Wave equations, examples and qualitative properties

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Contents

1 Introduction

There is a large number of real world phenomena that fits in the category of wave motion. We start with a simple example of *transport equation*

$$(\partial_t + c\partial_x)[u] = 0. \tag{1.1}$$

Obviously, the solutions of (1.1) also satisfy

$$\partial_{t,t}u(t,x) - c^2\partial_{x,x}u(t,x) = (\partial_t - c\partial_x) \circ (\partial_t + c\partial_x)[u] = 0. \tag{1.2}$$

Equation (1.2) is a simple example of *wave equation*; it may be used as a model of an infinite elastic string, propagation of sound waves in a linear medium, among other numerous applications. We shall discuss the basic properties of solutions to the wave equation (1.2), as well as its multidimensional and non-linear variants. To begin, we remark that (1.2) falls in the category of *hyperbolic equations*, in accordance with the form of its principal part in the frequency (Fourier) variables

$$\partial_{t,t}u(t,x) - c^2\partial_{x,x}u = \mathcal{F}^{-1}_{(\xi_0,\xi_1)\to(t,x)}\left[(\xi_0^2 - \xi_1^2)\mathcal{F}_{(t,x)\to(\xi_0,\xi_1)}[u]\right],$$

where $\mathcal{F}$ denotes the standard Fourier transform.

2 1-D linear wave equation

Writing

$$\begin{aligned}&\partial_{t,t}u(t,x) - c^2\partial_{x,x}u(t,x)\\ &= (\partial_t - c\partial_x) \circ (\partial_t + c\partial_x)[u] = (\partial_t + c\partial_x) \circ (\partial_t - c\partial_x)[u],\end{aligned}$$

we easily observe that solutions of (1.2) can be written in the form

$$u(t,x) = v(x+ct) + w(x-ct),\ t \in R,\ x \in R. \tag{2.1}$$

The general formula (2.1) yields solutions of (1.2) defined for both positive and *negative* values of the time t. The processes described by means of the wave equations like (1.2) are perfectly time reversible.

2.1 Uniqueness, finite speed of propagation

Multiplying the operator in (1.2) by u we obtain

$$\partial_t \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big) - c^2\partial_x\left(\partial_x u \partial_t u\right) = 0, \tag{2.2}$$

where the quantity

$$E = \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big)$$

represents the *energy*. Given an interval $[a,b] \subset R^1$ we may integrate (2.2) over the cone

$$C_{a,b,\tau} = \Big\{ t \in (0,\tau),\ x \in R^1 \ \Big|\ 0 < t < \tau,\ x \in (a+ct, b-ct) \Big\},$$

and use the Gauss-Green theorem to obtain

$$\int_{a+\tau c}^{b-\tau c} \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big)\ \mathrm{d}x = \int_a^b \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big)\ \mathrm{d}x \tag{2.3}$$

$$-\frac{1}{\sqrt{1+c^2}}\int_0^\tau \frac{c}{2}\Big(|\partial_t u(t,a+ct)|^2 + c^2|\partial_x u(t,a+ct)|^2\Big) + c^2\left(\partial_x u\partial_t u\right)(t,a+ct)\ \mathrm{d}t$$

$$-\frac{1}{\sqrt{1+c^2}}\int_0^\tau \frac{c}{2}\Big(|\partial_t u(t,b-ct)|^2 + c^2|\partial_x u(t,b-ct)|^2\Big) - c^2\left(\partial_x u\partial_t u\right)(t,b-ct)\ \mathrm{d}t$$

$$\leq \int_a^b \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big)\ \mathrm{d}x.$$

Thus the values of the solution in the wave cone $C_{a,b,\tau}$ are uniquely determined by the value of the "initial data" in terms of $\partial_t u$ and $\partial_x u$ at the initial time $t = 0$. The solutions of the wave equation (1.2) admit a finite speed of propagation $c > 0$. This is a characteristic feature of all hyperbolic problems, meaning the solutions propagate along characteristic curves (lines).

2.2 D'Alembert solution operator

As we have observed in the previous discussion, the solutions of the wave equation (1.2) are

- given by the formula (2.1),
- uniquely determined by u and $\partial_t u$ at the initial time $t = 0$.

Consequently, in terms of the functions v, w introduced in (2.1),

$$u(0,x) = u_0(x) = v(x) + w(x),$$

$$\partial_t u(0,x) = u_1(x) = cv'(x) - cw'(x);$$

whence, going back to (2.1), we deduce the so-called *D'Alembert solution formula*:

$$u(t,x) = \frac{1}{2}\Big[u_0(x+ct) + u_0(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(s)\ \mathrm{d}s. \tag{2.4}$$

It is easy to check that the function u given through (2.3) **(i)** solves the homogeneous wave equation (1.2) and **(ii)** satisfies the initial conditions

$$u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ x \in R \tag{2.5}$$

as long as u_0, u_1 are twice continuously differentiable in R.

We note immediately that solutions of the wave equation obtained from (2.4) inherit the regularity of the initial data (2.5). What is more, formula (2.4) could be used to provide a kind of "generalized" solution to the initial-value problem (1.2), (2.5) provided the data u_0, u_1 are not smooth enough. Indeed, for non-smooth data, say

$$u_0 \in L^1_{\mathrm{loc}}(R^1),\ u_1 \in L^1_{\mathrm{loc}}(R^1),$$

we can find a sequence of smooth functions $u_{0,\varepsilon} \in C_c^\infty(R)$, $u_{1,\varepsilon} \in C_c^\infty(R)$ such that

$$u_{0,\varepsilon} \to u_0,\ u_{1,\varepsilon} \to u_1 \text{ in } L^1(K) \text{ for any compact set } K \in R^1$$

and use (2.4) to conclude that the corresponding (unique) solutions u_ε of (1.2), (2.5) converge in $L^1_{\mathrm{loc}}([0,T] \times R^1)$ to a (unique) function u that may be viewed as a "weak" solution of the same problem with the initial data u_0, u_1.

2.3 Dispersion and local energy decay

Before starting our study of more complicated and even nonlinear analogues of the wave equation (1.2), we take advantage of the simplicity of D'Alembert's formula (2.4) to illustrate other characteristic features of wave propagation. We have seen in Section 2.1 that the *total energy*

$$\int_{R^1} E(t,x)\ \mathrm{d}x = \frac{1}{2}\int_{R^1}\left(|\partial_t u(t,\cdot)|^2 + |\partial_x u(t,\cdot)|^2\right)\ \mathrm{d}x$$

is a constant of motion, meaning *independent* of time for any solution of (1.2). Of course, we need the above integral to be finite at least at one time instant $t\ t_0$. This can be easily seen for *compactly supported* initial data u_0, u_1 by means of formula (2.3) and then extended via density argument to general u_0, u_1.

Consider the *local energy*

$$\mathcal{E}_{a,b}(t) = \int_a^b E(t,x)\ \mathrm{d}x \text{ for } -\infty < a < b < \infty.$$

Going back to D'Alembert's formula (2.4) we may compute

$$\int_{-T}^{T} \mathcal{E}_{a,b}(t)\ \mathrm{d}t = \frac{1}{2}\int_{-T}^{T}\int_a^b \left(|\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$\leq 2(c+1)\int_{-T}^{T}\int_a^b \left(|\partial_x u_0(x+ct)|^2 + |\partial_x u_0(x-ct)|^2\right)\mathrm{d}x\ \mathrm{d}t$$

$$+2(c+1)\int_{-T}^{T}\int_a^b \left(|\partial_x u_1(x+ct)|^2 + |\partial_x u_1(x-ct)|^2\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$\leq 4(b-a)\int_{R^1}\left(|\partial_x u_0(x)|^2 + |\partial_x u_1(x)|^2\right)\ \mathrm{d}x$$

for any $T > 0$.

Letting $T \to \infty$ we may therefore infer that

$$\int_{-\infty}^{\infty}\left[\int_a^b E(t,x)\right]\mathrm{d}x\ \mathrm{d}t \leq 4(b-a)\int_{R^1} E(0,x)\ \mathrm{d}x, \tag{2.6}$$

which may be interpreted as *local energy decay*. In accordance with (2.1), waves - solutions of (1.2) emanating from spatially localized initial data -

decay locally to zero in the integral sense (2.6). We may also observe *uniform* time decay, meaning

$$\left[\int_a^b E(t,x)\right] \to 0 \text{ as } t \to \infty$$

but only for *compactly supported* initial data. These phenomena are conditioned by unboundedness of the physical space R^1, where the waves have enough space to disperse. As we shall see later, the situation is completely different on *bounded* intervals, where the waves are reflected by the boundary.

Finally, we note that the local L^2-norm of a solution

$$\int_a^b |u(t,x)|^2 \, \mathrm{d}x$$

may remain bounded below away from zero as $t \to \infty$ for *certain* data as a direct consequence of D'Alembert's formula.

2.4 Wave equation on bounded intervals

Consider the 1-D wave equation

$$\partial^2_{t,t} u(t,x) - c^2 \partial^2_{x,x} u(t,x) = 0 \tag{2.7}$$

for x belonging to a *bounded* interval, say $(0,\pi)$, supplemented with the boundary conditions

$$u(t,0) = u(t,\pi) = 0, \ t > 0, \tag{2.8}$$

and the initial conditions,

$$u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x), \ x \in (0,\pi). \tag{2.9}$$

Our goal is to observe that D'Alembert's formula (2.4) can be adapted to the initial-boundary value problem (2.7 - 2.9). To this end, we first suppose that u_0, u_1 are spatially periodic with the period 2π. In such a case, it is easy to check that D'Alembert's formula yields a solution u with the same property, meaning periodic in x. In addition, assuming that both u_0 and u_1 are odd functions,

$$u_0(-x) = -u_0(x), \ u_1(-x) = -u_1(x), \ x \in R^1$$

we easily observe that u given by (2.4) is also odd. In particular, as a byproduct, we recover the boundary conditions (2.8).

We conclude that D'Alembert's formula yields a solution for the problem (2.7 - 2.9) provided the initial data u_0, u_1 were extended as odd, $2\pi-$periodic functions in R^1. Similarly, replacing *odd* by *even* we may deduce a solution formula for the problem with the so-called Neumann boundary condition

$$\partial_x u(t,0) = \partial_x u(t,\pi) = 0. \tag{2.10}$$

We may introduce the total energy

$$\frac{1}{2}\int_0^\pi \left(|\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2 \right)\, \mathrm{d}x,$$

exactly as in Section (2.1). However, unlike in the case of spatially localized solutions defined on the whole real line R^1, the total energy, though evaluated over a compact interval, *does not* decay to zero as $t \to \infty$. It can be easily seen that the total energy is actually *conserved*, meaning constant in time, as a consequence of our choice of the boundary conditions.

We conclude this part by a simple but rather interesting observation that *all* solutions to the initial-boundary value problem (2.7 - 2.9) are also $\frac{2\pi}{c}$*-time periodic*, a property that can be easily deduced from (2.4).

2.5 Riemann invariants, observability

There are several ways how to write the wave equation (1.2). One possibility is to introduce the so-called *Riemann invariants*

$$R = \partial_t u + c\partial_x u,\ \ S = \partial_t u - c\partial_x u \tag{2.11}$$

and rewrite (1.2) as a *system*

$$\partial_t R(t,x) - c\partial_x R(t,x) = 0,\ \ \partial_t S(t,x) + c\partial_x S(t,x) = 0 \tag{2.12}$$

of two independent transport equations. Accordingly, the quantity R is constant along the lines $t \mapsto [t, x - ct]$, while S is constant on $t \mapsto [t, x + ct]$ for $x \in R$.

Now, we exploit the relatively simple form of (2.12) to show *boundary observability* property for (1.2). To be more specific, we consider the situation described by the initial-boundary value problem (2.8), where the solutions

u and, consequently, $\partial_t u$ vanish on the boundary $x = 0, \pi$. Since (2.12) is a system of two independent transport equations, we easily deduce that

$$\int_0^\pi R^2\left(\frac{2\pi}{c}, x\right)\ \mathrm{d}x \leq c\int_0^{2\pi/c} R^2(t,0)\ \mathrm{d}t = c^3\int_0^{2\pi/c} |\partial_x u(t,0)|^2\ \mathrm{d}t, \quad (2.13)$$

and, similarly,

$$\int_0^\pi S^2\left(\frac{2\pi}{c}, x\right)\ \mathrm{d}x \leq c\int_0^{2\pi/c} S^2(t,0)\ \mathrm{d}t = c^3\int_0^{2\pi/c} |\partial_x u(t,0)|^2\ \mathrm{d}t. \quad (2.14)$$

We recall the convention that u may be viewed as a $2\pi-$spatially periodic odd function, while $\partial_x u$ is $2\pi-$spatially periodic even.

Consequently, relations (2.13), (2.14) give rise to

$$\int_0^\pi E(2\pi/c, x)\ \mathrm{d}x = \frac{1}{2}\int_0^\pi \left(|\partial_t u|^2 + |\partial_x u|^2\right)(2\pi/c, x)\ \mathrm{d}x$$

$$\leq \mathrm{const}(c)\int_0^{2\pi/c} |\partial_x u(0,t)|^2\ \mathrm{d}t.$$

However, the total energy is a constant of motion and we deduce the *observability inequality*:

$$\int_0^\pi E_0\ \mathrm{d}x = \frac{1}{2}\int_0^\pi \left(|\partial_x u_0|^2 + |u_1|^2\right)\ \mathrm{d}x \leq \mathrm{const}(c)\int_0^{2\pi/c} |\partial_x u(t,0)|^2\ \mathrm{d}t. \quad (2.15)$$

The message hidden in (2.15) reads that the behavior of solutions to the boundary value problem (2.7 - 2.9) is entirely controlled (determined) by the boundary values of $\partial_x u$ on the time interval of the length at least $2\pi/c$.

Repeating the same arguments we can show a more general inequality

$$\int_{-\pi}^\pi E_0\ \mathrm{d}x \leq \mathrm{const}(c)\int_0^{2\pi/c}\left(|\partial_t u(t,\xi)|^2 + |\partial_x u(t,\xi)|^2\right)\mathrm{d}t \text{ for any } \xi \in [-\pi, \pi] \quad (2.16)$$

that holds for any $2\pi-$spatially periodic solution u, in particular for any solution of the initial-boundary value problems (2.7), (2.8), (2.9) and (2.7), (2.9), (2.10).

2.6 Uniqueness and data dependence

As we have seen in Section 2.1, *smooth* solutions are uniquely determined by their initial values on any wave cone. We have used the Gauss-Green formula, and, in particular, the existence of suitably defined traces. Here, we show that solutions of the wave equation (1.2) are still uniquely determined by the initial data even if we suppose much less regularity. To begin, we extend the class of solutions saying that u is a *weak solution* of the wave equation (1.2) on the space-time cylinder $(0,T)\times B$ if the integral identity

$$\int_0^T \int_B u(t,x)\Big(\partial^2_{t,t}\varphi(t,x) - c^2\partial^2_{x,x}\varphi(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t = 0 \tag{2.17}$$

holds for any test function $\varphi \in C^\infty_c((0,T)\times B)$.

First, we observe that the "initial values" of u and $\partial_t u$ at the time $t=0$ can be well defined. To this end, we take a special test function $\varphi(t,x) = \psi(t)\phi(x)$ in (2.17). We easily check that the mapping

$$t \mapsto \int_B u(t,x)\phi(x)\ \mathrm{d}x\ \mathrm{d}t$$

has two derivatives with respect to the t variable integrable in $[0,T]$ provided $u \in L^1((0,T)\times B)$. In particular,

$$t \mapsto \int_B u(t,x)\phi(x)\ \mathrm{d}x\ \mathrm{d}t,\ \partial_t\left(t \mapsto \int_B u(t,x)\phi(x)\right)$$

may be viewed as *continuous* functions of $t \in [0,T]$. In particular, it makes sense to speal about the values of u and $\partial_t u$ at any time $t \in [0,T]$.

Assuming the solution u is more regular, say,

$$\partial_t u,\ \partial_x u \in L^1((0,T)\times B)$$

we deduce from (2.17) that

$$\left\{ \begin{array}{l} \int_0^T \int_B \left(\partial_t u + c\partial_x u\right)\left(\partial_t\varphi - c\partial_x\varphi\right)\ \mathrm{d}x\ \mathrm{d}t = 0, \\ \\ \int_0^T \int_B \left(\partial_t u - c\partial_x u\right)\left(\partial_t\varphi + c\partial_x\varphi\right)\ \mathrm{d}x\ \mathrm{d}t = 0. \end{array} \right\} \tag{2.18}$$

Thus the Riemann invariants R, S introduced in (2.11), being now solely integrable functions in $(0,T)\times B$, are still constant on the characteristic lines $t \mapsto [t, x-ct]$, $t \mapsto [t, x+ct]$, $x \in B$, respectively. In particular, the solution u is uniquely determined in the wave cone

$$C_{B,T} = \{t \in (0,T),\ y \in B \mid t \in (0,T),\ y = x+ct \text{ or } y = x-ct \text{ for a certain } x \in B\}$$

by the initial values u and $\partial_t u$ at $t=0$, $x \in B$.

3 1-D nonlinear wave equation

We discuss briefly the situation when the speed of propagation dependends on $\partial_x u$, specifically,

$$\partial^2_{t,t} u(t,x) - \partial_x \sigma\Big(\partial_x u(t,x)\Big) = 0. \tag{3.1}$$

Our main goal is to show that, in general, equation (3.1) does not admit smooth solutions no matter how regular and small the initial data are.

3.1 Riemann invariants

Similarly to Section 2.5, we rewrite equation (3.1) in terms of the Riemann invariants. Writing

$$U = \partial_t u, \ V = \partial_x u$$

we obtain

$$\partial_t V - \partial_x U = 0, \ \partial_t U - \partial_x \sigma(V) = 0. \tag{3.2}$$

Furthermore, introducing

$$h(Z) = \int_0^Z \sqrt{\sigma'(s)} \, \mathrm{d}s$$

we get

$$\partial_t h(V) - \sqrt{\sigma'(V)} \partial_x U = 0, \ \partial_t U - \sqrt{\sigma'(V)} \partial_x h(V) = 0;$$

whence

$$\partial_t \left[U + h(V) \right] - \sqrt{\sigma'(V)} \partial_x \left[U + h(V) \right] = 0, \tag{3.3}$$

and

$$\partial_t \left[U - h(V) \right] + \sqrt{\sigma'(V)} \partial_x \left[U - h(V) \right] = 0. \tag{3.4}$$

By analogy with Section 2.5, the quantities

$$R = U + h(V), \ S = U - h(V)$$

are termed *Riemann invariants*

3.2 Shock waves

It is easy to deduce from (3.3), (3.4) that the nonlinear equation (3.1) does not admit, in general, global in time smooth solutions. Indeed we can take that initial data so that $U = h(V)$, meaning $S = 0$. In accordance with (3.4), this property is preserved at any positive time as S is constant along characterisic curves

$$\mathbf{X}' = \sqrt{\sigma'(V(t, \mathbf{X})}, \ \mathbf{X}(0) = \mathbf{X}_0.$$

In particular, equation (3.3) reads

$$\partial_t U - \frac{1}{2}\sqrt{\sigma'(h^{-1}(U))}\partial_x U = 0, \tag{3.5}$$

which is nothing other than a quasilinear transport equation discuss. In particular, solutions of (3.5) may develop discontinuities (shock waves) in a finite time even if the initial data are taken smooth and small.

4 Semilinear equations

We finish our study of the wave equations by the semilinear problem

$$\partial^2_{t,t} u(t,x) - \partial^2_{x,x} u(t,x) + f(u(t,x)) = 0, \tag{4.1}$$

together with its multidimensional analogue

$$\partial^2_{t,t} u(t,x) - \Delta_x u(t,x) + f(u(t,x)) = 0, \tag{4.2}$$

supplemented with suitable boundary as well as initial conditions.

In contrast with the example of a *quasilinear* equation examined in the previous section, the equations (4.1), (4.2) are nonlinear only on the lower order terms. Thus we expect, at least under certain hypotheses imposed on f, that the solutions will inherit regularity of the initial data.

4.1 Finite time blow-up

Solutions of *non-linear* equations may not exist an arbitrary long time intervals. We have seen an example of a singular behavior in Section 3.2, where solutions of a quasilinear equation developed singularities in the form of shock waves in a finite time. For *semi-linear* equations like (4.1), (4.2), solutions may develop a *blow-up*, where the amplitude becomes infinite in a finite time. In contrast with the shock waves, where usually the solutions my be "continued" in some form, the blow up behavior may lead to the ultimate state with hypothetial "infinite" energy.

Consider regular solutions of the semilinear equation (4.1), supplemented, for definiteness, with the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0. \tag{4.3}$$

Multiplying the equation by $\partial_t u$, integrating by parts and making use of the boundary conditions, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\pi \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\partial_x u|^2 + F(u)\right)(t,\cdot)\ \mathrm{d}x = 0,$$

where we have set

$$F(u) = -\int_0^u f(z)\ \mathrm{d}z.$$

The quantity

$$E(t) = \int_\Omega \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\partial_x u|^2 + F(u)\right)(t,x)\ \mathrm{d}x$$

plays the role of *energy* for the semilinear wave equation (4.2), and, as we have just observed, it is a constant of motion. Note however that "energy" defined in such a way may be negative.

Introducing

$$I(t) = \frac{1}{2}\int_\Omega |u(t,x)|^2\ \mathrm{d}x$$

we easily compute

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) = \int_\Omega u(t,x)\partial_t u(t,x)\ \mathrm{d}x,$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}I(t) = \int_\Omega \left(|\partial_t u(t,x)|^2 + u(t,x)\partial^2_{t,t}u(t,x)\right)\ \mathrm{d}x,$$

where, by virtue of (4.2),

$$\int_\Omega u(t,x)\partial^2_{t,t}u(t,x)\ \mathrm{d}x = -\int_\Omega \left(|\nabla_x u(t,x)|^2 + f(u(t,x))u(t,x)\right)\ \mathrm{d}x.$$

Thus, combining the previous two identities, we arrive at

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I(t) = (2+2\lambda)\int_\Omega |\partial_t u(t,x)|^2 \ \mathrm{d}x + 2\lambda \int_\Omega |\nabla_x u(t,x)|^2 \ \mathrm{d}x \tag{4.4}$$
$$+ \int_\Omega \Big((2+4\lambda)F(u) - uf(u)\Big)(t,x) \ \mathrm{d}x - (2+4\lambda)E$$

for any $\lambda \geq 0$.

Suppose that

- there exists $\varepsilon > 0$ such that

$$(2+\varepsilon)F(u) \geq uf(u) \text{ for all } u \in R; \tag{4.5}$$

- the energy $E(t) = E < 0$ is negative.

Consequently, we can take $\lambda = \varepsilon/4$ and compute

$$I(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} I(t) \geq (1+\lambda)\int_\Omega |u(t,x)|^2 \ \mathrm{d}x \int_\Omega |\partial_t u(t,x)|^2 \ \mathrm{d}x \tag{4.6}$$
$$\geq (1+\lambda)\left|\frac{\mathrm{d}}{\mathrm{d}t} I(t)\right|^2,$$

where we have used the Cauchy-Schwartz inequality. Moreover, it follows from (4.5) that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I(t) \geq -(2+4\lambda)E > 0,$$

in particular, I is strictly convex and there is $\tau > 0$ such that

$$I(\tau) > 0, \ I'(\tau) > 0.$$

Thus, finally, dividing (4.6) on II', we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \log\left(\frac{\mathrm{d}}{\mathrm{d}t} I(t)\right) \geq \frac{\mathrm{d}}{\mathrm{d}t} \log\left(I^{1+\lambda}(t)\right) \text{ for all } t \geq \tau,$$

from which we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \geq \mu I^{1+\lambda}(t), \text{ with } \mu = \frac{\mathrm{d}}{\mathrm{d}t} I(\tau) I^{-1-\lambda}(\tau) > 0. \tag{4.7}$$

Relation (4.7) yields the existence of a *finite* number T such that

$$\lim_{t \to T-} I(t) = \infty. \tag{4.8}$$

We conclude that solutions of the problem (4.2), (4.3) with negative total energy E must blow-up in a finite time as soon as that nonlinearity f satisfy the convexity hypothesis (4.5). It is easy to check that the latter holds, for instance, if

$$f(u) = mu - |u|^{p-1}u, \ m \geq 0, \ p > 1.$$

Moreover, for such an f, we can find a couple of (smooth) functions $u_0 = u_0(x)$, $u_1 = u_1(x)$ satisfying

$$E_0 = \int_\Omega \left(\frac{1}{2}|u_1(x)|^2 + \frac{1}{2}|\partial_x u_0(x)|^2 + F(u_0(x)) \right) \, \mathrm{d}x < 0.$$

Accordingly, any classical solution u of the nonlinear wave equation (4.2) emanating from the initial data

$$u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x), \ x \in \Omega \tag{4.9}$$

must blow-up in a finite time T, specifically,

$$\int_\Omega |u(t,x)|^2 \, \mathrm{d}x \to \infty \text{ as } t \to T.$$

It is worth noting that the arguments used in the above discussion were based mainly on the structural properties of the non-linearity f. Accordingly, similar examples may be constructed for other types of boundary conditions and also on a large class of (unbounded) spatial domains, in particular for $\Omega = R^3$.

4.2 Soliton solutions, breathers

The example discussed in the previous section showed that the semilinear wave equation need not to possess a global-in-time solution for a certain class of convex nonlinearities. Here, we consider a seemingly similar problem, namely the so-called *Sine-Gordon equation*

$$\partial_{t,t}^2 u(t,x) - \partial_{x,x}^2 u(t,x) + \sin(u(t,x)) = 0,\ x \in R, \tag{4.10}$$

where the solutions are defined on the whole real line and decay for large x,

$$\lim_{|x|\to\infty} u(t,x) = 0. \tag{4.11}$$

Equation (4.10) possesses an explicit solution, namely

$$u(t,x) = 4\arctan\left(\frac{\sqrt{1-\omega^2}\cos(\omega t)}{\omega\cosh\left(\sqrt{1-\omega^2}x\right)}\right) \tag{4.12}$$

for $0 < \omega < 1$.

The solution given through (4.12) is called *breather*; it is time-periodic and spatially localized. Breathers belong to the class of solutions to nonlinear evolutionary equations termed *solitons*. Solitons are stable objects and may interact. There is a vast literature devoted to solitons and their basic properties. The evolutionary equations possessing soliton solutions are typically *completely integrable*, meaning, possessing and infinite family of conserved quantities. Here, we restrict ourselves to claiming that the Sine-Gordon equation (4.10) possesses this kind of spatially localized solutions.

4.3 A priori bounds

Unlike the equations with convex nonlinearities discussed in Section 4.1, the solutions of the Sine-Gordon equation (4.10) remain bounded on compact time intervals. Indeed we may write

$$\partial_{t,t}^2 u(t,x) - \partial_{x,x}^2 u(t,x) + \sin(u(t,x))$$

$$= \partial_{t,t}^2 u(t,x) - \partial_{x,x}^2 u(t,x) + u(t,x) + \sin(u(t,x)) - u(t,x);$$

whence, multiplying (4.10) on $\partial_t u$ and integrating by parts, we may infer that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_R \left(|\partial_t u|^2 + |\partial_x u|^2 + |u|^2\right)(t,x)\ \mathrm{d}t \tag{4.13}$$

$$\leq \int_R \left(\sin(u(t,x)) - u(t,x)\right)\partial_t u(t,x)\ \mathrm{d}t \leq 2\int_\Omega |u(t,x)||\partial_t u(t,x)|\ \mathrm{d}x$$

$$\leq \int_\Omega |u(t,x)|^2 \ \mathrm{d}x + \int_\Omega |\partial_t u(t,x)|^2 \ \mathrm{d}x.$$

Thus, by virtue of Gronwall's lemma,

$$\int_R \left(|\partial_t u|^2 + |\partial_x u|^2 + |u|^2 \right)(t,x) \ \mathrm{d}t \tag{4.14}$$

$$\leq \exp(2t) \int_R \left(|\partial_t u|^2 + |\partial_x u|^2 + |u|^2 \right)(0,x) \ \mathrm{d}t.$$

The relation (4.14) yields bounds, uniform with respect to compact time intervals, on the L^2-norms of $\partial_t u$, $\partial_x u$, and u in terms of the initial data. Specifically, denoting

$$u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x)$$

we get

$$\sup_{t\in[0,T]} \left(\|\partial_t u(t,\cdot)\|_{L^2(R)} + \|u(t,\cdot)\|_{W^{1,2}(R)} \right) \tag{4.15}$$

$$\leq c(T) \left(\|u_1\|_{L^2(R)} + \|u_0\|_{W^{1,2}(R)} \right).$$

We may wish to deduce similar bounds on higher order derivatives. To this end, we take the time derivative of the equation (4.10), and, denoting $\partial_t u = v$ we obtain

$$\partial^2_{t,t} v - \partial^2_{x,x} v + \cos(u) v = 0.$$

Since $|\cos(u)| \leq 1$, we can repeat the arguments leading to (4.15) to obtain

$$\sup_{t\in[0,T]} \left(\|\partial^2_{t,t} u(t,\cdot)\|_{L^2(R)} + \|\partial_t u(t,\cdot)\|_{W^{1,2}(R)} \right) \tag{4.16}$$

$$\leq c(T) \left(\|u_0\|_{W^{2,2}(R)} + \|u_1\|_{W^{1,2}(R)} \right),$$

where we have used (4.10) to express

$$\partial^2_{t,t} u(0,\cdot) = \partial^2_{x,x} u_0 - \sin(u_0).$$

Moreover, as

$$|\sin(u)| \leq |u|,$$

we may use once more the equation (4.10) to include $\partial^2_{x,x}$ in the left-hand side of (4.16):

$$\sup_{t\in[0,T]} \left(\|\partial^2_{t,t} u(t,\cdot)\|_{L^2(R)} + \|\partial_t u(t,\cdot)\|_{W^{1,2}(R)} + \|u(t,\cdot)\|_{W^{2,2}(R)} \right) \tag{4.17}$$

$$\leq c(T)\left(\|u_0\|_{W^{2,2}(R)} + \|u_1\|_{W^{1,2}(R)}\right).$$

Thanks to the standard embedding relations

$$W^{1,2}(R) \hookrightarrow BC(R), \tag{4.18}$$

the estimate (4.17) yields uniform bounds on u and its first derivatives in the space of bounded and continuous function on R. In particular, we may continue the above procedure be differentiating (4.10) in t and x to obtain uniform bounds on the solutions in the Sobolev space $W^{k,2}(R)$ of an arbitrary order $k = 0, 1, \ldots$. We note that, by virtue of (4.18), that solutions are classical, meaning twice continuously differentiable, if $k \geq 3$. Unlike the situation treated in Section 4.1, where the norm of solutions blows-up in a finite time, the solutions of the Sine-Gordon equation (4.10) are controlled by the initial data. This is obviously due to the specific properties of the nonlinearity, here represented by a uniformly Lipschitz function $\sin(u)$.

The estimates (4.15 - 4.17) are *formal*. They have been derived under the principal hypothesis that a sufficiently smooth solution u *exists*. Such a type of bounds is usually called *a priori estimates* in the literature. Intuitively, the available *a priori bounds* determine the function spaces framework suitable for a given *nonlinear* problem. From this point of view, the scale of Sobolev spaces $W^{k,2}$ resulting from the "energy" estimates (4.15 - 4.17) is more convenient for second-order problems like (4.10) rather than the classical framework of continuous functions.

5 Well-posedness for semilinear wave equations

We focus on a semilinear wave equation in the form

$$\partial^2_{t,t} u(t,x) - \Delta_x u(t,x) + f(u(t,x)) = 0,\ t > 0,\ x \in \Omega, \tag{5.1}$$

where $\Omega \subset R^N$ is a bounded domain with a regular boundary on which we prescribe the homogeneous Dirichlet condition

$$u(t,\cdot)|_{\partial\Omega} = 0. \tag{5.2}$$

Given the *initial data*

$$u(0,\cdot) = u_0,\ \partial_t u(0,\cdot) = u_1 \text{ in } \Omega, \tag{5.3}$$

our main goal will be to show that the resulting *initial-boundary value problem* (5.1 - 5.3) possesses a (possibly unique) solution on a given time interval $(0,T)$.

We follow the nowadays standard scheme based on

- *a priori* bounds;
- *compactness* or (weak) sequential stability;
- *approximate scheme* and convergence.

Such a way of a *constructive* proof of existence is easily adaptable when solving the real world problems, where the chosen approximate scheme coincides with the expected numerical implementation. Although it may seem at the first glance that *a priori* bounds as well as the property of *compactness* of the (hypothetical) family of solutions are superfluous in the proof of existence, they represent the natural preliminary steps in identifying the suitable function spaces framework as well as the approximate scheme.

In general, given a nonlinear problem, we first try to identify as many *a priori* bounds as possible in order to guarantee *compactness* or sequential stability of a hypothetical class of solutions. Sequential stability means that any sequence of smooth solutions bounded in terms of *a priori estimates* possesses at least a subsequence that converges to another solution of the same problem. Having clarified these two rather crucial issues, we may try to *construct* solutions by means of a suitable approximation scheme, the convergence of which can be established by the tools developed in the preceding two steps.

5.1 A priori bounds

Basically all *a priori* bounds available for solutions of the problem (5.1 - 5.3) follow from the so called *energy method*.

5.1.1 Basic energy estimates

We adopt the procedure introduced in Section 4.3. Multiplying the equation (5.1) on $\partial_t u$, integrating the resulting expression over Ω, and using the Gauss-Green theorem together with the boundary condition (5.2) to eliminate the boundary terms, we obtain the standard *energy balance*:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0,\ E(t) = \int_\Omega \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + F(u) \right)(t,x)\ \mathrm{d}x, \tag{5.4}$$

where we have denoted

$$F(u) = \int_0^u f(z)\ \mathrm{d}z.$$

As we have seen in Section 4.1, the relation (5.4) itself is not strong enough to yield *uniform* bounds unless we impose certain structural restrictions on f. Our aim is that the energy E represents a kind of "norm" in a suitable space. Since Ω is a bounded and regular domain, say of the class C^2, we have the *Poincaré inequality*:

$$\int_\Omega |\nabla_x v|^2\ \mathrm{d}x \geq \Lambda^2 \int_\Omega |v|^2\ \mathrm{d}x \text{ for any } v \in W_0^{1,2}(\Omega), \tag{5.5}$$

where $\Lambda > 0$ is the first (minimal) eigenvalue of the Dirichlet Laplacean in Ω,

$$-\Delta w = \Lambda w \text{ in } \Omega,\ w|_{\partial\Omega} = 0.$$

Inequality (5.5) motivates the following *hypothesis* imposed on f:

$$f(u) \geq -c(1 + |u|) \text{ for all } u \in R^1, \tag{5.6}$$

where c is a certain (positive) constant. Accordingly,

$$F(u) \geq -c(1 + |u|^2) \text{ for all } u \in R^1. \tag{5.7}$$

Here and hereafter, the symbol c or c_i denotes a generic positive real constant that specific value of which may vary from line to line.

With (5.7) in mind, we return to (5.4) to deduce

$$E(t) = \int_\Omega \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + F(u) \right)(t,x)\ \mathrm{d}x$$
$$\geq c_1 \left(\|\partial_t u(t,\cdot)\|^2_{L^2(\Omega)} + \|u(t,\cdot)\|^2_{W^{1,2}(\Omega)} \right) - c_2 \|u(t,\cdot)\|^2_{L^2(\Omega)};$$

whence, by Gronwall's lemma,

$$\sup_{t\in[0,T]} \left(\|\partial_t u(t,\cdot)\|^2_{L^2(\Omega)} + \|u(t,\cdot)\|^2_{W^{1,2}(\Omega)} \right) \leq c_1(T + E_0)\exp(c_2 T), \tag{5.8}$$

where

$$E_0 = \int_\Omega \left(\frac{1}{2}|u_1|^2 + \frac{1}{2}|\nabla_x u_0|^2 + F(u_0) \right)(x)\ \mathrm{d}x.$$

5.1.2 Higher order energy bounds

In order to derive estimates on higher order derivatives, we multiply the equation (5.1) on $-\Delta_x u$ and integrate by parts to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \left(|\Delta_x u|^2 + |\partial_t \nabla_x u|^2 \right)(t,x)\ \mathrm{d}x = \int_\Omega f(u(t,x)) \partial_t \Delta_x u(t,x)\ \mathrm{d}x. \tag{5.9}$$

The integral on the right-hand side needs extra treatment. We write

$$\int_\Omega f(u(t,x)) \partial_t \Delta_x u(t,x)\ \mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega f(u(t,x)) \Delta_x u(t,x)\ \mathrm{d}x - \int_\Omega f'(u(t,x)) \partial_t u(t,x) \Delta_x u(t,x)\ \mathrm{d}x,$$

where, by virtue of Hölder's inequality,

$$\left| \int_\Omega f(u(t,x)) \Delta_x u(t,x)\ \mathrm{d}x \right| \leq \|f(u(t,\cdot))\|_{L^2(\Omega)} \|\Delta_x u(t,\cdot)\|_{L^2(\Omega)}, \tag{5.10}$$

and

$$\left| \int_\Omega f'(u(t,x)) \partial_t u(t,x) \Delta_x u(t,x)\ \mathrm{d}x \right| \tag{5.11}$$
$$\leq \|f'(u(t,\cdot)\|_{L^p(\Omega)} \|\partial_t u(t,x)\|_{L^q(\Omega)} \|\Delta_x u(t,\cdot)\|_{L^2(\Omega)},\ \text{with } \frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$$

References