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Abstract

We study the separable complementation property (SCP) and its natural variations in Banach spaces of continuous functions over compacta $K_{\mathcal{A}}$ induced by almost disjoint families $\mathcal{A}$ of countable subsets of uncountable sets. For these spaces, we prove among others that $C(K_{\mathcal{A}})$ has the controlled variant of the separable complementation property if and only if $C(K_{\mathcal{A}})$ is Lindelöf in the weak topology if and only if $K_{\mathcal{A}}$ is monolithic. We give an example of $\mathcal{A}$ for which $C(K_{\mathcal{A}})$ has the SCP, while $K_{\mathcal{A}}$ is not monolithic and an example of a space $C(K_{\mathcal{A}})$ with controlled and continuous SCP which has neither a projectional skeleton nor a projectional resolution of the identity. Finally, we describe the structure of almost disjoint families of cardinality ω_1

which induce monolithic spaces of the form $K_{\mathcal{A}}$: They can be obtained from countably many ladder systems and pairwise disjoint families applying simple operations.

Contents

1 Introduction

In this paper we deal with Banach spaces of real-valued continuous functions defined on a compact Hausdorff space K with the supremum norm and denoted $C(K)$. Actually, the compact spaces we consider are induced by a combinatorial object—an almost disjoint family. Recall that a family of infinite sets $\mathcal{A}$ is *almost disjoint* if $A \cap B$ is finite whenever $A, B \in \mathcal{A}$ are different. Additionally, we shall only consider almost disjoint families of countable sets. Almost disjoint families of $\mathbb{N}$ have found many applications in the theory of Banach spaces, some classical cases include Whitley's short proof of Philips' theorem in [28], Haydon's constructions of Grothendieck spaces in [12], or Johnson and Lindenstrauss' counterexamples concerning weakly compactly generated Banach spaces in [14]. We, however, will focus on almost disjoint families of countable subsets of uncountable sets. Several of such families induce Banach spaces which could also be considered classical (counter)-examples like the Ciesielski-Pol spaces (see [4, 5]) or continuous functions on the ladder system spaces ([23], [4]).

Let us be more precise on how we obtain a compact space and then the Banach space from an almost disjoint family. Fix a family $\mathcal{A}$ of countable almost disjoint sets with $X = \bigcup \mathcal{A}$. We define a compact space $K_{\mathcal{A}}$ as follows. The points of $K_{\mathcal{A}}$ are

$$X \cup \{l_A \colon A \in \mathcal{A}\} \cup \{\infty\},$$

where each point of X is declared to be isolated, a basic neighborhood of l_A is of the form $\{l_A\} \cup (A \setminus F)$ where F is finite, and a basic neighborhood of ∞ is of the form

$$K_{\mathcal{A}} \setminus \bigcup_{i<n}(\{l_{A_i}\} \cup A_i)$$

where $\{A_i : i < n\} \subseteq \mathcal{A}$. The space $K_{\mathcal{A}}$ is compact scattered of height 3 with ∞ being the unique point at the maximal level. Furthermore, $K_{\mathcal{A}} \setminus \{\infty\}$ is first countable iff $\mathcal{A}$ consists of countable sets. In order to shorten some expressions, given $\mathcal{B} \subseteq \mathcal{A}$, we shall write $l_{\mathcal{B}}$ for the set $\{l_B : B \in \mathcal{B}\}$. For countable X these spaces can be traced to Alexandroff and Urysohn, and later were known as Mrówka's spaces, Ψ-spaces or Isbell spaces. One could also note that $K_{\mathcal{A}}$ is the Stone space of the Boolean algebra generated by $\mathcal{A}$ and the finite subsets of $\bigcup \mathcal{A}$.

We investigate the Banach spaces of the form $C(K_{\mathcal{A}})$ focusing on how some aspects of their geometry depend on the topological properties of $K_{\mathcal{A}}$ which in turn depend on the combinatorial properties of $\mathcal{A}$. To be more precise, we investigate the family of complemented separable subspaces of $C(K_{\mathcal{A}})$ depending on $\mathcal{A}$. The links between almost disjoint families and projections can be traced back to [28]. Recall that a subspace F of a Banach space X is *complemented* if there exists a bounded linear operator (called *projection*) $P : X \to X$ satisfying $P^2 = P$ and $\operatorname{im} P = F$, where $\operatorname{im} P$ denotes the image of P. We consider several properties of Banach spaces in terms of the structure of some families of projections on separable subspaces ranging from the existence of unconditional basis to the existence of projectional skeletons. Let us recall some of these properties here:

Definition 1.1. Let X be a Banach space.

1. X has the *separable complementation property* (SCP) if every separable subspace of X is contained in a separable complemented subspace.

2. X has the *controlled separable property* (briefly: *controlled SCP*) if for every separable subspace $F \subseteq X$ and for every separable subspace $G \subseteq X^*$ there exists a projection $P : X \to X$ such that $F \subseteq \operatorname{im} P$, $G \subseteq \operatorname{im} P^*$ and $\operatorname{im} P$ is separable.

3. X has the *continuous separable complementation property* (briefly: *continuous SCP*) if there is a family $\mathcal{F}$ of separable complemented subspaces of X such that $X = \bigcup \mathcal{F}$, $\mathcal{F}$ is up-directed and for every sequence $F_0 \subseteq F_1 \subseteq \ldots$ in $\mathcal{F}$ the closure of the union $\bigcup_{n \in \omega} F_n$ is in $\mathcal{F}$. A family $\mathcal{F}$ as above is called a skeleton of complemented subspaces of X.

4. X has a *projectional skeleton* if it has a skeleton as above such that there exist projections $P_E : X \to E$ which satisfy $P_E \circ P_F = P_F \circ P_E = P_E$ whenever $E \subseteq F$ with $E, F \in \mathcal{F}$.

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The SCP has appeared in the classical work of Amir and Lindenstrauss ([3]) where it is proved that weakly compactly generated Banach spaces share this property and it motivated some main lines of research concerning non-separable Banach spaces. Probably the widest well-investigated, at the present moment, class of Banach spaces with this property is the class of weakly Lindelöf determined spaces ([11]). It is closed under taking quotients and subspaces, however SCP is not inherited by these operations. This follows from a recent example of Figiel, Johnson and Pełczyński ([10]) for subspaces and from a simple observation that $l_1(\kappa)$ has the SCP for any cardinal κ. The terminology concerning the controlled SCP and the continuous SCP is taken from [29] and [15] respectively, however these properties were considered in the literature earlier (e.g. 4.4. [24], 6, 7 [20]); controlled SCP has been recently in [8, 9]; projectional skeletons were introduced and studied in [21]. By definitions, the SCP is implied by the other properties, and the existence of a projectional skeleton implies the continuous SCP. The remaining implications are not valid even for $C(K)$ spaces for quite particular classes. This follows from our examples (1.8, 1.6) as well as from the main examples of [15, 20]. One of the simplest examples of a $C(K)$ space with a projectional skeleton and without the controlled SCP comes by taking $K = \{0, 1\}^{\omega_1}$. It is well known that this K belongs to the class of Valdivia compact spaces and therefore $C(K)$ has a projectional skeleton (see, e.g., [16, 21]). On the other hand, K is not monolithic, so $C(K)$ cannot have the controlled SCP.

Perhaps at the beginning of the discussion of the results of our paper we should mention a result proved at the end of Section 2:

Theorem 1.2. *Suppose that $\mathcal{A}$ is an almost disjoint family of countable sets. $C(K_{\mathcal{A}})$ has a continuous SCP if and only if it has the SCP.*

In Section 2 we analyze the simplest case of an almost disjoint family, namely, a *disjointifiable* family, in other words, a family that is equivalent (modulo finite sets) to a pairwise disjoint one (see Definition 2.1). It turns out that in this trivial case corresponding to weakly compactly generated Banach space we have unconditional basis and hence the richest family of commuting projections. We obtain:

Theorem 1.3. *Suppose $\mathcal{A}$ is an almost disjoint family of countable sets, then the following are equivalent:*

1. *$\mathcal{A}$ is disjointifiable,*

2. *$K_{\mathcal{A}}$ is an Eberlein compact,*

3. *$C(K_{\mathcal{A}})$ is isomorphic to $c_0(\kappa)$ for $\kappa = |\mathcal{A}| + \omega$,*

4. *$C(K_{\mathcal{A}})$ has an unconditional basis,*

5. *$C(K_{\mathcal{A}})$ has a projectional skeleton.*

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Proof. Apply 2.3, 2.4, 2.6. □

In Section 3 we consider the case corresponding to Banach spaces which are Lindelöf in the weak topology. We obtain the following:

Theorem 1.4. *Suppose $\mathcal{A}$ is an almost disjoint family of countable sets, then the following are equivalent*

1. $\mathcal{A}$ *is locally small,*

2. $K_{\mathcal{A}}$ *is monolithic,*

3. $C(K_{\mathcal{A}})$ *has the controlled SCP,*

4. $C(K_{\mathcal{A}})$ *is Lindelöf in the weak topology,*

5. $B_{C(K_{\mathcal{A}})^*}$ *is monolithic in the weak* topology.*

Proof. Apply 3.2 and 3.5. □

A locally small almost disjoint family is defined in 3.1, a compact space K is *monolithic* if all of its separable subsets are second countable. In case K is scattered, this just means that separable subsets are countable. The notion of a monolithic space is due to Arkhangel'skii (see [2]). A monolithic space is often called ω-*monolithic*, since this notion clearly generalizes to higher cardinals. Typical examples of monolithic spaces are *Corson compacta*, namely, compact subsets of Σ-products $\Sigma(\Gamma)$, where $\Sigma(\Gamma)$ is the subspace of $\mathbb{R}^{\Gamma}$ consisting of all functions $x\colon \Gamma \to \mathbb{R}$ whose support $\operatorname{supp}(x) = \{\alpha\colon x(\alpha) \neq 0\}$ is countable. It is obvious that the closure of every countable subset of $\Sigma(\Gamma)$ is compact and second countable (contained in $\mathbb{R}^S$ for some countable $S \subseteq \Gamma$).

The above results and our examples 1.8 and 1.6 decide completely which implications hold among the properties of Definition 1.1 namely (1)⇔(3), (4)⇒(3), (2)⇒(1) and (4)⇒(2). The last implication follows from a well known fact that Eberlein compacta are monolithic, being Corson compact.

In general the controlled SCP does not imply the Lindelöf property in the weak topology. To see this, take any non-metrizable monolithic compact linearly ordered space K (e.g., K could be a compact Aronszajn line) and apply Nakhmanson's theorem [22], which says that for a linearly ordered compact K the Lindelöf property of $C(K)$ implies that K is second countable; by [15, Thm. 1.6], $C(K)$ has the controlled SCP. It is interesting that a sufficient condition given in [7, Thm. 12.33] for the Lindelöf property due to Orihuela has a flavor of the controlled SCP.

In Section 4 we obtain some structural results concerning locally small almost disjoint families of countable sets, of cardinality ω_1, to express them we need the notion of a ladder system (see the beginning of Section 4) and we need to introduce

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some simple definition: Suppose $\mathcal{A} \cup \mathcal{B}$ is an almost disjoint family $\mathcal{A}' \subseteq \mathcal{A}$ and $\varphi : \mathcal{A}' \to \mathcal{B}$ is an injection. Then

$$(\mathcal{A} \setminus \mathcal{A}') \cup \{A' \cup \varphi(A') : A' \in \mathcal{A}'\} \cup (\mathcal{B} \setminus \varphi[\mathcal{A}'])$$

is denoted $\mathcal{A} +_\varphi \mathcal{B}$. The main result of Section 4 is the following structural theorem:

Theorem 1.5. *Suppose that $\mathcal{A}$ is a locally small almost disjoint family of countable sets of cardinality ω_1. Then $\mathcal{A}$ is equivalent to the family of the form*

$$\bigcup_{n \in \mathbb{N}} (\mathcal{A}_n +_{\varphi_n} \mathcal{B}_n)$$

where all $\mathcal{A}_n$s are ladder systems, all $\mathcal{B}_n$s are disjointifiable and φ_ns are injections defined on subfamilies of $\mathcal{A}_n$s.

In other words, locally small almost disjoint families are obtained in a canonical way from countably many ladder systems and pairwise disjoint families. As noted at the end of Section 4, if all the families $\mathcal{A}_n$ are disjointifiable the $\mathcal{A}$ is disjointifiable as well. One should note that the structure of non locally small almost disjoint families of sets, even in the case of subsets of $\mathbb{N}$ is a much harder task with extensive literature devoted to it.

Analyzing Banach spaces induced by the ladder system spaces in Section 4, we obtain the following

Theorem 1.6. *There is an almost disjoint family $\mathcal{A}$ such that $C(K_\mathcal{A})$ has controlled SCP and continuous SCP but it has neither a PRI nor a projectional skeleton.*

Actually the above result provides the first example of a Banach space with the continuous SCP, not isomorphic to a space with a PRI. This question was raised in [20] and was motivated by an observation that in a Banach space of density $\aleph_1$, a skeleton of 1-complemented separable subspaces naturally leads to a PRI. We should note here that in general the existence of a PRI in the spaces $C(K_\mathcal{A})$ does not imply even the SCP:

Example 1.7. Let $\mathcal{B} \subseteq [\omega]^\omega$ be an almost disjoint family of size ω_1 and let $\mathcal{A} \subseteq [\omega_2 \setminus \omega]^\omega$ be a disjoint family of size ω_2. Let $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. The space $C(K_\mathcal{B})$ is a complemented subspace of $C(K_\mathcal{C})$ and the canonical copy of c_0 is not complemented in $C(K_\mathcal{B})$ (see the proof of Proposition 6.1 for details). Thus, $C(K_\mathcal{C})$ fails the SCP. On the other hand, it is easy to construct a PRI on $C(K_\mathcal{C})$. In fact, let $\{P_\alpha\}_{\alpha < \omega_2}$ be a PRI on $C(K_\mathcal{A})$ and, knowing that $C(K_\mathcal{C}) = C(K_\mathcal{A}) \oplus C(K_\mathcal{B})$, extend P_{ω_1} by enlarging its kernel to $\ker P_{\omega_1} \oplus C(K_\mathcal{B})$. Finally, replace each P_α for $\alpha > \omega_1$ by the appropriate projection onto $\operatorname{im} P_\alpha \oplus C(K_\mathcal{B})$.

Having in mind the results of Section 3, the only property from 1.1 which can hold for non-monolithic $K_\mathcal{A}$s is the SCP (and equivalent continuous SCP). It turns out that such an example exists, its construction is the subject matter of Section 6:

Theorem 1.8. *There is an almost disjoint family $\mathcal{A}$ which is not locally small, i.e., none of the conditions of 1.4 like the controlled SCP are satisfied by $C(K_{\mathcal{A}})$, yet the space $C(K_{\mathcal{A}})$ has the SCP.*

This is a different behavior than for linearly ordered compact spaces K, as it was proved in [15] that for these $C(K)$s the SCP is equivalent to controlled SCP and to the monolithicity of K (Theorem 1.6 of [15]). Also this cannot happen if K is separable because then $\bigcup \mathcal{A}$ is countable and we are in the realm of subsets of $\mathbb{N}$ (6.1).

In the different direction from our present research is the question how few complemented separable subspaces could appear in the space of the form $C(K_{\mathcal{A}})$. Of course then we focus on spaces without SCP. Some separable subspaces must be complemented as $K_{\mathcal{A}}$s have nontrivial convergent sequences because they are scattered, so we have many complemented copies of c_0. In [17] the second author assuming the continuum hypothesis constructed an almost disjoint family such that the only decompositions of $C(K_{\mathcal{A}}) = E \oplus F$ are for $E \sim c_0$ and $F \sim C(K_{\mathcal{A}})$ or vice versa. This space does not have SCP nor is weakly Lindelöf.

We shall use standard set-theoretic notation. For example, an ordinal α will be treated as the interval of ordinals $[0, \alpha)$. According to this notation, the set of natural numbers is ω. The symbol $\approx$ will denote the relation of being homeomorphic (when dealing with compact spaces) and $\equiv$ and $\sim$ the relations of being isometric or isomorphic as Banach spaces, respectively. $|X|$ denotes the cardinality of X. The symbol $\triangle$ denotes the symmetric difference, i.e., $A \triangle B = (A \setminus B) \cup (B \setminus A)$. At some point we shall use Fodor's Pressing Down Lemma, so we state it below for the sake of completeness.

Let κ be a regular cardinal. Recall that a set $A \subseteq \kappa$ is *closed* if $\sup C \in A$ whenever $C \subseteq A \cap \alpha$ for some $\alpha < \kappa$ and A is *unbounded* if $\sup A = \kappa$. A closed and unbounded set is called briefly a *club*. A set S is *stationary* in κ if $S \cap A \neq \emptyset$ whenever $A \subseteq \kappa$ is closed and unbounded. All these notions, as well as the proof of Fodor's Lemma, can be found, e.g., in [13].

Lemma 1.9 (Fodor's Pressing Down Lemma)**.** *Let κ be an uncountable regular cardinal and let $S \subseteq \kappa$ be a stationary set. Let $f \colon S \to \kappa$ be a regressive function, that is, $f(\xi) < \xi$ for every $\xi \in S$. Then there exists a stationary set $T \subseteq S$ such that $f \restriction T$ is constant.*

We shall use Fodor's Lemma for $\kappa = \omega_1$ only. Roughly speaking, when dealing with a family of (usually countable) structures living in ω_1 and indexed by countable ordinals, the Pressing Down Lemma allows us to refine the family (still having an uncountable one) fixing finitely many parameters defined as regressive functions.

2   Weakly compactly generated spaces $C(K_{\mathcal{A}})$

Definition 2.1. We say that two almost disjoint families of countable sets $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if the Boolean algebra generated by $\mathcal{A}$ and finite subsets of $\bigcup \mathcal{A}$ is

isomorphic to the Boolean algebra generated by $\mathcal{B}$ and the finite subsets of $\bigcup \mathcal{B}$. We say that an almost disjoint family of countable sets is *disjointifiable* if $\mathcal{A}$ is equivalent to a pairwise disjoint family.

Note that if i is a Boolean isomorphism as in the definition above, then i is a bijection between $\mathcal{A}$ and $\mathcal{B}$ as well as i induces a bijection between $\bigcup \mathcal{A}$ and $\bigcup \mathcal{B}$.

Lemma 2.2. *Suppose that $\mathcal{A}$ is an almost disjoint family of countably infinite sets.*

(1) *$\mathcal{A}$ is disjointifiable if and only if for every $A \in \mathcal{A}$ there is a finite set F_A such that $\{A \setminus F_A : A \in \mathcal{A}\}$ is a pairwise disjoint family.*

(2) *$\mathcal{A}$ and $\mathcal{B}$ are equivalent, if and only if $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$ are homeomorphic.*

(3) *If $\mathcal{A}$ is point countable, then it is disjointifiable.*

(4) *Suppose $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ is an almost disjoint family and each subfamily $\mathcal{A}_n$ is disjointifiable, then $\mathcal{A}$ is disjointifiable.*

Proof. (1) Suppose $i \colon K_{\mathcal{A}} \to K_{\mathcal{B}}$ is a homeomorphism. Then i induces a bijection from $\mathcal{A}$ onto $\mathcal{B}$ so that $i(A) \triangle A$ is finite for every $A \in \mathcal{A}$. Thus, it is obvious that a disjointifiable family can be "corrected" to a disjoint family by subtracting finite sets. Now suppose $\mathcal{A} = \{A_\alpha\}_{\alpha < \kappa}$ is such that $\mathcal{B} = \{A \setminus F_\alpha\}_{\alpha < \kappa}$ is a disjoint family, where $F_\alpha \subseteq A_\alpha$ is finite for each $\alpha < \kappa$. By induction, we can decrease each F_α so that the family remains disjoint. By this way, we shall get $\bigcup \mathcal{B} = \bigcup \mathcal{A}$, that is, the spaces $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$ have the same sets of isolated points. There is a unique continuous map $i \colon K_{\mathcal{A}} \to K_{\mathcal{B}}$ which is identity on the set of isolated points. It is clear that i is a homeomorphism.

(2) This is the Stone duality.

(3) For $A, A' \in \mathcal{A}$ we say that $A \sim A'$ if and only if there are $A = A_0, ..., A_n = A'$ such that $A_i \cap A_{i+1} \neq \emptyset$ for each $0 \leq i < n$. The point countability of $\mathcal{A}$ implies that the equivalence classes of $\sim$ are countable and have disjoint unions. As countable families are disjointifiable, we conclude that $\mathcal{A}$ is disjointifiable.

(4) We may assume that the families $\mathcal{A}_n$ do not have elements in common. Each of $\mathcal{A}_n$'s is equivalent to a pairwise disjoint family $\mathcal{B}_n$, so $\bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ is point countable, and so by (3) it is equivalent to a pairwise disjoint family. It is clear that $\bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ is equivalent to $\mathcal{A}_n$ by (1).

$\square$

Proposition 2.3. *$K_{\mathcal{A}}$ is an Eberlein compact if and only if $\mathcal{A}$ is disjointifiable.*

Proof. We will use Rosenthal's characterization of Eberlein compacta which says that K is an Eberlein compactum if and only if it has a point separating σ-point finite family of open F_σ sets (see [26], [16, Thm. 19.30] or [7, Thm. 12.13]). It is clear that in the case of a totally disconnected space we may assume that the F_σ sets are clopen.

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If $\mathcal{A}$ is disjointifiable, then $K_\mathcal{A}$ is homeomorphic to $K_\mathcal{B}$ for $\mathcal{B}$ pairwise disjoint. Let $\varphi_B : \mathbb{N} \to [B]^{<\omega}$ be an enumeration of all finite subsets of B for $B \in \mathcal{B}$. If we define $\mathcal{B}_n = \{B \setminus \varphi(n) : B \in \mathcal{B}\}$ we obtain another pairwise disjoint family. These families together with the family of all one element subsets of $\bigcup \mathcal{B}$ form the required point separating σ-point finite family of clopen sets.

Now suppose that $\mathcal{F}_n$ are point separating σ-point finite family of clopen sets of $K_\mathcal{A}$. Only countably many of these sets may contain ∞ and those sets miss only finitely many points of the form l_A for $A \in \mathcal{A}$. So by 2.2(3) it is enough to use 2.2(4) to conclude that $\mathcal{A}$ is disjointifiable. $\qquad\square$

Assume $\mathcal{A}$ is a disjoint family of countably infinite sets and $|\mathcal{A}| = \kappa \geqslant \omega$. Then $K_\mathcal{A}$ is naturally homeomorphic to

$$K_{\langle \kappa, \omega \rangle} := \{\emptyset\} \cup [\kappa]^1 \cup \{\{\alpha, \kappa + n\} : \alpha \in \kappa, n \in \omega\},$$

endowed with the Cantor cube topology inherited from $\wp(\kappa) \approx 2^\kappa$. Thus, given an infinite cardinal κ, there exists a unique topological type of a compact space $K_\mathcal{A}$ coming from an almost disjoint family of countable sets such that $|\mathcal{A}| = \kappa$.

Proposition 2.4. *If $\mathcal{A}$ is an almost disjoint family of countable sets such that $K_\mathcal{A}$ is Eberlein compact, then $C(K_\mathcal{A}) \sim c_0(\kappa)$ for $\kappa = |\mathcal{A}| + \omega$, in particular $C(K_\mathcal{A})$ has an unconditional basis.*

Proof. We may assume that $\mathcal{A}$ is pairwise disjoint. Consider

$$X = \{f \in C(K_\mathcal{A}) : f(l_A) = 0 = f(\infty) \text{ for each } A \in \mathcal{A}\}.$$

Define $P : C(K_\mathcal{A}) \to X$ by $P(f)(x) = f(x) - f(l_A)$ where $x \in \bigcup \mathcal{A}$ and A is the unique element of $\mathcal{A}$ such that $x \in A$. It is easy to check that P is a projection onto X. But $X \equiv c_0(\bigcup \mathcal{A})$ and $C(K_\mathcal{A}/X) \sim c_0(|\mathcal{A}|) \oplus \mathbb{R}$ which completes the proof of the proposition. $\qquad\square$

Lemma 2.5. *Suppose that $\mathcal{A}$ is an almost disjoint family of countable sets, then $[0, \omega_1]$ is not a subspace of the dual ball $B_{C(K_\mathcal{A})}$ with the weak* topology.*

Proof. Suppose that $\varphi : [0, \omega_1] \to B_{C(K_\mathcal{A})}$ is a homeomorphic embedding. As convergence in the dual norm implies the weak* convergence, we may cover the copy of $[0, \omega_1)$ by norm-open sets U_α such that

$$\varphi(\alpha) \in \varphi[[0, \omega_1)] \cap U_\alpha \subseteq \varphi[[0, \alpha]].$$

Let $\mathcal{V}$ be a locally point-finite refinement of U_αs which exists as metrizable spaces are paracompact ([6]). Whenever there is $\beta < \alpha$ and $V \ni \varphi(\alpha)$ such that $\varphi(\beta) \in V \in \mathcal{V}$ let $f(\alpha)$ be one of them. If the set of such αs were stationary, by the Pressing Down Lemma, f would be constant on an uncountable set which would contradict the point-finiteness of $\mathcal{V}$. Hence, for a club set of αs there is no such β and so such

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$\varphi(\alpha)$ can be separated by a norm open set from $\varphi(\beta)$s for $\beta < \alpha$. $\varphi(\alpha)$ can also be separated by a norm open set from the $\varphi(\beta)$s for $\beta > \alpha$. As club subsets of ω_1 are homeomorphic to ω_1, we may assume that $\varphi([0, \omega_1))$ is norm discrete. By the fact that the ideal of non-stationary subsets of ω_1 is σ-complete (see [13]), we can conclude that there is a stationary set $S \subseteq \omega_1$ such that there is $\delta > 0$ such that for every distinct $\alpha, \beta \in S$ we have $||\varphi(\alpha) - \varphi(\beta)|| > \delta$, where the norm is the dual norm in $C(K_{\mathcal{A}})^*$. It is well known that stationary sets contain closed subsets of arbitrarily big countable order type. This implies that for every countable ordinal α the ball $B_{C(K_{\mathcal{A}})}$ with the weak* topology contains homeomorphic copies K_α of ordinal intervals $[0, \alpha]$ such that the distance between any two points is not smaller than δ. Given $\alpha < \omega_1$, as $[0, \alpha]$ is countable and supports of the measures corresponding to the elements of the dual space are countable, there is a countable $\mathcal{B} \subseteq \mathcal{A}$ such that taking the restriction of elements of K_α to

$$X(\mathcal{B}) = \{1_x : x \in \bigcup \mathcal{B}\} \cup \{1_{A \cup \{l_A\}} : A \in \mathcal{B}\},$$

gives an injective weak* continuous function, and so a homeomorphism onto its image. Moreover, we may assume that the norms of elements of K_α do not change when we take the restrictions. This leads to a contradiction as $X(\mathcal{B})$ is isomorphic to c_0 by Proposition 2.4 and so the Szlenk index of it is ω ([25, Prop. 2.27]). By [25, Cor. 2.20], the Szlenk index must be bigger than the Cantor-Bendixson height of all countable ordinal intervals, once we have the above δ-separated copies of them in the dual ball of $X(\mathcal{B})$. This completes the proof of the lemma. □

Proposition 2.6. *If $C(K_{\mathcal{A}})$ has a projectional skeleton, then $K_{\mathcal{A}}$ is an Eberlein compact.*

Proof. Assume $C(K_{\mathcal{A}})$ has a projectional skeleton $\mathcal{P}$. There is a renorming of $C(K_{\mathcal{A}})$ for which $\mathcal{P}$ becomes a 1-projectional skeleton (see [21, Cor. 16]). Let G be the dual unit ball with respect to this renorming. Then G belongs to the class $\mathcal{R}$ introduced in [19]. By [19, Thm. 4.3] a member of class $\mathcal{R}$ is either a Corson compact or contains a copy of $[0, \omega_1]$ which is excluded by Lemma 2.5, because G is contained in a multiple of the standard dual ball. Since $K_{\mathcal{A}}$ is contained in a multiple of G, it follows that $K_{\mathcal{A}}$ is Corson. But $K_{\mathcal{A}}$ is scattered, and hence Eberlein by Alster's theorem [1] which says that a scattered Corson compact space is Eberlein compact. □

Proof of Theorem 1.2. First, recall Sobczyk's Theorem [27]: Every isomorphic copy of c_0 is complemented in every separable Banach space. It follows that, if $C(K_{\mathcal{A}})$ has SCP, then every isomorphic copy of c_0 in $C(K_{\mathcal{A}})$ is complemented. Consider the subspaces $X(\mathcal{B})$ which are closures of linear spaces generated by constant functions and by

$$\{1_x : x \in \bigcup \mathcal{B}\} \cup \{1_{A \cup \{l_A\}} : A \in \mathcal{B}\},$$

where $\mathcal{B}$ is a countable subset of $\mathcal{A}$. It is enough to note that

$$\{X(\mathcal{B}) : \mathcal{B} \text{ is a countable subset of } \mathcal{A}\}$$

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forms a skeleton of separable complemented subspaces. They are complemented being isomorphic to spaces $C(K_\mathcal{B})$ respectively, and so isomorphic to c_0 by Proposition 2.4 since $K_\mathcal{B}$ is metrizable, and so Eberlein compact. $\square$

3   Weakly Lindelöf spaces of the form $C(K_\mathcal{A})$

Definition 3.1. We say that an almost disjoint family of countable sets $\mathcal{A}$ is *locally small* if

$$\{A \in \mathcal{A} : A \cap B \text{ is infinite}\}$$

is at most countable for each countable $B \subseteq \bigcup \mathcal{A}$.

Proposition 3.2. $K_\mathcal{A}$ *is monolithic if and only if* $\mathcal{A}$ *is locally small if and only if* $C(K_\mathcal{A})$ *is Lindelöf in the weak topology.*

Proof. The first equivalence is immediate from the definitions.

   If $K_\mathcal{A}$ is not monolithic, then $C(K_\mathcal{A})$ is not Lindelöf in the weak topology by Lemma 2.2. of [18] which is based on Theorem 2 of [23]. The shortest way of proving the other implication is to use forcing method. Suppose $K_\mathcal{A}$ is monolithic but $C(K_\mathcal{A})$ is not Lindelöf in the weak topology. Let $\kappa = |\mathcal{A}|$. Let $\{V_\xi : \xi < \kappa\}$ be an open cover of $C(K_\mathcal{A})$ without a countable subcover. and let P be a forcing notion which collapses κ to ω_1 and is countably closed. Note that $K_\mathcal{A}$ is the same in the generic extension and so $\{V_\xi : \xi < \kappa\}$ remains an open cover of $C(K_\mathcal{A})$ without a countable subcover because P does not introduce any new countable sets. Moreover $\mathcal{A}$ remains locally small by the same argument and the absoluteness of intersections. In the generic extension we can however apply Theorem 2.5. of [18] which is based on the results of [23] whose generalization to higher cardinals seems at least tedious. Applying the above theorem, we obtain a contradiction. $\square$

   The following fact follows straight from the definition:

Proposition 3.3. *Let a Banach space* E *have the controlled SCP. Then the dual unit ball* $\overline{B}_{E^*}$ *with the weak star topology is monolithic.*

Proof. Fix a countable set $B \subseteq \overline{B}_{E^*}$ and choose a projection $P \colon E \to E$ such that PE is separable and $B \subseteq P^*E^*$. Then $\operatorname{cl} B$ is contained in the second countable space P^*E^* considered with the weak star topology. $\square$

   It is not known whether monolithicity of the dual unit ball (perhaps plus some extra property of the Banach space) implies the controlled SCP. We prove such a result for $C(K_\mathcal{A})$ spaces.

Lemma 3.4. *Let* $\mathcal{A}$ *be an almost disjoint family of countable sets with* $X = \bigcup \mathcal{A}$ *and let* $S \subseteq X$ *and* $\mathcal{B} \subseteq \mathcal{A}$ *be countable, such that* $\operatorname{cl} S \cap l_\mathcal{A} \subseteq l_\mathcal{B}$. *Then there exists a countable set* $N \subseteq X$ *such that* $S \subseteq N$, $B \subseteq^* N$ *for every* $B \in \mathcal{B}$ *and* $A \cap N$ *is finite for every* $A \in \mathcal{A} \setminus \mathcal{B}$.

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Proof. Let $T = \bigcup \mathcal{B}$ and let $\mathcal{D} \subseteq \mathcal{A} \setminus \mathcal{B}$ be such that $\operatorname{cl} T = T \cup l_{\mathcal{B} \cup \mathcal{D}} \cup \{\infty\}$. Since $K_{\mathcal{A}}$ is monolithic, the family $\mathcal{D}$ is necessarily countable (possibly finite or empty). Write $\mathcal{B} = \{B_n\}_{n \in \omega}$ and $\mathcal{D} = \{D_n\}_{n \in \omega}$, where we agree that $D_n = \emptyset$ for $n \in \omega$ in case $\mathcal{D} = \emptyset$. Let $C_n = B_n \setminus (D_0 \cup \cdots \cup D_n)$ and define

$$N = S \cup \bigcup_{n \in \omega} C_n.$$

We claim that N is as required.

Indeed, if $A \in \mathcal{B}$ then $A = B_n$ for some $n \in \omega$ and therefore $A \subseteq^* C_n \subseteq N$. If $A \in \mathcal{D}$ then, by assumption, $A \cap S$ is finite and $A \cap C_n = \emptyset$ for all but finitely many $n \in \omega$. Furthermore, $A \cap C_n \subseteq A \cap B_n$ is finite for $n \in \omega$, therefore $A \cap N$ is finite. Finally, if $A \in \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{D})$ then $A \cap S$ is finite and $A \cap \bigcup_{n \in \omega} C_n \subseteq A \cap T$ is finite, because $l_A \notin \operatorname{cl} T$. $\qquad \square$

Proposition 3.5. *Let $\mathcal{A}$ be an almost disjoint family consisting of countable sets. Then $C(K_{\mathcal{A}})$ has the controlled SCP if and only if $K_{\mathcal{A}}$ is monolithic.*

Proof. In view of Proposition 3.3, we need to prove the "if" part only. So fix an almost disjoint family $\mathcal{A}$ and assume $X = \bigcup \mathcal{A}$.

Recall that, given $f \in C(K_{\mathcal{A}})$, there exists a countable set $S_f \subseteq X$ such that f is constant on $K_{\mathcal{A}} \setminus \operatorname{cl} S_f$. Every functional $\mu \in C(K_{\mathcal{A}})^*$ can be identified with a real-valued Borel measure whose support $\operatorname{supp}(\mu)$ is separable and therefore countable, because $K_{\mathcal{A}}$ is monolithic.

Fix countable sets $A \subseteq C(K_{\mathcal{A}})$ and $B \subseteq C(K_{\mathcal{A}})^*$. Let $S \subseteq X$ be a countable set such that $S_f \subseteq S$ for $f \in A$ and $\operatorname{supp}(\mu) \subseteq \operatorname{cl} S$ for every $\mu \in B$. Since $K_{\mathcal{A}}$ is monolithic, $\operatorname{cl} S$ is countable, therefore there exists a countable $\mathcal{C} \subseteq \mathcal{A}$ such that $\operatorname{cl} S = S \cup l_{\mathcal{C}} \cup \{\infty\}$.

By Lemma 3.4, there exists a countable set $N \subseteq X$ such that $S \subseteq N$ and $A \subseteq^* N$ for $A \in \mathcal{C}$ and $A \cap N$ is finite for every $A \in \mathcal{A} \setminus \mathcal{C}$. Let $V = \operatorname{cl} N$ and $W = \operatorname{cl}(X \setminus N)$. Note that each of the sets V and W is closed and $V \cap W = \{\infty\}$. Define a retraction $r \colon K_{\mathcal{A}} \to V$ by collapsing the set W to $\{\infty\}$. Define $P \colon C(K_{\mathcal{A}}) \to C(K_{\mathcal{A}})$ by

$$Pf = (f \restriction V) \circ r.$$

In other words, $(Pf) \restriction V = f \restriction V$ and $Pf(x) = f(\infty)$ for $x \in W$. Clearly, P is a linear projection of norm one and its range consists of all functions $f \in C(K_{\mathcal{A}})$ that are constant on W. Since V is second countable, $\operatorname{im} P$ is separable.

Finally, we have that $A \subseteq \operatorname{im} P$ because $S_f \subseteq N$ for $f \in A$, and $B \subseteq \operatorname{im} P^*$ because $\operatorname{supp}(\mu) \subseteq V$ for $\mu \in B$. $\qquad \square$

4 The structure of locally small almost disjoint families of cardinality ω_1

The purpose of this section is to prove Theorem 1.5. This is a structural result where ladder system spaces play an important role, so we turn to them first. Let κ be an

uncountable cardinal. An indexed family $\mathcal{L} = (L_\alpha)_{\alpha \in S}$ for $S \subseteq E_\kappa(\omega) = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \omega\}$ is called a *ladder system* on S if each element L_α consisting of ordinals in $\kappa \setminus E_\kappa(\omega)$ is of order type ω, $L_\alpha \subseteq \alpha$ and $\sup(L_\alpha) = \alpha$.

It is clear that $\mathcal{L} = (L_\alpha)_{\alpha \in S}$ is an almost disjoint family of countable sets. The space $K_{\mathcal{L}}$ is called a *ladder system compact space*. We shall call it *non-trivial* if S is stationary. This is explained by the following folklore fact:

Proposition 4.1. *Suppose that $\mathcal{L} = (L_\alpha : \alpha \in S)$ is a ladder system on $S \subseteq \omega_1$ Then $K_{\mathcal{L}}$ is Eberlein if and only if S is non-stationary.*

Proof. We will use 2.2. Suppose that $K_{\mathcal{L}}$ is disjointifiable. Let F_α for $\alpha \in S$ be such that $L_\alpha \setminus F_\alpha$s form a pairwise disjoint family. Let $f(\alpha) = \min(L_\alpha \setminus F_\alpha)$. This is a regressive function defined on a stationary set, so it is constant on a stationary set, contradicting the disjointness.

Now, if S is non-stationary, let $C \subseteq \omega_1$ be a club set disjoint from S. For $\alpha \in S$ let $f(\alpha) = \max\{C \cap \alpha\} < \alpha$, hence $F_\alpha = L_\alpha \setminus (f(\alpha) + 1)$ is finite. Note that this means that $L_\alpha \setminus F_\alpha$ is disjoint from $L_{\alpha'} \setminus F_{\alpha'}$ whenever $f(\alpha) \neq f(\alpha')$. But pre-images under f of one element sets are countable, hence $\mathcal{L}$ is a union of countably any disjointifiable families, and so disjointifiable by Lemma 2.2(4). $\qquad\square$

In our formalism, the non-isolated points of $K_{\mathcal{A}}$ where $\mathcal{A}$ is a ladder system are $\{l_A : A \in \mathcal{A}\}$ however we can associate them with limit ordinals $\alpha \in \omega_1$, namely l_{A_α} is associated with α. Also ∞ is associated with ω_1. This way we obtain a standard form of the ladder system space whose set of points is $[0, \omega_1]$. Ladder system spaces were used by R. Pol as the first example of a Lindelöf space $C_p(K)$ for which K is not Corson compact, see [23], they were also exploited in [4] in the context of function spaces. It is clear that if $\mathcal{L}$ is a ladder system, then $K_{\mathcal{L}}$ is monolithic, therefore its space of continuous functions has the controlled SCP and all the properties of Theorem 1.4. We will show that every locally small almost disjoint family of cardinality ω_1 is equivalent to a family made of ladder systems and pairwise disjoint families.

Proof of Theorem1.5. Let $\mathcal{A}$ be a family of countable sets of cardinality ω_1. We may assume that $\bigcup \mathcal{A}$ consists of successor countable ordinals, i.e., $\omega_1 \setminus E_{\omega_1}(\omega)$.

Note that using the local smallness of $\mathcal{A}$ by the standard closure argument for every $\alpha < \omega_1$ there is a limit ordinal $\alpha < \delta < \omega_1$ such that if $A \cap \xi$ is infinite for some $\xi < \delta$, then $\sup(A) < \delta$. Let us call this property of δ as $\mathcal{P}$. Note that by the above observation the set of ordinals with property $\mathcal{P}$ is closed and unbounded in ω_1. Let $\{\delta_\alpha : \alpha < \omega_1\}$ be an increasing (and necessarily continuous) enumeration of this club set. For $A \in \mathcal{A}$ let $\eta(A) = \min\{\alpha : |A \cap \delta_\alpha| \text{ is infinite}\}$. Directly from property $\mathcal{P}$ we obtain the following:

Lemma 4.2. $\sup(A) < \delta_{\eta(A)+1}$ *for every $A \in \mathcal{A}$.*

Lemma 4.3. *The family*

$$\mathcal{A}_1 = \{A \in \mathcal{A} : \sup(A) < \delta_{\eta(A)}\}$$

is disjointifiable.

Proof. If $\sup(A) < \delta_{\eta(A)}$, then by the continuity of the sequence of δ_αs there is $\xi(A)$ such that $\xi(A) + 1 = \eta(A)$, $A \cap \delta_{\xi(A)}$ is finite and $A \subseteq \delta_{\eta(A)}$. On the other hand, by local smallness of $\mathcal{A}$ there are only countably many elements A of $\mathcal{A}$ such that $\eta(A)$ is fixed. $\mathcal{A}_1' = \{A \setminus \delta_{\xi(A)} : A \in \mathcal{A}_1\}$ is a union of uncountably many countable families each with its sum included in an interval of the form $[\delta_\alpha, \delta_{\alpha+1})$. So it is a union of countably many pairwise disjoint families, and so it is disjointifiable by Lemma 2.2(4), and so $\mathcal{A}_1$ is disjointifiable. $\square$

Define $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$.

Lemma 4.4. *If $A \in \mathcal{A}_2$, then the order type of $A \cap \delta_{\eta(A)}$ is ω.*

Proof. If the order type is different, then there is $\xi < \delta_{\eta(A)}$ such that $A \cap \xi$ is infinite. Then by property $\mathcal{P}$ of $\delta_{\eta(A)}$ we would have that $A \in \mathcal{A}_1$. $\square$

Let S be the set of all $\delta_\alpha < \omega_1$ such that there exists $A \in \mathcal{A}_2$ with $\eta(A) = \alpha$. For each $\delta_\alpha \in S$ let $\{A_n^{\delta_\alpha} : n \in \mathbb{N}\}$ be an enumeration (possibly with repetitions) of all $A \in \mathcal{A}_2$ with $\eta(A) = \alpha$; it exists because $\mathcal{A}$ is locally small. For $\alpha \in S$ let

$$B_n^{\delta_\alpha} = A_n^{\delta_\alpha} \cap \delta_\alpha \quad \text{and} \quad C_n^{\delta_\alpha} = A_n^{\delta_\alpha} \setminus \delta_\alpha.$$

It is clear that $\mathcal{B}_n = \{B_n^{\delta_\alpha} : \alpha \in S\}$ are ladder systems on S. So it is enough to prove that each

$$\mathcal{C}_n = \{C_n^{\delta_\alpha} : \alpha \in S\}$$

is disjointifiable. But for this note that it is actually a disjoint family, as

$$C_n^{\delta_\alpha} \subseteq [\delta_{\eta(A_n^{\delta_\alpha})}, \delta_{\eta(A_n^{\delta_\alpha})+1}),$$

which follows from Lemma 4.2. $\square$

It should be clear by 2.2(3) that $\mathcal{A} +_\varphi \mathcal{B}$ is disjointifiable if both $\mathcal{A}$ and $\mathcal{B}$ are disjointifiable. So 2.2(4) implies that if $\mathcal{A}$ of 1.5 is not disjointifiable, i.e., $K_{\mathcal{A}}$ is not Eberlein compact, then at least one of the ladder systems of 1.5 is nontrivial, in our construction this translates to the stationarity of the set S by Proposition 4.1.

Note that $\mathcal{A} +_\varphi \mathcal{B}$ is not equivalent to a nontrivial ladder system $\mathcal{L}$ if $\mathcal{A}$ is pairwise disjoint, φ is a bijection and $\mathcal{B}$ is arbitrary, possibly a nontrivial ladder system. To see this, let i be the Boolean isomorphism from the algebra generated by $\mathcal{A} +_\varphi \mathcal{B}$ and finite subsets of $\bigcup(\mathcal{A} +_\varphi \mathcal{B})$ and the algebra generated by $\mathcal{L}$ and finite subsets of $\bigcup \mathcal{L}$. As elements of $\mathcal{B}$ are infinite, for each $L \in \mathcal{L}$ there will be distinct $x_L \in \bigcup \mathcal{B}$ such that $i(\{x_L\}) \in L$. But this contradicts the Pressing Down Lemma and injectivity of i.

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We should note that assuming $\square_{\omega_1}$ there are non-disjointifiable locally small almost disjoint families of countable sets $\mathcal{A}$ of cardinality ω_2 such that $\mathcal{B} \subseteq \mathcal{A}$ is disjointifiable for every $\mathcal{B}$ of cardinality ω_1. This is just a ladder system on a stationary $E \subsetneq \{\alpha \in \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ such that $E \cap \alpha$ is non-stationary in any $\alpha \in \omega_2$ of cofinality ω_1 (23.6 [13]). So the structure of bigger families may not depend on the subfamilies of cardinality ω_1.

Question 4.5. Describe the structure of locally small almost disjoint families of cardinalities bigger than ω_1.

5   A space $C(K_{\mathcal{A}})$ with controlled and continuous SCP but without a projectional skeleton

Definition 5.1. Let $L \subseteq K$ be compact Hausdorff spaces. We define

$$C(K\|L) = \{f \in C(K) : f \text{ is constant on } L\}.$$

The subject of this section is the

Proof of Theorem 1.6. Let $\mathcal{C} = \{C_\alpha\}_{\alpha \in S}$ be a ladder system on ω_1 with S stationary. We will prove that the Banach space $C(K_{\mathcal{C}})$ is the union of a continuous chain of 2-complemented separable subspaces, yet it has no projectional skeleton. The second statement follows from Proposition 2.6 and Proposition 4.1 above. We shall check that canonical separable subspaces of $E := C(K_{\mathcal{C}})$ are 2-complemented. By a canonical separable subspace we mean

$$E_\alpha = C(K_{\mathcal{C}}\|L_\alpha),$$

where $\alpha < \omega_1$ and $L_\alpha = (\omega_1 \setminus \alpha) \cup \{l_{C_\xi} : \xi \in S \setminus \alpha\} \cup \{\infty\}$. It is easy to check that $\{E_\alpha\}_{\alpha < \omega_1}$ is a continuous chain of separable spaces whose union is E. Define

$$K_\alpha = \alpha \cup \{l_{C_\xi} : \xi \in S \cap (\alpha + 1)\} \cup \{\infty\}.$$

Note that K_α is the closure of $\alpha = [0, \alpha)$ in $K_{\mathcal{C}}$ and $K_\alpha \cap L_\alpha = \{\infty, l_{C_\alpha}\}$ if $\alpha \in S$, while $K_\alpha \cap L_\alpha = \{\infty\}$ for every $\alpha \in \omega_1 \setminus S$. Let $r_\alpha \colon K_{\mathcal{C}} \to K_{\mathcal{C}}$ be such that $r_\alpha \upharpoonright K_\alpha$ is the identity and either $r^{-1}(\infty) = L_\alpha \setminus \{l_{C_\alpha}\}$ or $r^{-1}(\infty) = L_\alpha$, depending on whether $\alpha \in S$ or $\alpha \notin S$. Then r_α is a continuous retraction onto K_α. In case $\alpha \in S$, the point l_{C_α} is isolated in L_α, therefore r_α is indeed continuous.

Fix $\alpha < \omega_1$. If $\alpha \notin S$ then L_α is a clopen set and E_α is 1-complemented in E, which is witnessed by the formula

$$Pf = f \circ r_\alpha.$$

Now assume that $\alpha \in S$ and let $U = C_\alpha \cup \{\infty\}$. Define $Q \colon E \to E_\alpha$ by setting

$$Qf = f \circ r_\alpha + (f(l_{C_\alpha}) - f(\infty)) \cdot \chi_U.$$

It is clear that Q is a linear operator with $\|Q\| \leqslant 2$. Observe that $Qf(l_{C_\alpha}) = f(l_{C_\alpha})$ and $Qf(\infty) = f(l_{C_\alpha})$. Furthermore, Qf is constant on L_α, that is, $Qf \in E_\alpha$. On the other hand, if $f \in E_\alpha$ then $f = f \circ r_\alpha$ and $f(\infty) = f(l_{C_\alpha})$, therefore $Qf = f$. Hence Q is a projection onto E_α. It follows that E_α is 2-complemented in E. This completes the proof. $\qquad\square$

6 A non-weakly Lindelöf space $C(K_\mathcal{A})$ which has the SCP

The subject of this section is the proof of theorem 1.8. First we note that there cannot be a space like in this theorem with only countably many isolated points:

Proposition 6.1. *Let $\mathcal{A}$ be an uncountable almost disjoint family such that $\bigcup \mathcal{A}$ is countable. Then $C(K_\mathcal{A})$ fails the SCP.*

Proof. The space $C(K_\mathcal{A})$ has a canonical isometric copy of c_0 which is not complemented by Whitley's argument [28], as in the case of $c_0 \subseteq \ell_\infty$ (see also [16, Thm. 17.3] for a more general statement). This fact, together with Sobczyk's theorem implies that $C(K_\mathcal{A})$ cannot have the SCP. $\qquad\square$

Lemma 6.2. *Suppose $\mathcal{A}$ is an infinite almost disjoint family of subsets of a set X. Then every $f \in C(K_\mathcal{A})$ is eventually constant, that is, it belongs to $C(K_\mathcal{A} \| \operatorname{cl}(X \setminus Y))$ for some countable $Y \subseteq X$.*

Proof. Clearly, if f is constant on $X \setminus Y$, then f is constant on $\operatorname{cl}(X \setminus Y))$. So, if the lemma is false, then there is $\varepsilon > 0$ and uncountably many pairs x_ξ, y_ξ of elements of X such that $|f(x_\xi) - f(y_\xi)| > \varepsilon$. So, as neighborhoods of ∞ are complements of countable sets, every neighborhood of ∞ contains a pair as above. This contradicts the continuity of f at ∞. $\qquad\square$

Lemma 6.3. *Suppose that $\mathcal{A}$ is an infinite almost disjoint family of countable sets with $\bigcup \mathcal{A} = X$. Let $Y \subseteq X$ be countably infinite, co-infinite such that $X \setminus Y$ is not covered by finitely many elements of $\mathcal{A}$ and let $\mathcal{B} = \{A \in \mathcal{A} : A \subseteq^* Y\}$. Suppose that there exists an injection $F : Y \to X \setminus Y$ satisfying for each $A \in \mathcal{A}$:*

$(*)$ *If $A \notin \mathcal{B}$, then $F[A \cap Y] =^* A \cap F[Y]$.*

Then there is a retraction $r : K_\mathcal{A} \to \operatorname{cl}(X \setminus Y)$ onto $\operatorname{cl}(X \setminus Y)$ and in particular the space $C(K_\mathcal{A} \| \operatorname{cl}(X \setminus Y))$ is complemented in $C(K_\mathcal{A})$.

Proof. First note that $\infty \in \operatorname{cl}(X \setminus Y)$ and that

$$\operatorname{cl}(X \setminus Y) = (X \setminus Y) \cup l_{\mathcal{A} \setminus \mathcal{B}} \cup \{\infty\}.$$

Define $r : K_\mathcal{A} \to \operatorname{cl}(X \setminus Y)$ considering several cases: If $x \in Y$, then $r(x) = F(x)$; if $x = l_B$ where $B \in \mathcal{B}$, then $r(x) = \infty$ and of course, $r(x) = x$ when $x \in \operatorname{cl}(X \setminus Y)$.

We need to check the continuity of r at each $x \in K_{\mathcal{A}}$. Of course, we need to consider only the non-isolated points from $l_{\mathcal{A}} \cup \{\infty\}$. First note that

$(**)$ If $A \in \mathcal{B}$, then $F[A \cap Y] \cap A'$ is finite for each $A' \in \mathcal{A}$.

This is because by $(*)$ if $A' \in \mathcal{A} \setminus \mathcal{B}$, then almost disjointness of A and A' implies the almost disjointness of $F[A]$ and A' since F is an injection. On the other hand if $A' \in \mathcal{B}$, then $A' \subseteq^* Y$ which is disjoint from $F[A]$. Going to the continuity of r, we have several cases.

If $x = l_B$ for some $B \in \mathcal{B}$, then the convergent sequence B which converges to l_B is sent by r to a convergent sequence $F[B]$ which by $(**)$ converges to ∞ which is $r(l_B)$ giving us the continuity of r at x.

If $x = l_A$ for some $A \in \mathcal{A} \setminus \mathcal{B}$, then the convergent sequence A which converges to l_A is sent by r to a convergent sequence $F[A \cap B] \cup (A \setminus B)$ which by $(*)$ is included in the convergent sequence $A \setminus B$ which converges to $l_A = r(l_A)$ giving us the continuity of r at x.

Finally, to prove the continuity of r at ∞, it is enough to consider $Z \subseteq X$ such that $\infty \in \operatorname{cl} Z$ and prove that $\infty \in \operatorname{cl}(r[Z])$. The general case can be reduced to two sub-cases $Z \subseteq X \setminus Y$ and $Z \subseteq Y$. The first one is trivial as r is the identity on $X \setminus Y$. In the second case if ∞ were not in the closure of $r[Z]$, then without loss of generality we could assume that $r[Z] = F[Z]$ is included in some $A \in \mathcal{A} \setminus \mathcal{B}$, by $(*)$ this gives

$$F[Z] \subseteq A \cap F[Y] =^* F[A \cap Y],$$

which gives $Z \subseteq^* A \cap Y$ since F is an injection and this is impossible as $\infty \in \operatorname{cl}(Z)$.

Define the projection $P : C(K_{\mathcal{A}}) \to C(K_{\mathcal{A}} || \operatorname{cl}(X \setminus Y))$ by

$$P(f) = f - f \circ r + f(\infty).$$

As r is constant on $X \setminus Y$, any function of the form $P(f)$ is constantly equal to $P(\infty)$ on $\operatorname{cl}(X \setminus Y)$. If $f \in C(K_{\mathcal{A}} || \operatorname{cl}(X \setminus Y))$, then it is equal to $f(\infty)$ on $\operatorname{cl}(X \setminus Y)$ (as $\infty \in \operatorname{cl}(X \setminus Y)$), so $P(f) = f - f(\infty) + f(\infty) = f$, hence P is the required projection. $\qquad\square$

Let us now describe the main difficulty in finding an almost disjoint family satisfying 1.8 and the way we overcome it. The most well known space of the form $K_{\mathcal{A}}$ which is not monolithic is obtained from an uncountable almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$. However, it does not have the SCP. The reason why $c(K_{\mathcal{A}} || K_{\mathcal{A}} \setminus \mathbb{N}) \sim c_0$ is not included in any separable complemented subspace of $C(K_{\mathcal{A}})$ is that in this case, it would be complemented in $C(K_{\mathcal{A}})$ by Sobczyk's theorem. The latter is impossible for the same reason that c_0 is not complemented in ℓ_{∞}.

Looking for a non-monolithic space of the form $K_{\mathcal{A}}(X)$ we note that, for such an $\mathcal{A}$ there must be a countable set $Y \subseteq X$ such that $\{A \cap Y : A \in \mathcal{A}\}$ is uncountable, hence the situation is quite similar to that of an uncountable almost disjoint family of subsets of $\mathbb{N}$, that is obtaining the SCP seems impossible. Lemma 6.3 provides a

solution to this problem. Let $\mathcal{A}$ be an uncountable almost disjoint family of subsets of $\mathbb{N}$. Let $\mathbb{N}' = \{n' : n \in \mathbb{N}\}$ where n's are distinct elements not belonging to $\mathbb{N}$. Define $F : \mathbb{N} \to \mathbb{N}'$ by $F(n) = n'$ and define

$$\mathcal{A}' = \{A \cup F[A] : A \in \mathcal{A}\}.$$

Applying Lemma 6.3 for $X = \mathbb{N} \cup \mathbb{N}'$, $Y = \mathbb{N}$, F, $\mathcal{A} = \mathcal{A}'$ as above, we obtain that $C(K_{\mathcal{A}'}||K_{\mathcal{A}'} \setminus \mathbb{N})$ is complemented in $C(K_{\mathcal{A}'})$ even though $C(K_{\mathcal{A}}||K_{\mathcal{A}} \setminus \mathbb{N})$ was not complemented in $C(K_{\mathcal{A}})$. This remark explains how a non-monolithic space $K_{\mathcal{A}}$ with an uncomplemented subspace $C(K_{\mathcal{A}}||K_{\mathcal{A}} \setminus \mathbb{N})$ can be extended to a bigger space $K_{\mathcal{A}'}$ where already $C(K_{\mathcal{A}'}||K_{\mathcal{A}'} \setminus \mathbb{N})$ is complemented. This is done by bifurcating the points of $\mathbb{N}$ and enlarging the sets of the family $\mathcal{A}$. Of course an important point which makes this enlargement useful here is that $C(K_{\mathcal{A}'}||K_{\mathcal{A}'} \setminus \mathbb{N})$ and $C(K_{\mathcal{A}}||K_{\mathcal{A}} \setminus \mathbb{N})$ are isometric in a canonical way via the restriction of a continuous function from $K_{\mathcal{A}'}$ to $K_{\mathcal{A}}$.

For the rest of this section fix an increasing enumeration $(\lambda_\eta : \eta \in \omega_1)$ of all limit countable ordinals, in particular $\lambda_0 = 0$. For every $\eta > 0$ with $\eta \in \omega_1$ we fix a bijection $F_\eta : [0, \lambda_\eta) \to [\lambda_\eta, \lambda_{\eta+1})$. The main idea of the proof of 1.8 is to construct an almost disjoint family $\mathcal{A} = \{A_\eta : 0 < \eta < \omega_1\}$ of subsets of ω_1 such that $A_\eta \subseteq \lambda_\eta$ for each $0 < \eta < \omega_1$ so that the injections $F_\eta : \lambda_\eta \to (\omega_1 \setminus \lambda_\eta)$ satisfy the hypothesis of Lemma 6.3. Then, some essential parts of the enlargements of the families $\{A \cap \lambda_\eta : A \in \mathcal{A}\}$ described in the previous paragraph are already included in $\mathcal{A}$. This gives that $C(K_{\mathcal{A}}|| \operatorname{cl}(\omega_1 \setminus \lambda_\eta))$ are complemented in $C(K_{\mathcal{A}})$ for each $0 < \eta < \omega_1$. As any separable subspace of $C(K_{\mathcal{A}})$ is included in $C(K_{\mathcal{A}}|| \operatorname{cl}(\omega_1 \setminus \lambda_\eta))$ for some $\eta < \omega_1$ by Lemma 6.2, we obtain the SCP. So we need the following:

Lemma 6.4. *There is a family $(A_\eta : \eta < \omega_1)$ of subsets of $\wp(\omega_1)$ such that:*

1. *$A_\eta \subseteq \lambda_\eta$ for each $\eta \in (0, \omega_1)$,*

2. *$A_\eta \cap [\lambda_\xi, \lambda_{\xi+1})$ is infinite for all $\xi < \eta < \omega_1$,*

3. *$A_\eta \cap [\lambda_\xi, \lambda_{\xi+1}) =^* F_\xi[A_\eta \cap \lambda_\xi]$ for all $\xi < \eta < \omega_1$,*

4. *$A_\eta \cap A_\xi$ is finite for all $\xi < \eta < \omega_1$.*

5. *No finite union of the sets of the form $A_\xi \cap [0, \omega)$ cover a co-finite subset of $[0, \omega)$.*

Proof. We put $A_0 = \emptyset$, $A^1 \subseteq [0, \omega) \subseteq [\lambda_0, \lambda_1)$ any infinite co-infinite subset of $[0, \omega)$ and suppose that A_αs were constructed so that (1) - (5) are satisfied. To construct $A_\eta = A_\eta^\eta$ we will construct inductively A_η^βs for $\beta \leq \eta$ so that the following are satisfied:

a) *$A_\eta^\beta \subseteq \lambda_\beta$ for each $\beta \leq \eta$,*

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b) $A_\eta^\beta \cap [\lambda_\xi, \lambda_{\xi+1})$ is infinite for all $\xi < \beta \leq \eta$,

c) $A_\eta^\beta \cap [\lambda_\xi, \lambda_{\xi+1}) =^* F_\xi[A_\eta^\beta \cap \lambda_\xi]$ for all $\xi < \beta \leq \eta$,

d) $A_\eta^\beta \cap A_\xi$ is finite for all $\xi < \eta$ and all $\beta \leq \eta$.

e) $A_\eta^\beta \cap \lambda_{\beta'} =^* A_\eta^{\beta'}$ for all $\beta' \leq \beta \leq \eta$.

We put $A_\eta^0 = \emptyset$, Using the inductive assumption (5) find an infinite co-infinite $A_\eta^1 \subseteq [0, \omega) \subseteq [\lambda_0, \lambda_1)$ which is almost disjoint from all $A_\xi \cap [0, \omega)$ for $\xi < \eta$ and so that (5) still holds. To construct $A_\eta^{\beta+1}$ given an A_η^β just put

$$A_\eta^{\beta+1} = A_\eta^\beta \cup F_{\lambda_\beta}[A_\eta^\beta].$$

This gives a) - c), e) immediately from the inductive hypothesis a) - c), e). The item d) follows from the inductive hypothesis d) and from items (1), (3) for $\xi < \eta$.

Only the limit stage is nontrivial. Given a limit $\beta \leq \eta$ choose a strictly increasing sequence $(\beta_n)_{n \in \mathbb{N}}$ with $\beta_0 = 0$ and its supremum equal to β. Enumerate $\{\xi : \xi < \eta\}$ as $\{\xi_n : n \in \mathbb{N}\}$. Define

$$A_\eta^\beta = \bigcup_{n \in \mathbb{N} \setminus \{0\}} [A_\eta^{\beta_n} \cap [\lambda_{\beta_{n-1}}, \lambda_{\beta_n})] \setminus \bigcup_{i \leq n} A_{\xi_i}].$$

Note that the above definition implies that $A_\eta^{\beta_n} \cap \lambda_{\beta_n} =^* A_\eta^\beta \cap \lambda_{\beta_n}$ by the inductive assumption e) since there are only finitely many integers less than n, hence e) follows since β_ns are unbounded in β. a) is immediate. c) follows from the fact that

$$A_\eta^\beta \cap [\lambda_\xi, \lambda_{\xi+1}) = A_\eta^{\beta_{n+1}} \cap [\lambda_\xi, \lambda_{\xi+1}) =^* F_\xi[A_\eta^{\beta_{n+1}} \cap \lambda_\xi] =^* F_\xi[A_\eta^\beta \cap \lambda_\xi]$$

for $n \in \mathbb{N}$ such that $\xi \in [\beta_n, \beta_{n+1})$ by the inductive assumption c) since in the definition of A_η^β we subtract finite sets by the inductive assumption d) and from the fact that $A_\eta^{\beta_{n+1}} \cap \lambda_{\beta_{n+1}} =^* A_\eta^\beta \cap \lambda_{\beta_{n+1}}$. b) follows from c) and the fact that A_η^1 is infinite. d) follows from the fact that we subtract $\bigcup_{i \leq n} A_{\xi_i}$ in the construction of A_η^β.

Now we see that $A_\eta = A_\eta^\eta$ satisfies (1) - (5). $\qquad \square$

Proof of Theorem 1.8. Let $\mathcal{A} = \{A_\eta : \eta < \omega_1\}$ be as in Lemma 6.4. First note that $K_\mathcal{A}$ is not monolithic. This follows from the fact that $[0, \omega) = [\lambda_0, \lambda_1)$ has $l_\mathcal{A}$ in its closure, and this is because $A \cap [\lambda_0, \lambda_1)$ is infinite for each $A \in \mathcal{A}$ which is 2) of Lemma 6.4. By Lemma 6.3 for every $0 < \eta < \omega_1$ the space $C(K_\mathcal{A} || \operatorname{cl}(\omega_1 \setminus \lambda_\eta))$ is complemented in $C(K_\mathcal{A})$. This together with Lemma 6.2 completes the proof of Theorem 1.8. $\qquad \square$

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