Relative entropies and singular limits of compressible fluids

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Relative entropy (energy)

Dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = A(t, u(t)),\ u(t) \in X,\ u(0) = u_0$$

Relative entropy

$U : t \mapsto U(t) \in Y$ a "trajectory" in the phase space $Y \subset X$

$$\mathcal{E}\left\{ u(t) \middle| U(t) \right\},\ \mathcal{E} : X \times Y \to R$$

Distance

$\mathcal{E}\{u\,|U\}$ is a "distance" between u, and U, meaning $\mathcal{E}(u, U) \geq 0$ and $\mathcal{E}\{u|U\} = 0$ only if $u = U$

Lyapunov function

$\mathcal{E}\left\{u(t)|\tilde{U}\right\}$ is a Lyapunov function provided $\tilde{U}$ is a equilibrium, meaning

$$A(t, \tilde{U}) = 0 \text{ for all } t.$$

$$t \mapsto \mathcal{E}\left\{u(t)\middle|\tilde{U}\right\} \text{ is non-increasing}$$

Stability

$$\mathcal{E}\left\{u(\tau)\middle|U(\tau)\right\} \leq \mathcal{E}\left\{u(s)\middle|U(s)\right\} + c(T)\int_s^T \mathcal{E}\left\{u(t)\middle|U(t)\right\}\,\mathrm{d}t$$

if U is a solution of the same system (ranging in a "better" space) Y

Applications

Stability of equilibria

Any solution ranging in X stabilizes to an equilibrium belonging to Y (to be proved!)

Weak-strong uniqueness

Solutions ranging in the "better" space Y are unique among solutions in X

Singular limits

Stability and convergence of a family of solutions u_ε corresponding to A_ε to a solution $U = u$ of the limit problem with generator A

Viscous, compressible, and heat conducting fluid in motion

- mass density $\varrho = \varrho(t, x)$
- absolute temperature $\vartheta = \vartheta(t, x)$
- velocity field $\mathbf{u} = \mathbf{u}(t, x)$

Thermodynamic functions

- pressure $p = p(\varrho, \vartheta)$
- internal energy $e = e(\varrho, \vartheta)$
- entropy $s = s(\varrho, \vartheta)$

Transport

- viscous stress $\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$
- heat flux $\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta)$

Navier-Stokes-Fourier system

EQUATION OF CONTINUITY:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

MOMENTUM BALANCE:

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

$$\mathbf{u}|_{\partial\Omega} = 0, \ \text{or } [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

ENTROPY PRODUCTION:

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma, \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

TOTAL ENERGY CONSERVATION:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x = 0$$

Constitutive relations

Gibbs' relation:

$$\vartheta D s(\varrho,\vartheta) = D e(\varrho,\vartheta) + p(\varrho,\vartheta) D \frac{1}{\varrho}$$

Newton's rheological law:

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law:

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Entropy production rate:

$$\sigma = (\geq) \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Equilibrium (static) solutions

Equilibrium solutions minimize the entropy production:

$$\mathbf{u} \equiv 0, \ \vartheta \equiv \overline{\vartheta} > 0 \text{ a positive constant}$$

Static problem

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F$$

Total mass and energy are constants of motion:

$$\int_\Omega \tilde{\varrho} \ \mathrm{d}x = M_0, \ \int_\Omega \tilde{\varrho} e(\tilde{\varrho}, \overline{\vartheta}) - \tilde{\varrho} F \ \mathrm{d}x = E_0$$

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) - \frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta}) \right) \ \mathrm{d}x$$

$$+ \int_\Omega \overline{\vartheta} \sigma \ \mathrm{d}x = 0$$

Ballistic free energy

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho \Big(e(\varrho, \vartheta) - \overline{\vartheta} s(\varrho, \vartheta) \Big)$$

Thermodynamic stability

Positive compressibility

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$$

Positive specific heat at constant volume

$$\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Coercivity of the ballistic free energy

- $\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})$ convex
- $\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta)$ attains its global minimum (zero) at $\vartheta = \overline{\vartheta}$

Relative entropy

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \, dx$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Calculations

$$\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 \ \mathrm{d}x = \int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 - \varrho\mathbf{u}\cdot\mathbf{U} + \frac{1}{2}\varrho|\mathbf{U}|^2 \ \mathrm{d}x$$

$$\int_\Omega H_\Theta(\varrho,\vartheta) \ \mathrm{d}x = \int_\Omega \left(\varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta)\right) \ \mathrm{d}x$$

$$\int_\Omega \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) \ \mathrm{d}x$$

Applications

Unconditional stability of the equilibrium solutions

Any (weak) solution of the Navier-Stokes-Fourier system stabilizes to an equilibrium (static) solution for $t \to \infty$

Weak-strong uniqueness

Weak and strong solutions emanating from the same initial data coincide as long as the latter exists. Strong solutions are unique in the class of weak solutions

Singular limit in the incompressible, inviscid regime

Solutions of the Navier-Stokes-Fourier system converge in the limit of low Mach and high Reynolds and Péclet number to the Euler-Boussinesq system.

Scaled Navier-Stokes-Fourier system

EQUATION OF CONTINUITY

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

BALANCE OF MOMENTUM

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2}\nabla_x p(\varrho, \vartheta)} = \boxed{\varepsilon^a} \mathrm{div}_x \mathbb{S}$$

ENTROPY PRODUCTION

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \boxed{\varepsilon^b} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+a}} \mathbb{S} : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Target system

INCOMPRESSIBILITY

$$\mathrm{div}_x \mathbf{v} = 0$$

EULER SYSTEM

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0$$

TEMPERATURE TRANSPORT

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0$$

BASIC ASSUMPTION

The incompressible Euler system possesses a strong solution $\mathbf{v}$ *on a time interval* $(0, T_{\rm max})$ *for the initial data* $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$.

Prepared data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3),\ \mathbf{v}_0 \in W^{k,2}(\Omega; R^3),\ k > \frac{5}{2}$$

Boundary conditions

Navier's complete slip condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Convergence

$$b > 0, \ 0 < a < \frac{10}{3}$$

$$\mathrm{ess} \sup_{t \in (0,T)} \| \varrho_\varepsilon(t, \cdot) - \overline{\varrho} \|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon c$$

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \ \mathbf{v} \text{ in } \boxed{L^\infty_{\rm loc}((0, T]; L^2_{\rm loc}(\Omega; R^3))}$$

and weakly-(*) in $L^\infty(0, T; L^2(\Omega; R^3))$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T \text{ in } \boxed{L^\infty_{\rm loc}((0, T]; L^q_{\rm loc}(\Omega; R^3)), \ 1 \leq q < 2},$$

and weakly-(*) in $L^\infty(0, T; L^2(\Omega))$

The uniform bounds independent of ε are obtained by means of the choice

$$r = \overline{\varrho},\ \Theta = \overline{\vartheta},\ \mathbf{U} = 0$$

in the relative entropy inequality:

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2 + L^{5/3}(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \sqrt{\varrho} \mathbf{u}_\varepsilon \right\|_{L^2(\Omega; R^3)} \leq c$$

Linearization

$$\varepsilon \partial_t \boxed{\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right)} + \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\varepsilon \partial_t (\varrho_\varepsilon \mathbf{u}_\varepsilon) + \nabla_x \left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} \right) = \varepsilon \mathbf{f}_1$$

$$\partial_t \left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right)$$

$$+ \mathrm{div}_x \left[\left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right) \mathbf{u}_\varepsilon \right] = \varepsilon f_2$$

Another application of the relative entropy inequality

Take

$$r_\varepsilon = \overline{\varrho} + \varepsilon R_\varepsilon, \ \Theta_\varepsilon = \overline{\vartheta} + \varepsilon \mathcal{T}_\varepsilon, \ \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality

Acoustic equation

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon) = 0$$

Transport equation

$$\partial_t (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) + (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0$$

Lighthill's acoustic equation

Wave equation

$$\varepsilon\partial_t Z + \Delta\Phi = 0,\ \ \varepsilon\partial_t\Phi + Z = 0,$$

Neumann boundary condition

$$\nabla_x\Phi \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

Initial conditions

$$\Phi(0\cdot) = \Phi_0,\ \ Z(0,\cdot) = Z_0,$$

Solution formula

Acoustic potential

$$\Phi(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 - \frac{\mathrm{i}}{\sqrt{-\Delta_N}}Z_0\right]$$

$$+\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 + \frac{\mathrm{i}}{\sqrt{-\Delta_N}}Z_0\right]$$

Time derivative

$$Z(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\mathrm{i}\sqrt{-\Delta_N}[\Phi_0] + Z_0\right]$$

$$+\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[-\mathrm{i}\sqrt{-\Delta_N}[\Phi_0] + Z_0\right]$$

Strichartz estimates for the flat Laplacean

Strichartz estimates

$$\int_{-\infty}^{\infty} \left\| \exp\left(\pm \mathrm{i} \sqrt{-\Delta} t \right) [h] \right\|_{L^q(R^3)}^p \ \mathrm{d}t \leq \|h\|_{H^{1,2}(R^3)}^p$$

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty$$

Local energy decay

$$\int_{-\infty}^{\infty} \left\| \chi \exp\left(\pm \mathrm{i} \sqrt{-\Delta} t \right) [h] \right\|_{H^{\alpha,2}(R^3)}^2 \ \mathrm{d}t \leq c(\chi) \|h\|_{H^{\alpha,2}(R^3)}^2$$

$$\alpha \leq \frac{3}{2}, \ \chi \in C_c^\infty(R^3)$$

Limiting absorption principle

The cut-off resolvent operator

$$(1+|x|^2)^{-s/2} \circ [-\Delta_N - \mu \pm \mathrm{i}\delta]^{-1} \circ (1+|x|^2)^{-s/2}, \ \delta > 0, \ s > 1$$

can be extended as a bounded linear operator on $L^2(\Omega)$ *for* $\delta \to 0$ *and* μ *belonging to compact subintervals of* $(0, \infty)$.

Kato's theorem

Theorem

Let A be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R,\ v \in \mathcal{D}(A^*),\ \|v\|_X = 1} \|A \circ R_H[\lambda] \circ A^*[v]\|_X < \infty.$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|A \exp(-\mathrm{i}tH)[w]\|_X^2 \ \mathrm{d}t \leq \Gamma^2.$$

Decay estimates

$$\int_0^\infty \left\| \chi G(-\Delta_N) \exp\left(\pm \mathrm{i} \sqrt{-\Delta_N} t \right) [h] \right\|_{L^2(\Omega)}^2 \ \mathrm{d}t \leq c \|h\|_{L^2(\Omega)}^2$$

Scaled estimates

$$\int_0^T \left\| \chi G(-\Delta_N) \exp\left(\pm \mathrm{i} \sqrt{-\Delta_N} \frac{t}{\varepsilon} \right) [h] \right\|_{L^2(\Omega)}^2 \ \mathrm{d}t \leq \varepsilon c(T) \|h\|_{L^2(\Omega)}^2$$

$$G \in C_c^\infty(0, \infty), \ \chi \in C_c^\infty(\overline{\Omega})$$

Admissible domains

Limiting absorption principle

The operator Δ_N satisfies the limiting absorption principle in Ω.

Strichartz estimates on "larger" domain

There is a domain such that $D \cap \{|x| > R\} = \Omega \cap \{|x| > R\}$ and Δ_N satisfies the Strichartz estimates in D.

Local decay on "larger" domain

The operator Δ_N satisfies the local energy decay estimates in D.

Frequency localized Strichartz estimates

$$\int_{-\infty}^{\infty} \left\| G(-\Delta_N) \exp\left(\pm \mathrm{i}\sqrt{-\Delta_N} t\right)[h] \right\|_{L^q(\Omega)}^p \leq c(G) \|h\|_{L^2(\Omega)}^p$$

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty, \ G \in C_c^\infty(0, \infty)$$