Scaling and singular limits in fluid mechanics

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

joint work with

A. Novotný (Toulon), H. Petzeltová (Prague)

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Motto

Johann von
Neumann
[1903-1957]

In mathematics you
don't understand things.
You just get used to
them.

Mathematical model

Viscous, compressible, and heat conducting fluid in motion

- mass density $\varrho = \varrho(t, x)$
- absolute temperature $\vartheta = \vartheta(t, x)$
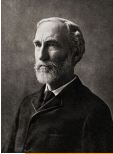- velocity field $\mathbf{u} = \mathbf{u}(t, x)$

Thermodynamic functions

- pressure $p = p(\varrho, \vartheta)$
- internal energy $e = e(\varrho, \vartheta)$
- entropy $s = s(\varrho, \vartheta)$

Transport

- viscous stress $\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$
- heat flux $\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta)$

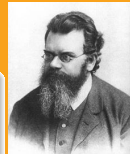# Gibbs' equation and thermodynamic stability

Willard Gibbs
[1839-1903]

Gibbs' equation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Ludwig Boltzmann
[1844-1906]

Constitutive relations

Isaac Newton
[1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x\mathbf{u}\mathbb{I}$$

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x\vartheta$$

François Marie
Charles Fourier
[1772-1837]

Navier-Stokes-Fourier system

Claude Louis
Marie Henri Navier
[1785-1836]

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

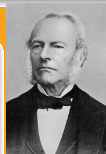## Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Entropy balance

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta}\right) = \sigma$$

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

George
Gabriel
Stokes
[1819-1903]

Scaled Navier-Stokes-Fourier system

Mass conservation

$$[\mathrm{Sr}]\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$
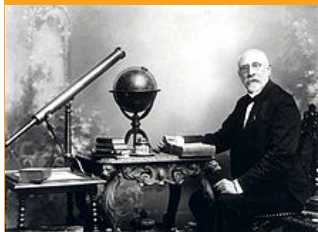
Momentum balance

$$[\mathrm{Sr}]\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \left[\frac{1}{\mathrm{Ma}^2}\right] \nabla_x p(\varrho, \vartheta) = \left[\frac{1}{\mathrm{Re}}\right] \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Entropy balance

$$[\mathrm{Sr}]\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \left[\frac{1}{\mathrm{Pe}}\right] \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

$$\sigma = \frac{1}{\vartheta} \left(\left[\frac{\mathrm{Ma}^2}{\mathrm{Re}}\right] \mathbb{S} : \nabla_x \mathbf{u} - \left[\frac{1}{\mathrm{Pe}}\right] \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Čeněk Strouhal
[1850-1922]

Strouhal number

$$[Sr] = \frac{\text{length}_{\text{char}}}{\text{time}_{\text{char}}\,\text{velocity}_{\text{char}}}$$

Scaling by means of Strouhal number is used in the study of the long-time behavior of the fluid system, where the characteristic time is large

Mach number

Ernst Mach [1838-1916]

Mach number

$$[\text{Ma}] = \frac{\text{velocity}_{\text{char}}}{\sqrt{\text{pressure}_{\text{char}}/\text{density}_{\text{char}}}}$$

Mach number is the ratio of the characteristic speed to the speed of sound in the fluid. Low Mach number limit, where, formally, the speed of sound is becoming infinite, characterizes incompressibility

Osborne Reynolds
[1842-1912]

Reynolds number

$$[\mathrm{Re}] = \frac{\mathrm{density_{char}\, velocity_{char}\, length_{char}}}{\mathrm{viscosity_{char}}}$$

High Reynolds number is attributed to turbulent flows, where the viscosity of the fluid is negligible

Péclet number

Jean Claude
Eugène Péclet
[1793-1857]

Péclet number

$$[\mathrm{Pe}] = \frac{\mathrm{pressure_{char}\, velocity_{char}\, length_{char}}}{\mathrm{heat\ conductivity_{char}\, temperature_{char}}}$$

High Péclet number corresponds to low heat conductivity of the fluid that may be attributed to turbulent flows

Inviscid incompressible limit

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta)} = \boxed{\varepsilon^a} \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \boxed{\varepsilon^b} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+a}} \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Boundary conditions and total energy conservation

Navier's slip

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \varepsilon^c [\mathbb{S}(\vartheta, \nabla_x \mathbf{u})\mathbf{n}]_{\tan} + \beta(\vartheta)\mathbf{u}|_{\partial\Omega} = 0, \ c, \beta > 0$$

Energy insulation

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\partial\Omega} = -\beta(\vartheta)\varepsilon^d |\mathbf{u}|^2|_{\partial\Omega}, \ d = 2 + a - c - b$$

Total mass and energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho \ \mathrm{d}x = 0, \ \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\varepsilon^2 \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \ \mathrm{d}x = 0$$

Target (limit) system

Leonhard Paul
Euler [1707-1783]

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Transport equation for temperature deviation

$$\partial_t \mathcal{T} + \mathbf{v} \cdot \nabla_x \mathcal{T} = 0$$

Ballistic free energy

Jerald LaVerne
Ericksen [*1924]

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta)\Big)$$

Coercivity

$$\partial^2_{\varrho,\varrho} H_\Theta(\varrho, \Theta) = \frac{1}{\varrho}\partial_\varrho p(\varrho, \Theta)$$

$$\partial_\vartheta H_\Theta(\varrho, \vartheta) = (\vartheta - \Theta)\frac{1}{\vartheta}\partial_\vartheta e(\varrho, \vartheta)$$

Coercivity

Coercivity of the ballistic free energy

$$H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta)$$

$$\geq c(K)\left(|\varrho - r|^2 + |\vartheta - \Theta|^2\right) \text{ for all } (\varrho, \vartheta) \in K$$

$$H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta)$$

$$\geq c(K)\left(1 + \varrho e(\varrho, \vartheta) + \varrho s(\varrho, \vartheta)\right) \text{ whenever } (\varrho, \vartheta) \in [0, \infty)^2 \setminus K.$$

$K \subset (0, \infty)^2$ a compact containing r, Θ

Relative entropy (scaled)

Relative entropy

$$\mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) =$$

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^2} \left(H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \right]$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon^{b-2} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \; \mathrm{d}x \, \mathrm{d}t$$

$$+ \varepsilon^{a-c} \int_0^\tau \int_{\partial\Omega} \frac{\Theta \beta}{\vartheta} |\mathbf{u}|^2 \; \mathrm{d}S_x \, \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}_\varepsilon(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \, \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

$$\boxed{\mathcal{R}_\varepsilon(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_0^\tau \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \varepsilon^{a-c} \int_0^\tau \int_{\partial\Omega} \beta \mathbf{u} \cdot \mathbf{U} \, \mathrm{d}\mathrm{S}_x \, \mathrm{d}t$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r}(\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \, \mathrm{d}x \mathrm{d}t$$

$$- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \Big(\partial_t \Theta + \mathbf{u} \cdot \nabla_x \Theta \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \varepsilon^{b-2} \int_0^\tau \int_\Omega \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \, \mathrm{d}x \mathrm{d}t$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \, \mathrm{d}x \, \mathrm{d}t$$

Initial data and far field behavior

Initial data

$$\varrho(0, \cdot) = \overline{\varrho} + \varepsilon \boxed{\varrho_{0,\varepsilon}^{(1)}}, \ \varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\vartheta(0, \cdot) = \overline{\vartheta} + \varepsilon \boxed{\vartheta_{0,\varepsilon}^{(1)}}, \ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon} \to \mathbf{v}_0 \text{ in } L^2(\Omega; R^3), \ \mathbf{v}_0 \in W^{k,2}(\Omega; R^3), \ k > \frac{5}{2}$$

Far field

$$\varrho \to \overline{\varrho}, \ \vartheta \to \overline{\vartheta}, \ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

Convergence

$$b > 0, \ 0 < c < a < \frac{10}{3},$$

Asymptotic incompressibility

$$\operatorname{ess} \sup_{t \in (0,T)} \| \varrho_\varepsilon(t, \cdot) - \overline{\varrho} \|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon c$$

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \ \mathbf{v} \ \text{in} \ \boxed{L^\infty_{\mathrm{loc}}((0, T]; L^2_{\mathrm{loc}}(\Omega; R^3))}$$

and weakly-(*) in $L^\infty(0, T; L^2(\Omega; R^3))$

Temperature deviation

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \mathcal{T} \ \text{in} \ \boxed{L^\infty_{\mathrm{loc}}((0, T]; L^q_{\mathrm{loc}}(\Omega; R^3)), \ 1 \leq q < 2},$$

and weakly-(*) in $L^\infty(0, T; L^2(\Omega))$

Uniform bounds

The uniform bounds independent of ε are obtained by taking

$$r = \overline{\varrho}, \ \Theta = \overline{\vartheta}, \ \mathbf{U} = 0$$

in the relative entropy inequality

Energy bounds

$$\operatorname{ess\,sup}_{t \in (0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2 + L^{5/3}(\Omega)} \leq c,$$

$$\operatorname{ess\,sup}_{t \in (0,T)} \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,$$

$$\operatorname{ess\,sup}_{t \in (0,T)} \left\| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \right\|_{L^2(\Omega; R^3)} \leq c$$

Dissipation

Integral bounds

$$\varepsilon^a \int_0^T \int_\Omega \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c$$

$$\varepsilon^{a-c} \int_0^T \int_{\partial\Omega} |\mathbf{u}_\varepsilon|^2 \ \mathrm{dS}_x \ \mathrm{d}t \leq c$$

$$\varepsilon^{b-2} \int_0^T \int_\Omega |\nabla_x \vartheta_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c$$

First order approximation

Linearization

$$\varepsilon \partial_t \boxed{\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right)} + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\varepsilon \partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \nabla_x \left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} \right) = \mathbf{f}_{1,\varepsilon}$$

$$\partial_t \left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right)$$

$$+ \mathrm{div}_x \left[\left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right) \mathbf{u}_\varepsilon \right] = f_{2,\varepsilon}$$

Stability

Another application of the relative entropy inequality

Take

$$r_\varepsilon = \overline{\varrho} + \varepsilon R_\varepsilon, \ \Theta_\varepsilon = \overline{\vartheta} + \varepsilon T_\varepsilon, \ \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality

Acoustic equation

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta T_\varepsilon) = 0$$

Transport equation

$$\partial_t (\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0$$

Lighthill's acoustic equation

Wave equation

$$\varepsilon \partial_t Z + \Delta \Phi = 0, \ \varepsilon \partial_t \Phi + Z = 0,$$

Michael James
Lighthill
[1924-1998]

Neumann boundary condition

$$\nabla_x \Phi \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

Initial conditions

$$\Phi(0\cdot) = \Phi_0, \ Z(0,\cdot) = Z_0,$$

Neumann Laplacean

Domain of definition

$$\mathcal{D}(-\Delta_N) = \Big\{ w \in W^{1,2}(\Omega) \ \Big| \ \int_\Omega \nabla_x w \cdot \nabla_x \phi \ \mathrm{d}x = \int_\Omega g\phi \ \mathrm{d}x$$

$$\text{for a certain } g \in L^2(\Omega) \text{ and all } \phi \in C_c^\infty(\overline{\Omega}) \Big\}$$

$$-\Delta_N w = g.$$

Neumann Laplacean

The Neumann Laplacean $-\Delta_N$ is a non-negative self-adjoint operator on the Hilbert space $L^2(\Omega)$

Duhamel's formula

Acoustic potential

$$\Phi(t, \cdot) = \frac{1}{2} \exp\left(i\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 - \frac{i}{\sqrt{-\Delta_N}}Z_0\right]$$

$$+ \frac{1}{2}\exp\left(-i\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 + \frac{i}{\sqrt{-\Delta_N}}Z_0\right]$$

Jean-Marie
Constant Duhamel
[1797-1872]

Time derivative

$$Z(t, \cdot) = \frac{1}{2}\exp\left(i\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[i\sqrt{-\Delta_N}[\Phi_0] + Z_0\right]$$

$$+ \frac{1}{2}\exp\left(-i\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[-i\sqrt{-\Delta_N}[\Phi_0] + Z_0\right]$$

Strichartz estimates for the flat Laplacean

Strichartz estimates

$$\int_{-\infty}^{\infty} \left\| \exp\left(\pm i\sqrt{-\Delta}\, t \right) [h] \right\|_{L^q(R^3)}^p \, dt \le \|h\|_{H^{1,2}(R^3)}^p$$

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty$$

Local energy decay

$$\int_{-\infty}^{\infty} \left\| \chi \exp\left(\pm i\sqrt{-\Delta}\, t \right) [h] \right\|_{H^{\alpha,2}(R^3)}^2 \, dt \le c(\chi) \|h\|_{H^{\alpha,2}(R^3)}^2$$
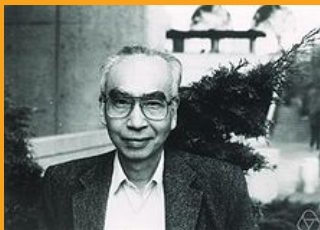
$$\alpha \le \frac{3}{2}, \ \chi \in C_c^{\infty}(R^3)$$

Limiting absorption principle

The cut-off resolvent operator

$$(1 + |x|^2)^{-s/2} \circ [-\Delta_N - \mu \pm i\delta]^{-1} \circ (1 + |x|^2)^{-s/2}, \ \delta > 0, \ s > 1$$

can be extended as a bounded linear operator on $L^2(\Omega)$ for $\delta \to 0$ and μ belonging to compact subintervals of $(0, \infty)$.

Tosio Kato [1917-1999]

Robert S. Strichartz

Kato's theorem

Theorem

Let A be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R,\ v \in \mathcal{D}(A^*),\ \|v\|_X = 1} \|A \circ R_H[\lambda] \circ A^*[v]\|_X < \infty.$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|A \exp(-\mathrm{i}tH)[w]\|_X^2 \ \mathrm{d}t \leq \Gamma^2.$$

Frequency localized local decay estimates

Decay estimates

$$\int_0^\infty \left\| \chi G(-\Delta_N) \exp\left(\pm i \sqrt{-\Delta_N}\, t\right)[h] \right\|_{L^2(\Omega)}^2 \, \mathrm{d}t \leq c \|h\|_{L^2(\Omega)}^2$$

Scaled estimates

$$\int_0^T \left\| \chi G(-\Delta_N) \exp\left(\pm i \sqrt{-\Delta_N}\, \frac{t}{\varepsilon}\right)[h] \right\|_{L^2(\Omega)}^2 \, \mathrm{d}t \leq \varepsilon c(T) \|h\|_{L^2(\Omega)}^2$$

$$G \in C_c^\infty(0, \infty),\ \chi \in C_c^\infty(\overline{\Omega})$$

Admissible domains

Limiting absorption principle

The operator Δ_N satisfies the limiting absorption principle in Ω.

Strichartz estimates on "larger" domain

There is a domain such that $D \cap \{|x| > R\} = \Omega \cap \{|x| > R\}$ and Δ_N satisfies the Strichartz estimates in D.

Local decay on "larger" domain

The operator Δ_N satisfies the local energy decay estimates in D.

Frequency localized Strichartz estimates

$$\int_{-\infty}^{\infty} \left\| G(-\Delta_N) \exp\left(\pm i\sqrt{-\Delta_N}\,t\right)[h] \right\|_{L^q(\Omega)}^p \leq c(G)\|h\|_{L^2(\Omega)}^p$$

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty, \ G \in C_c^\infty(0, \infty)$$