Relative entropies and weak/strong solutions to the Navier-Stokes-Fourier system

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Relative entropy (energy)

Dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = A(t, u(t)),\ u(t) \in X,\ u(0) = u_0$$

Relative entropy

$U : t \mapsto U(t) \in Y$ a "trajectory" in the phase space $Y \subset X$

$$\mathcal{E}\left\{ u(t) \middle| U(t) \right\},\ \mathcal{E} : X \times Y \to R$$

Basic properties

- $\mathcal{E}\left\{u\ |U\right\}$ is a "distance" between u, and U, meaning $\mathcal{E}(u,U) \geq 0$ and $\mathcal{E}\left\{u|U\right\} = 0$ only if $u = U$
- $\mathcal{E}\left\{u(t)|\tilde{U}\right\}$ is a Lyapunov function provided $\tilde{U}$ is a equilibrium, meaning

$$A(t,\tilde{U}) = 0 \text{ for all } t.$$

$$t \mapsto \mathcal{E}\left\{u(t)\Big|\tilde{U}\right\} \text{ is non-increasing}$$

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$$\mathcal{E}\left\{u(\tau)\Big|U(\tau)\right\} \leq \mathcal{E}\left\{u(s)\Big|U(s)\right\} + c(T)\int_s^\tau \mathcal{E}\left\{u(t)\Big|U(t)\right\}\ \mathrm{d}t$$

if U is a solution of the same system (ranging in a "better" space) Y

Applications

Stability of equilibria

Any solution ranging in X stabilizes to an equilibrium belonging to Y (to be proved!)

Weak-strong uniqueness

Solutions ranging in the "better" space Y are unique among solutions in X

Singular limits

Stability and convergence of a family of solutions u_ε corresponding to A_ε to a solution $U = u$ of the limit problem with generator A

Model description

State variables:

Mass density

$$\varrho = \varrho(t,x)$$

Absolute temperature

$$\vartheta = \vartheta(t,x)$$

Velocity field

$$\mathbf{u} = \mathbf{u}(t,x)$$

Thermodynamic functions:

Pressure

$$p = p(\varrho,\vartheta)$$

Internal energy

$$e = e(\varrho,\vartheta)$$

Entropy

$$s = s(\varrho,\vartheta)$$

Gibbs' law

$$\vartheta D s(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)$$

Thermodynamic stability

$$\frac{\partial p(\varrho,\vartheta)}{\partial\varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} = c_v(\varrho,\vartheta) > 0$$

Transport and conservative boundary conditions

Transport:

Newton's law

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

Conservative boundary conditions:

No-slip

$$\mathbf{u}|_{\partial\Omega} = 0$$

No-flux

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Energy balance formulation

Total energy

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right)$$

$$+ \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e + p - \varrho F \right) \mathbf{u} \right] + \mathrm{div}_x \mathbf{q} - \mathrm{div}_x \left(\mathbb{S} \mathbf{u} \right) = 0$$

Thermal energy

$$\varrho c_v(\varrho, \vartheta) \left(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta \right) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u}$$

Entropy

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right)$$

$$= \boxed{ \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) }$$

Navier-Stokes-Fourier system (classical formulation)

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

Thermal energy equation

$$\varrho c_v(\varrho, \vartheta) \left(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta \right) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u}$$

Navier-Stokes-Fourier system (weak formulation)

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

Entropy equation

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma \boxed{\geq} \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \, \mathrm{d}x = 0$$

Initial state

Initial data (regular)

$$\varrho(0,\cdot) = \varrho_0, \ \vartheta(0,\cdot) = \vartheta_0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

- $$\varrho_0 \in W^{3,2}(\Omega), \ 0 < \underline{\varrho} \leq \varrho_0 < \overline{\varrho}$$
- $$\vartheta_0 \in W^{3,2}(\Omega), \ 0 < \underline{\vartheta} \leq \vartheta_0 < \overline{\vartheta}$$
- $$\mathbf{u}_0 \in W^{3,2}(\Omega; R^3)$$

Compatibility conditions

$$\nabla_x \vartheta_0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{u}_0|_{\partial\Omega} = 0$$

$$\nabla_x p(\varrho_0, \vartheta_0) = \mathrm{div}_x \mathbb{S}(\vartheta_0, \nabla_x \mathbf{u}_0) - \varrho_0 \nabla_x F|_{\partial\Omega} = 0$$

Existence of smooth solutions (classical theory)

A. Valli [1982] Existence of classical *local-in-time* solutions in the class:

$$\varrho \in C([0, T_{\max}); W^{3,2}(\Omega)), \ \vartheta_0 \in C([0, T_{\max}); W^{3,2}(\Omega))$$
$$\mathbf{u} \in C([0, T_{\max}); W^{3,2}(\Omega; R^3))$$

A.Matsumura, T.Nishida [1980,1983] Existence of classical *global-in-time* solutions in the same class for the initial data sufficiently close to a static state

Static states

$$\mathbf{u} \equiv 0, \ \vartheta = \tilde{\vartheta} > 0 - \text{a positive constant}, \ \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = \tilde{\varrho} \nabla_x F$$

Weak solutions

- Equation of continuity holds in the sense of distributions (renormalized equation also satisfied)
- Momentum balance holds in the sense of distributions
- Entropy production equation holds in the sense of distributions, entropy production rate satisfies the inequality
- The system is augmented by the total energy balance

Global existence of weak solutions

Hypotheses

Pressure

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4 \ , \ a > 0$$

$$\lim_{Y \to \infty} \frac{P(Y)}{Y^{5/3}} = p_\infty > 0$$

Internal energy

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta \left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4 \ , \ a > 0$$

Viscosity coefficients

$$\mu(\vartheta) \approx (1 + \vartheta^\alpha), \ 0 \leq \eta(\vartheta) \leq c(1 + \vartheta^\alpha), \ \alpha \in [2/5, 1]$$

$$\kappa(\vartheta) \approx (1 + \vartheta^3)$$

Total dissipation balance

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho \Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \Big)$$

$$\partial^2_{\varrho,\varrho} H_\Theta(\varrho, \Theta) = \frac{1}{\varrho} \partial_\varrho p(\varrho, \Theta)$$

$$\partial_\vartheta H_\Theta(\varrho, \vartheta) = \varrho(\vartheta - \Theta)\partial_\vartheta s(\varrho, \vartheta)$$

Coercivity

$\varrho \mapsto H_\Theta(\varrho, \Theta)$ is convex

$\vartheta \mapsto H_\Theta(\varrho, \vartheta)$ attains its global minimum (zero) at $\vartheta = \Theta$

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\tilde\vartheta}(\varrho,\vartheta) - \frac{\partial H_{\tilde\vartheta}(\tilde\varrho,\tilde\vartheta)}{\partial\varrho}(\varrho-\tilde\varrho) - H_{\tilde\vartheta}(\tilde\varrho,\tilde\vartheta)\right)\ \mathrm{d}x$$

$$+\int_\Omega \tilde\vartheta\sigma\ \mathrm{d}x = 0$$

$\tilde\varrho,\ \tilde\vartheta$ – static solution

$$\int_\Omega \tilde\varrho\ \mathrm{d}x = \int_\Omega \varrho\ \mathrm{d}x = M_0,\ \int_\Omega \tilde\varrho e(\tilde\varrho,\tilde\vartheta) - \tilde\varrho F\ \mathrm{d}x = E_0$$

Relative entropy

Relative entropy functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho} (\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Results

Global existence of weak solutions

Dissipative (weak) solutions exist (under the constitutive restrictions specified above) globally in time for any choice of the initial data.

Stability

Any (weak) solution of the Navier-Stokes-Fourier system stabilizes to an equilibrium (static) solution for $t \to \infty$.

Weak $\Rightarrow$ dissipative

Any weak solution is a dissipative solution

Weak-strong uniqueness

Dissipative (weak) and strong solutions emanating from the same initial data coincide as long as the latter exists. Strong solutions are unique in the class of weak solutions.

Conditional regularity criterion

Theorem (Conditional regularity)

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Under the structural hypotheses specified above, suppose that $\{\varrho, \vartheta, \mathbf{u}\}$ is a dissipative (weak) solution of the Navier-Stokes-Fourier system on the set $(0,T) \times \Omega$ emanating from regular initial data satisfying the relevant compatibility conditions.
Assume, in addition, that

$$\mathrm{ess} \sup_{t \in (0,T)} \|\nabla_x \mathbf{u}(t,\cdot)\|_{L^\infty(\Omega;R^{3\times 3})} < \infty$$

The $\{\varrho, \vartheta, \mathbf{u}\}$ is a classical solution determined uniquely in the class of all dissipative (weak) solutions to the problem.

Other applications

- Inviscid incompressible limits for the system with Navier-type boundary conditions
- Vanishing viscosity and/or heat conductivity, convergence to (inviscid) Boussinesq system