Sectral measures, dispersive estimates and propagation of acoustic waves in incompressible limits

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Motivation - incompressible limits

State variables

$\varrho = \varrho(t,x)$ .... mass density
$\mathbf{u} = \mathbf{u}(t,x)$ .... velocity field

Navier-Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho) = \mu \Delta \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}$$

Helmholtz decomposition

$$\varrho \mathbf{u} = \mathbf{H}[\varrho \mathbf{u}] + \nabla_x \Phi$$

$\mathbf{H}[\varrho \mathbf{u}]$ .... solenoidal component
Φ .... acoustic potential

Lighthill's acoustic analogy

Acoustic equation

$$\varepsilon \partial_t \left(\frac{\varrho - \overline{\varrho}}{\varepsilon} \right) + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\varepsilon \partial_t (\varrho \mathbf{u}) + p'(\overline{\varrho}) \nabla_x \left(\frac{\varrho - \overline{\varrho}}{\varepsilon} \right) = \varepsilon \mathrm{div}_x \mathbb{L}$$

Lighthill's tensor

$$\mathbb{L} = \mu \nabla_x \mathbf{u} + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I} - \varrho \mathbf{u} \otimes \mathbf{u} - \frac{p(\varrho) - p'(\overline{\varrho})(\varrho - \overline{\varrho}) - p(\overline{\varrho})}{\varepsilon^2}$$

Wave equation for the acoustic potential

$$\varepsilon \partial_t R + \Delta \Phi = 0, \ \varepsilon \partial_t \Phi + p'(\overline{\varrho}) R = \varepsilon F$$

$$\nabla \Phi \cdot \mathbf{n}|_{\partial \Omega} = 0, \ \Phi(x) \to 0 \text{ as } |x| \to \infty$$

Neumann Laplacean and decay estimates

Neumann Laplacean

$$-\Delta_N[u] = g$$

$$\mathcal{D}(\Delta_N) = \left\{ u \in W^{1,2}(\Omega) \ \middle| \ \langle \nabla u; \nabla \psi \rangle_{L^2(\Omega)} = \langle g; \psi \rangle_{L^2(\Omega)} \right.$$

$$\left. \text{for some } g \in L^2(\Omega) \text{ and any } \psi \in C^\infty_c(\overline{\Omega}) \right\}$$

Desired decay estimates

$$\int_0^T \left\| \xi G(-\Delta_N) \exp\left(\pm \mathrm{i} \sqrt{-\Delta_N} \frac{t}{\varepsilon} \right) [v] \right\|^2_{L^2(\Omega)} \leq \omega(\varepsilon, \xi, G) \|v\|^2_{L^2(\Omega)}$$

$$\xi \in C^\infty_c(\overline{\Omega}), \ \ G \in C^\infty_c(0, \infty)$$

$$\omega(\varepsilon, \xi, G) \to 0 \text{ as } \varepsilon \to 0$$

Positive results

RAGE theorem

absence of point spectrum (no trapped modes)

$$\Leftrightarrow$$

$$\omega(\varepsilon, \xi, G) \to 0 \text{ as } \varepsilon \to 0$$

Kato's theorem - Limiting Absorption Principle (LAP)

the cut-off resolvent operator

$$(1 + |x|^2)^{-s/2} \circ [-\Delta_N - \mu \pm \mathrm{i}\delta]^{-1} \circ (1 + |x|^2)^{-s/2},\ \delta > 0,\ s > 1$$

can be extended as a bounded linear operator on $L^2(\Omega)$ for $\delta \to 0$ and μ belonging to *compact* subintervals of $(0, \infty)$.

$$\Rightarrow$$

$$\omega(\varepsilon, \xi, G) \leq \varepsilon c(\xi, G)$$

Reformulation via spectral measures

Decay estimates

$$\int_0^T \left| \left\langle \exp\left(\pm \mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)[v]; G(-\Delta_N)[\xi] \right\rangle \right|^2 \, \mathrm{d}t \leq \omega(\varepsilon, \xi, G)\|v\|^2_{L^2(\Omega)}$$

Reformulation via spectral measures

$$\left\langle \exp\left(\pm \mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)[v]; G(-\Delta_n)[\xi] \right\rangle$$

$$= \int_0^\infty \exp\left(\pm \mathrm{i}\sqrt{\lambda}\frac{t}{\varepsilon}\right) G(\lambda)\tilde{v}(\lambda) \, \mathrm{d}\mu_\xi(\lambda)$$

where μ_ξ is the spectral measure associated to the function ξ

$$\tilde{v} \in L^2(\Omega; \mathrm{d}\mu_\xi), \ \|\tilde{v}\|_{L^2_{\mu_\xi}(\Omega)} \leq \|v\|_{L^2(\Omega)}$$

Results in terms of the spectral measures

RAGE theorem

spectral measure μ_ξ does not charge points in $[0, \infty)$

$$\Leftrightarrow$$

$$\omega(\varepsilon, \xi, G) \to 0 \text{ as } \varepsilon \to 0$$

Kato's theorem

spectral measure μ_ξ
locally Lipschitz continuous with respect to the Lebesgue measure

$$\mu_\xi[I] \leq c_\delta |I| \text{ for any open interval } I \subset [\delta, \frac{1}{\delta}],\ \delta > 0$$

$$\Rightarrow$$

$$\omega(\varepsilon, \xi, G) \leq \varepsilon c(\xi, G)$$

Applications

Stone's formula

$$\mu_\xi(a,b)$$

$$= \lim_{\delta\to 0+} \lim_{\eta\to 0+} \int_{a+\delta}^{b-\delta} \left\langle \left(\frac{1}{-\Delta_N - \lambda - \mathrm{i}\eta} - \frac{1}{-\Delta_N - \lambda + \mathrm{i}\eta} \right) \xi ; \xi \right\rangle \, \mathrm{d}\lambda$$