2\rho_{\widetilde{A}}(g),\quad g\in\mathfrak{M}(X),$$

with

$$\widetilde{A}(t):=\int_0^t a^{-1}(s)ds,\quad t\in\mathbb{R}_+,$$

being called the Young function complementary to A.

3. Proof of Theorem 1.1

We will require the following inequalities, which are analogues of ones for the Hardy-Littlewood maximal function, Mf, in [6, pp.6-7 and p.27]. Namely, for all $\tau > 0$

$$\frac{1}{\tau}\int_{\{t\in I_\mu : f^*(t)>\tau\}} f^*(t)dt \le |\{t\in I_\mu : (Pf^*)(t)>\tau\}| \le \frac{2}{\tau}\int_{\{t\in I_\mu : f^*(t)>\frac{\tau}{2}\}} f^*(t)dt. \tag{3.1}$$

Their proofs are even simpler than the ones for Mf. Thus, let t_0 be the least t for which $(Pf^*)(t) = \tau$. (The inequalities are trivial if there is no such t_0). Then,

$$|\{t\in I_\mu : (Pf^*)(t)>\tau\}| = t_0 = \frac{1}{\tau}\int_0^{t_0} f^*(t)dt \ge \frac{1}{\tau}\int_{\{t\in I_\mu : f^*(t)>\tau\}} f^*(t)dt.$$

Again, defining

$$f_\tau(t) := \min\,[f^*(t), \frac{\tau}{2}]$$

and

$$f^\tau(t) := f^*(t) - f_\tau(t),$$

one has

$$\begin{aligned}|\{t\in I_\mu : (Pf^*)(t)>\tau\}| &\le |\{t\in I_\mu : (Pf^\tau)(t)>\frac{\tau}{2}\}|\\ &\le \frac{2}{\tau}\int_{I\mu} f^\tau(t)dt\\ &\le \frac{2}{\tau}\int_{\{t\in I_\mu : f^*(t)>\frac{\tau}{2}\}} f^\tau(t)dt\\ &\le \frac{2}{\tau}\int_{\{t\in I_\mu : f^*(t)>\frac{\tau}{2}\}} f^*(t)dt.\end{aligned}$$

Next, we observe that

$$A(t) = t\int_0^t c(s)\frac{ds}{s}.$$

Now, the first inequality in (3.1) ensures that for all $\lambda > 0$,

$$\int_{\mathbb{R}_+}\int_{\{t\in I_\mu : f^*(t)>\tau\}} f^*(t)dtc(\tau)\frac{d\tau}{\tau} \le \int_{\mathbb{R}_+} |\{t\in I_\mu : (Pf^*)(t)>\tau\}|c(\tau)d\tau;$$

that is,

$$\begin{aligned}\int_{I_\mu} A\left(\frac{f^*(t)}{\lambda}\right)dt &= \int_{I_\mu} \frac{f^*(t)}{\lambda}\int_0^{\frac{f^*(t)}{\lambda}} c(\tau)\frac{d\tau}{\tau}dt \\ &= \int_{\mathbb{R}_+}\int_{\{t\in I_\mu: f^*(t)>\tau\}} \frac{f^*(t)}{\lambda}dtc(\tau)\frac{d\tau}{\tau} \\ &\le \int_{\mathbb{R}_+} |\{t\in I_\mu : \frac{(Pf^*)(t)}{\lambda} > \tau\}|c(\tau)d\tau \\ &= \int_{I_\mu} C\left(\frac{(Pf^*)(t)}{\lambda}\right)dt.\end{aligned}$$

Again, the second inequality in (3.1) yields

$$\begin{aligned}\int_{I_\mu} C\left(\frac{(Pf^*)(t)}{\lambda}\right)dt &= \int_{\mathbb{R}_+} |\{t\in I_\mu : \frac{(Pf^*)(t)}{\lambda} > \tau\}|c(\tau)d\tau \\ &\le 2\int_{\mathbb{R}_+}\int_{\{t\in I_\mu: \frac{f^*(t)}{\lambda}>\frac{\tau}{2}\}} \frac{f^*(t)}{\lambda}dtc(\tau)\frac{d\tau}{\tau} \\ &= \int_{I_\mu} \frac{2f^*(t)}{\lambda}\int_0^{\frac{2f^*(t)}{\lambda}} c(\tau)\frac{d\tau}{\tau}dt \\ &= \int_{I_\mu} A\left(\frac{2f^*(t)}{\lambda}\right)dt.\end{aligned}$$

We conclude

$$\frac{1}{2}\rho_{\Gamma_C}(f) \le \rho_{A}(f^*) \le \rho_{\Gamma_C}(f),$$

which completes the proof of Theorem 1.1, since $\rho_A(f) = \rho_A(f^*)$. □

Corollary 3.1. *Let $A(t) = \int_0^t a(s)ds$, $t\in\mathbb{R}_+$, be a Young function, for which*

$$\int_{\mathbb{R}_+} A\left(\frac{k}{1+t}\right)dt < \infty,$$

for some constant $k>0$, so, in particular,

$$\int_0^t a(s)\frac{ds}{s} < \infty, \quad t\in\mathbb{R}_+.$$

Set

$$\mathcal{A}(t) := t\int_0^t a(s)\frac{ds}{s}, \quad t\in\mathbb{R}_+. \tag{3.2}$$

Then, $\mathcal{A}(t)$ is a Young function such that

$$a(t) = \mathcal{A}'(t) - \frac{\mathcal{A}(t)}{t}, \quad t\in\mathbb{R}_+,$$

whence, for any σ-finite measure space (X,μ),

$$\frac{1}{2}\rho_{\Gamma_{\mathcal{A}}}(f) \le \rho_{\mathcal{A}}(f) \le \rho_{\Gamma_{\mathcal{A}}}(f), \quad f \in \mathfrak{M}(X).$$

Remark 3.2. The complementary Young function, $\widetilde{\mathcal{A}}$, of $\mathcal{A}$ satisfies

$$\frac{3}{2}\int_0^{\frac{t}{3}} b^{-1}(s)ds \le \widetilde{\mathcal{A}}(t) \le \int_0^t b^{-1}(s)ds,$$

where

$$b(t) := \int_0^t a(s)\frac{ds}{s}, \quad t \in \mathbb{R}_+.$$

For, observe that

$$\begin{aligned} b(t) \le \mathcal{A}'(t) &= \int_0^t a(s)\frac{ds}{s} + a(t) \\ &\le \frac{\ln 2 + 1}{\ln 2}\int_0^{2t} a(s)\frac{ds}{s} \\ &\le 3b(2t), \end{aligned}$$

and so

$$\frac{1}{2}b^{-1}\left(\frac{t}{3}\right) \le \left(\mathcal{A}'\right)^{-1}(t) \le b^{-1}(t), \quad t \in \mathbb{R}_+.$$

Example 3.3. Consider the Young function

$$A(t) = \int_0^t \ln^\beta(1+s)ds, \quad 0<\beta<1, \quad t \in \mathbb{R}_+.$$

Then,

$$c(t) = \frac{\beta}{t}\int_0^t \frac{s}{1+s}\ln^{\beta-1}(1+s)ds \sim \beta\ln^{\beta-1}(1+t), \quad \text{as} \quad t \to \infty,$$

from which we see that $c(t)$ essentially decreases rather than increases. That is, C is *not* convex.

4. An application to interpolation theory

Given Banach spaces X_1 and X_2 imbedded in a common Hausdorff topological vector space, $\mathcal{H}$, the $\mathcal{K}$-method of interpolation provides a concrete way to construct new Banach spaces X which lie between them, in the sense that, for any linear operator T satisfying $T : X_i \to X_i$, $i = 1, 2$, one has $T : X \to X$.

The key element in the method is the Peetre K-functional defined at $x \in X_1 + X_2$ and $t \in \mathbb{R}_+$ by

$$K(t,x;X_1,X_2) := \inf_{x=x_1+x_2}\left[\|x_1\|_{X_1} + t\|x_2\|_{X_2}\right].$$

For our purposes, each of the so-called interpolation spaces, X, will correspond to an r.i. norm ρ on $\mathfrak{M}_+(\mathbb{R}_+)$, with $\rho\left(\frac{1}{1+t}\right) < \infty$; more specifically, the norm of X is defined as

$$\|x\|_X := \rho\left(\frac{K(t,x;X_1,X_2)}{t}\right), \quad x \in X_1 + X_2.$$

The following is a special case of a result proved in [3, Theorem 7.2] for X_1 and X_2 r.i. spaces and $\rho = \rho_A$ an Orlicz norm. It elaborates, in a particular instance, the deep duality theorem of Brudnyi and Krugljak [2].

Theorem 4.1. *Let (X,μ) be a σ-finite measure space and suppose ρ_1 and ρ_2 are r.i. norms on $\mathfrak{M}_+(X)$. Assume, further, that*

$$L_{\rho_1'}(X) \cap L_{\rho_2'}(X) \quad \text{is dense in} \quad L_{\rho_2'}(X)$$

and

$$\rho_2'(\chi_{E_k}) \downarrow 0 \quad \text{as} \quad E_k \downarrow \emptyset, \quad E_k \subset X.$$

Consider a Young function $A(t) = \int_0^t a(s)ds$, $t \in \mathbb{R}_+$, satisfying

$$\int_{\mathbb{R}_+} A\left(\frac{k}{1+t}\right) dt < \infty,$$

for some constant $k > 0$. Then, the functional

$$\rho(f) := \rho_A\left(\frac{K(t,f;L_{\rho_1}(X),L_{\rho_2}(X))}{t}\right), \quad f \in L_{\rho_1}(X) + L_{\rho_2}(X),$$

is an r.i. norm on $\mathfrak{M}_+(X)$ and the r.i. space, $L_\rho(X)$, to which it gives rise is an interpolation space between $L_{\rho_1}(X)$ and $L_{\rho_2}(X)$.

Moreover, if, in addition,

$$\int_{\mathbb{R}_+} \widetilde{A}\left(\frac{k}{1+t}\right) dt < \infty,$$

for some constant $k > 0$, and if $\mathcal{A}$ is the Young function defined in (3.2) and $\widetilde{\mathcal{A}}$ is its complementary function, one has

$$\rho'(g) \approx \rho_{\widetilde{\mathcal{A}}}\left(\frac{d}{dt}K(t,g;L_{\rho_2'}(X),L_{\rho_1'}(X))\right), g \in L_{\rho_2'}(X) + L_{\rho_1'}(X).$$

Now $\frac{d}{dt}K(t,x,X_1,X_2)$ can be computed only in the case when the K-functional is known exactly. More often, the latter is only known to within constant multiples. The motivation behind Theorem 1.1 is the following consequence of Theorem 4.1. A version of this result involving further assumptions on the Young function A is given in [3, Theorem 8.2].

Theorem 4.2. *Let X, ρ_1, ρ_2, A, ρ and $\mathcal{A}$ be as in Theorem 4.1, with $a(t)$ absolutely continuous. Define the the increasing function C by*

$$C(t) := \int_0^t c(s)ds,$$

in which

$$c(t) := \widetilde{\mathcal{A}}'(t) - \frac{\widetilde{\mathcal{A}}(t)}{t} = \frac{1}{t}\int_0^t s\widetilde{\mathcal{A}}''(s)ds, \quad t \in \mathbb{R}_+.$$

Then, provided

$$\int_{\mathbb{R}_+} C\left(\frac{k}{1+t}\right) dt < \infty,$$

for some constant $k > 0$, one has

$$\rho'(g) \approx \inf\left\{\lambda > 0 : \int_{\mathbb{R}_+} C\left(\frac{K(t, g; L_{\rho_2'}(X), L_{\rho_1'}(X))}{\lambda t}\right) dt \leq 1\right\},$$

$g \in L_{\rho_2'}(X) + L_{\rho_1'}(X)$. In particular, $g \in L_{\rho_2'}(X) + L_{\rho_1'}(X)$ belongs, to $L_{\rho'}(X)$ if and only if there exists a constant $\lambda_g \in \mathbb{R}_+$ such that

$$\int_{\mathbb{R}_+} C\left(\frac{K(t, g; L_{\rho_1}(X), L_{\rho_2}(X))}{\lambda_g t}\right) dt < \infty.$$

Remark 4.3. Theorem 4.2 is essential to the characterization of the optimal r.i. imbedding space of an Orlicz-Sobolev space found in [4, Theorem 6.3].

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(Amiran Gogatishvili) Institute of Mathematics, Academy of Science of the Czech Republic, Žitná 25, 11567 Prague 1, Czech Republic

E-mail address: gogatish@math.cas.cz

(Ron Kerman) Department of Mathematics, Brock University, 500 Glendridge Ave. St. Catharines, Ontario, Canada L2S 3A1

E-mail address: rkerman@brocku.ca