Scaling and singular limits in fluid mechanics

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joint work with

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Johann von Neumann [1903-1957]

In mathematics you don't understand things. You just get used to them.

Mathematical model

Viscous, compressible, and heat conducting fluid in motion

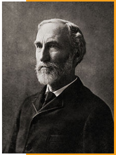
- mass density $\varrho = \varrho(t, x)$
- absolute temperature $\vartheta = \vartheta(t, x)$
- velocity field $\mathbf{u} = \mathbf{u}(t, x)$

Thermodynamic functions

- pressure $p = p(\varrho, \vartheta)$
- internal energy $e = e(\varrho, \vartheta)$
- entropy $s = s(\varrho, \vartheta)$

Transport

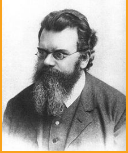
- viscous stress $\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$
- heat flux $\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta)$



Willard Gibbs
[1839-1903]

Gibbs' equation

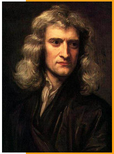
$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)$$



Ludwig Boltzmann
[1844-1906]

Thermodynamic stability

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0$$



Isaac Newton
[1643-1727]

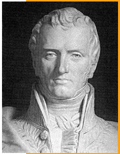
Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}$$

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta$$

Joseph Fourier
[1768-1830]



Claude Louis Marie Henri Navier [1785-1836]

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

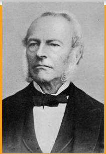
Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Entropy balance

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma$$

$$\sigma = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$



George Gabriel Stokes [1819-1903]

Scaled Navier-Stokes-Fourier system

Mass conservation

$$[\mathrm{Sr}]\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

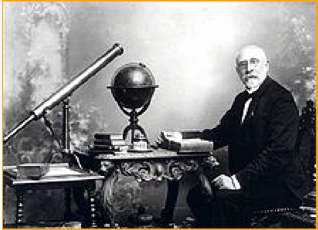
$$[\mathrm{Sr}]\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \left[\frac{1}{\mathrm{Ma}^2}\right] \nabla_x p(\varrho, \vartheta) = \left[\frac{1}{\mathrm{Re}}\right] \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Entropy balance

$$[\mathrm{Sr}]\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \left[\frac{1}{\mathrm{Pe}}\right] \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

$$\sigma = \frac{1}{\vartheta}\left(\left[\frac{\mathrm{Ma}^2}{\mathrm{Re}}\right] \mathbb{S} : \nabla_x \mathbf{u} - \left[\frac{1}{\mathrm{Pe}}\right] \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

Characteristic numbers - Strouhal number



Čeněk Strouhal
[1850-1922]

Strouhal number

$$[\mathrm{Sr}] = \frac{\mathrm{length}_{\mathrm{char}}}{\mathrm{time}_{\mathrm{char}} \mathrm{velocity}_{\mathrm{char}}}$$

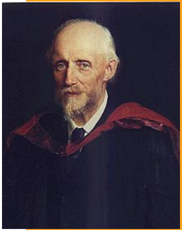
Scaling by means of Strouhal number is used in the study of the long-time behavior of the fluid system, where the characteristic time is large

Ernst Mach [1838-1916]

Mach number

$$[\mathrm{Ma}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\sqrt{\mathrm{pressure}_{\mathrm{char}}/\mathrm{density}_{\mathrm{char}}}}$$

Mach number is the ratio of the characteristic speed to the speed of sound in the fluid. Low Mach number limit, where, formally, the speed of sound is becoming infinite, characterizes incompressibility



Osborne Reynolds
[1842-1912]

Reynolds number

$$[\mathrm{Re}] = \frac{\mathrm{density}_{\mathrm{char}} \mathrm{velocity}_{\mathrm{char}} \mathrm{length}_{\mathrm{char}}}{\mathrm{viscosity}_{\mathrm{char}}}$$

High Reynolds number is attributed to turbulent flows, where the viscosity of the fluid is negligible

Jean Claude
Eugène Péclet
[1793-1857]

Péclet number

$$[\mathrm{Pe}] = \frac{\mathrm{pressure}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}\mathrm{length}_{\mathrm{char}}}{\mathrm{heat\ conductivity}_{\mathrm{char}}\mathrm{temperature}_{\mathrm{char}}}$$

High Péclet number corresponds to low heat conductivity of the fluid that may be attributed to turbulent flows

Inviscid incompressible limit

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta)} = \boxed{\varepsilon^a} \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \boxed{\varepsilon^b} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+a}} \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Boundary conditions and total energy conservation

Navier's slip

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ \varepsilon^c [\mathbb{S}(\vartheta, \nabla_x \mathbf{u})\mathbf{n}]_{\rm tan} + \beta(\vartheta)\mathbf{u}|_{\partial\Omega} = 0, \ \ c, \beta > 0$$

Energy insulation

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\partial\Omega} = -\beta(\vartheta)\varepsilon^d |\mathbf{u}|^2|_{\partial\Omega}, \ \ d = 2 + a - c - b$$

Total mass and energy conservation

$$\frac{\rm d}{{\rm d}t} \int_\Omega \varrho \ {\rm d}x = 0, \ \ \frac{\rm d}{{\rm d}t} \int_\Omega \left(\varepsilon^2 \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \ {\rm d}x = 0$$

Target (limit) system

Leonhard Paul Euler [1707-1783]

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Transport equation for temperature deviation

$$\partial_t \mathcal{T} + \mathbf{v} \cdot \nabla_x \mathcal{T} = 0$$

Jerald LaVerne Ericksen [*1924]

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho \Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \Big)$$

Coercivity

$$\partial^2_{\varrho,\varrho} H_\Theta(\varrho, \Theta) = \frac{1}{\varrho} \partial_\varrho p(\varrho, \Theta)$$

$$\partial_\vartheta H_\Theta(\varrho, \vartheta) = (\vartheta - \Theta) \frac{1}{\vartheta} \partial_\vartheta e(\varrho, \vartheta)$$

Coercivity

Coercivity of the ballistic free energy

$$H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta)$$

$$\geq c(K)\left(|\varrho - r|^2 + |\vartheta - \Theta|^2\right) \ \text{for all } (\varrho,\vartheta) \in K$$

$$H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta)$$

$$\geq c(K)\left(1 + \varrho e(\varrho,\vartheta) + \varrho s(\varrho,\vartheta)\right) \ \text{whenever } (\varrho,\vartheta) \in [0,\infty)^2 \setminus K.$$

$K \subset (0,\infty)^2$ a compact containing r, Θ

Relative entropy (scaled)

Relative entropy

$$\mathcal{E}_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right)=$$

$$\int_\Omega\left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+\frac{1}{\varepsilon^2}\left(H_\Theta(\varrho,\vartheta)-\frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho-r)-H_\Theta(r,\Theta)\right)\right]$$

Relative entropy inequality

$$\left[\mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon^{b-2} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \varepsilon^{a-c} \int_0^\tau \int_{\partial\Omega} \frac{\Theta \beta}{\vartheta} |\mathbf{u}|^2 \, \mathrm{dS}_x \, \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}_\varepsilon(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \, \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

$$\boxed{\mathcal{R}_\varepsilon(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_0^\tau \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$+\varepsilon^{a-c} \int_0^\tau \int_{\partial\Omega} \beta \mathbf{u} \cdot \mathbf{U} \, \mathrm{dS}_x \, \mathrm{d}t$$

$$+\frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \, \mathrm{d}x \mathrm{d}t$$

$$-\frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \Big(\partial_t \Theta + \mathbf{u} \cdot \nabla_x \Theta \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$+\varepsilon^{b-2} \int_0^\tau \int_\Omega \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \, \mathrm{d}x \mathrm{d}t$$

$$+\frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \, \mathrm{d}x \, \mathrm{d}t$$

Initial data and far field behavior

Initial data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \boxed{\varrho^{(1)}_{0,\varepsilon}},\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon \boxed{\vartheta^{(1)}_{0,\varepsilon}},\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon} \to \mathbf{v}_0 \text{ in } L^2(\Omega;R^3),\ \mathbf{v}_0 \in W^{k,2}(\Omega;R^3),\ k > \frac{5}{2}$$

Far field

$$\varrho \to \overline{\varrho},\ \vartheta \to \overline{\vartheta},\ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

$$b > 0,\ 0 < c < a < \frac{10}{3},$$

Asymptotic incompressibility

$$\mathrm{ess}\sup_{t\in(0,T)} \| \varrho_\varepsilon(t,\cdot) - \overline{\varrho} \|_{L^2+L^{5/3}(\Omega)} \leq \varepsilon c$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\ \mathbf{v} \text{ in } \boxed{L^\infty_{\rm loc}((0,T];L^2_{\rm loc}(\Omega;R^3))}$$

and weakly-(*) in $L^\infty(0,T;L^2(\Omega;R^3))$

Temperature deviation

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \mathcal{T} \text{ in } \boxed{L^\infty_{\rm loc}((0,T];L^q_{\rm loc}(\Omega;R^3)),\ 1 \leq q < 2},$$

and weakly-(*) in $L^\infty(0,T;L^2(\Omega))$

Uniform bounds

The uniform bounds independent of ε are obtained by taking

$$r = \overline{\varrho},\ \Theta = \overline{\vartheta},\ \mathbf{U} = 0$$

in the relative entropy inequality

Energy bounds

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2 + L^{5/3}(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \right\|_{L^2(\Omega; R^3)} \leq c$$

Dissipation

Integral bounds

$$\varepsilon^{a} \int_0^T \int_\Omega \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c$$

$$\varepsilon^{a-c} \int_0^T \int_{\partial\Omega} |\mathbf{u}_\varepsilon|^2 \ \mathrm{dS}_x \ \mathrm{d}t \leq c$$

$$\varepsilon^{b-2} \int_0^T \int_\Omega |\nabla_x \vartheta_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c$$

First order approximation

Linearization

$$\varepsilon \partial_t \boxed{\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right)} + \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\varepsilon \partial_t (\varrho_\varepsilon \mathbf{u}_\varepsilon) + \nabla_x \left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} \right) = \mathbf{f}_{1,\varepsilon}$$

$$\partial_t \left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right)$$

$$+ \mathrm{div}_x \left[\left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right) \mathbf{u}_\varepsilon \right] = f_{2,\varepsilon}$$

Another application of the relative entropy inequality

Take

$$r_\varepsilon = \overline{\varrho} + \varepsilon R_\varepsilon, \ \Theta_\varepsilon = \overline{\vartheta} + \varepsilon T_\varepsilon, \ \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality

Acoustic equation

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0$$

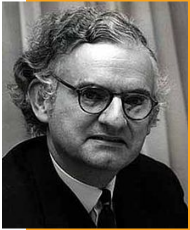
$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon) = 0$$

Transport equation

$$\partial_t (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) + (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0$$

Wave equation

$$\varepsilon \partial_t Z + \Delta \Phi = 0, \ \varepsilon \partial_t \Phi + Z = 0,$$



Michael James Lighthill [1924-1998]

Neumann boundary condition

$$\nabla_x \Phi \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

Initial conditions

$$\Phi(0\cdot) = \Phi_0, \ Z(0,\cdot) = Z_0,$$

Domain of definition

$$\mathcal{D}(-\Delta_N) = \left\{ w \in W^{1,2}(\Omega) \ \middle| \ \int_\Omega \nabla_x w \cdot \nabla_x \phi \ \mathrm{d}x = \int_\Omega g \phi \ \mathrm{d}x \right.$$

$$\left. \text{for a certain } g \in L^2(\Omega) \text{ and all } \phi \in C_c^\infty(\overline{\Omega}) \right\}$$

$$-\Delta_N w = g.$$

Neumann Laplacean

The Neumann Laplacean $-\Delta_N$ is a non-negative self-adjoint operator on the Hilbert space $L^2(\Omega)$

Duhamel's formula

Acoustic potential

$$\Phi(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 - \frac{\mathrm{i}}{\sqrt{-\Delta_N}}Z_0\right] + \frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 + \frac{\mathrm{i}}{\sqrt{-\Delta_N}}Z_0\right]$$

Jean-Marie Constant Duhamel [1797-1872]

Time derivative

$$Z(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\mathrm{i}\sqrt{-\Delta_N}[\Phi_0] + Z_0\right] + \frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[-\mathrm{i}\sqrt{-\Delta_N}[\Phi_0] + Z_0\right]$$

Strichartz estimates

$$\int_{-\infty}^{\infty} \left\| \exp\left(\pm \mathrm{i} \sqrt{-\Delta} t \right) [h] \right\|_{L^q(R^3)}^p \ \mathrm{d}t \leq \|h\|_{H^{1,2}(R^3)}^p$$

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty$$

Local energy decay

$$\int_{-\infty}^{\infty} \left\| \chi \exp\left(\pm \mathrm{i} \sqrt{-\Delta} t \right) [h] \right\|_{H^{\alpha,2}(R^3)}^2 \ \mathrm{d}t \leq c(\chi) \|h\|_{H^{\alpha,2}(R^3)}^2$$

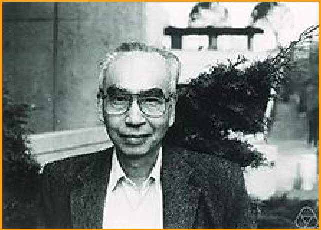
$$\alpha \leq \frac{3}{2}, \ \chi \in C_c^\infty(R^3)$$

Limiting absorption principle

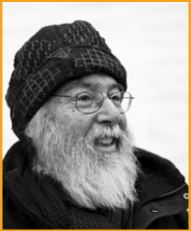
The cut-off resolvent operator

$$(1+|x|^2)^{-s/2} \circ [-\Delta_N - \mu \pm \mathrm{i}\delta]^{-1} \circ (1+|x|^2)^{-s/2}, \ \ \delta > 0, \ \ s > 1$$

can be extended as a bounded linear operator on $L^2(\Omega)$ *for* $\delta \to 0$ *and* μ *belonging to compact subintervals of* $(0, \infty)$.



Tosio Kato [1917-1999]



Robert S. Strichartz

Kato's theorem

Theorem

Let A be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R,\ v \in \mathcal{D}(A^*),\ \|v\|_X = 1} \| A \circ R_H[\lambda] \circ A^*[v] \|_X < \infty.$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \| A \exp(-\mathrm{i}tH)[w] \|_X^2 \ \mathrm{d}t \leq \Gamma^2.$$

Frequency localized local decay estimates

Decay estimates

$$\int_0^\infty \left\| \chi G(-\Delta_N) \exp\left(\pm \mathrm{i} \sqrt{-\Delta_N} t \right) [h] \right\|_{L^2(\Omega)}^2 \ \mathrm{d}t \leq c \|h\|_{L^2(\Omega)}^2$$

Scaled estimates

$$\int_0^T \left\| \chi G(-\Delta_N) \exp\left(\pm \mathrm{i} \sqrt{-\Delta_N} \frac{t}{\varepsilon} \right) [h] \right\|_{L^2(\Omega)}^2 \ \mathrm{d}t \leq \varepsilon c(T) \|h\|_{L^2(\Omega)}^2$$

$$G \in C_c^\infty(0, \infty), \ \chi \in C_c^\infty(\overline{\Omega})$$

Admissible domains

Limiting absorption principle

The operator Δ_N satisfies the limiting absorption principle in Ω.

Strichartz estimates on "larger" domain

There is a domain such that $D \cap \{|x| > R\} = \Omega \cap \{|x| > R\}$ and Δ_N satisfies the Strichartz estimates in D.

Local decay on "larger" domain

The operator Δ_N satisfies the local energy decay estimates in D.

Frequency localized Strichartz estimates

$$\int_{-\infty}^{\infty} \left\| G(-\Delta_N) \exp\left(\pm \mathrm{i}\sqrt{-\Delta_N}\, t\right)[h] \right\|_{L^q(\Omega)}^p \leq c(G) \|h\|_{L^2(\Omega)}^p$$

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ \ q < \infty, \ \ G \in C_c^\infty(0, \infty)$$