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A NOTE ON J-SETS OF LINEAR OPERATORS

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ABSTRACT. We construct a Banach space operator $T \in B(X)$ such that the set $J_T(0)$ has a nonempty interior but $J_T(0) \neq X$. This gives a negative answer to a problem raised by G. Costakis and A. Manoussos.

1. INTRODUCTION AND PRELIMINARIES

Let X be an infinite dimensional complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. For $T \in B(X)$ and $x \in X$ let $\mathrm{Orb}(T,x) = \{x, Tx, T^2x, ...\}$ be the orbit of T at x.

By a result of Bourdon and Feldman [?], if the closure $\overline{\mathrm{Orb}(T,x)}$ has a non-empty interior, then $\overline{\mathrm{Orb}(T,x)} = X$, and so x is a hypercyclic vector for T.

In [?], a weaker concept to that of the limit set of an orbit was introduced and studied. For $T \in B(X)$ and $x \in X$, let $J_T(x)$ be the set of all vectors $y \in X$ such that there exist a strictly increasing sequence $(k_n) \subseteq \mathbb{N}$ and a sequence $(x_n) \subseteq X$ with $x_n \to x$ and $T^{k_n}x_n \to y$ as $n \to \infty$. It is easy to see that the set $J_T(x)$ is always closed.

In [?], Problem 1, it was asked whether there is an analogue of the Bourdon-Feldman theorem in the case of J-sets: if the set $J_T(x)$ has a nonempty interior, does it imply that $J_T(x) = X$?

The goal of this paper is to give a negative answer to this question.

Let X be a Banach space, $x \in X$ and $r > 0$. We denote by $B(x,r) = \{y \in X : \|y - x\| \leq r\}$ the closed ball with radius r and center x. We denote by $\mathrm{int}\, A$ the interior of any subset $A \subset X$.

2. MAIN RESULT

Example. There exist a Banach space X and an operator $T \in B(X)$ such that $\mathrm{int}\, J_T(0) \neq \emptyset$ and $J_T(0) \neq X$.

Construction. Let $(k_n)_{n=1}^{\infty}$ be a fixed fast increasing sequence of positive integers. It is sufficient to assume that $k_{n+1} \geq 5k_n^2$ for all $n \in \mathbb{N}$. Let X be the ℓ_1 space with the standard basis

$$\{u_i : i = 0, 1, ...\} \cup \{v_{n,j} : n \in \mathbb{N}, 1 \leq j \leq k_n\}.$$

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More precisely, the elements of X can be expressed as

$$x=\sum_{i=0}^\infty \alpha_i u_i+\sum_{n=1}^\infty\sum_{j=1}^{k_n}\beta_{n,j}v_{n,j}$$

with complex coefficiens $\alpha_i,\beta_{n,j}$ such that

$$\|x\|:=\sum_{i=0}^\infty|\alpha_i|+\sum_{n=1}^\infty\sum_{j=1}^{k_n}|\beta_{n,j}|<\infty.$$

Let $\{w_n : n\in\mathbb{N}\}$ be a countable dense set in $B(0,\frac14)$. Without loss of generality we may assume that each w_n belongs to the space $\bigvee\bigl\{u_0,u_1,...,u_n,v_{m,j} : 1\le m<n,\ 1\le j\le k_m\bigr\}$.

We are going to construct an operator T with $J_T(0)\supset B(u_0,1/4)$. To this end it is sufficient to have $u_0+w_n\in T^{k_n}B(0,1/n)$ for each n. The purpose of the finite-dimensional subspace $\bigvee\{v_{n,j} : 1\le j\le k_n\}$ is to achieve this relation. The infinite-dimensional subspace $\bigvee\{u_i : i=0,1,\dots\}$ will ensure that $J_T(0)\ne X$.

Let $T\in B(X)$ be defined by

$$\begin{aligned}Tu_i&=2u_{i+1}\qquad(i=0,1,...),\\ Tv_{n,j}&=2v_{n,j+1}\qquad(n\in\mathbb{N},1\le j\le k_n-1),\\ Tv_{n,k_n}&=\frac{n}{2^{k_n-1}}(u_0+w_n)\qquad(n\in\mathbb{N}).\end{aligned}$$

It is easy to see that $\|T\|=2$. For each $n\in\mathbb{N}$ we have

$$T^{k_n}(n^{-1}v_{n,1})=2^{k_n-1}n^{-1}Tv_{n,k_n}=u_0+w_n.$$

This implies that $B(u_0,\frac14)\subset J_T(0)$. Indeed, let $z\in X$ with $\|z\|\le\frac14$ and let (n_i) be an increasing sequence in $\mathbb{N}$ satisfying $w_{n_i}\to z$ as $i\to\infty$. Then $n_i^{-1}v_{n_i,1}\to0$ and $\lim_{i\to\infty}T^{k_{n_i}}(n_i^{-1}v_{n_i,1})=\lim_{i\to\infty}(u_0+w_{n_i})=u_0+z$. In particular, $\operatorname{int}J_T(0)\ne\emptyset$.

It remains to show that $J_T(0)\ne X$. Suppose on the contrary that $J_T(0)=X$. In particular, it means that $v_{1,1}\in J_T(0)$, and so there exist $k\in\mathbb{N}$ and $y\in X,\|y\|\le1$ with

$$\|T^ky-v_{1,1}\|<\frac14. \tag{1}$$

Moreover, we may assume that $k>k_2+k_1$. Write $m_n=k_n+k_{n-1}+\dots+k_1$. Since $k_{i+1}\ge5k_i^2\ge5k_i$, we have $m_n\le\frac{5k_n}{4}$, and so $k_n\le m_n\le\frac54k_n$.

Let $n\in\mathbb{N}$ satisfy $m_{n-1}\le k<m_n$. By assumption, $n\ge3$. Write $X_0=\bigvee\{u_i : i=0,1,\dots\}$. For $n\in\mathbb{N}$ let $X_n=\bigvee\{v_{n,i} : 1\le i\le k_n\}$. Let P_j be the natural projection onto X_j, i.e., $\ker P_j=\bigvee_{i\ne j}X_i$. Clearly $\|P_j\|=1$ for each j.

Write $y=y_0+y_1+x+y_2$, where $y_0=P_0y,\ y_1=\Bigl(\sum_{i=1}^{n-1}P_i\Bigr)y,\ x=P_ny$ and $y_2=\Bigl(\sum_{i=n+1}^\infty P_i\Bigr)y$. We have $\|y_0\|+\|y_1\|+\|x\|+\|y_2\|=\|y\|\le1$. Obviously $T^ky_0\in X_0$ and

$$T^ky_1\in T^k\Bigl(\bigvee_{i=1}^{n-1}X_i\Bigr)\subset T^{k-k_{n-1}}\Bigl(\bigvee_{i=0}^{n-2}X_i\Bigr)\subset\dots\subset T^{k-k_{n-1}-\dots-k_1}\bigl(X_0\bigr)\subset X_0.$$

Finally, $\Bigl\|\Bigl(\sum_{i=0}^{n-1}P_i\Bigr)T^ky_2\Bigr\|\le\frac{2^k(n+1)}{2^{k_{n+1}-1}}\le\frac{n+1}{2^{k_{n+1}-m_n}}\le\frac{n+1}{2^{k_n}}<\frac14.$

If $m_{n-1} \le k < m_n - 2m_{n-1} = k_n - m_{n-1}$, then

$$\|P_1 T^k y\| \le \|P_1 T^k x\| + \|P_1 T^k y_2\| \le \frac{2^k n}{2^{k_n - 1}} + \frac{1}{4} \le \frac{n}{2^{m_{n-1}}} + \frac{1}{4} < \frac{1}{2}.$$

So $\|T^k y - v_{1,1}\| \ge \|P_1(T^k y - v_{1,1})\| \ge 1 - \frac{1}{2} = \frac{1}{2}$, a contradiction with (1).

So we may assume that $k_n - m_{n-1} \le k \le k_n + m_{n-1} = m_n$. Write for short $m = m_{n-1}$. For $j = 1, 2, \dots$ let $Y_j = \bigvee\{u_{(j-1)m}, \dots, u_{jm-1}\}$. Write also $Y_0 = \bigcup_{i=1}^{n-1} X_i$. Let Q_j be the natural projection onto Y_j $(j = 0, 1, \dots)$. Note that $k - m \ge k_n - 2m \ge 5k_{n-1}^2 - 2m \ge \frac{16}{5}m^2 - 2m \ge m^2$, and so $T^k(y_0 + y_1) \in \bigvee\{u_i : i \ge m^2\}$. Thus $\Big(\sum_{i=0}^m Q_j\Big)T^k(y_0 + y_1) = 0$ and

$$\begin{aligned} \Big\|\Big(\sum_{j=0}^m Q_j\Big)(T^k x - v_{1,1})\Big\| &= \Big\|\Big(\sum_{j=0}^m Q_j\Big)(T^k(y_0 + y_1 + x) - v_{1,1})\Big\| \\ &\le \Big\|\Big(\sum_{j=0}^m Q_j\Big)(T^k y - v_{1,1})\Big\| + \Big\|\Big(\sum_{j=0}^m Q_j\Big)T^k y_2\Big\| \le \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}. \end{aligned} \tag{2}$$

Let $x = \sum_{i=1}^{k_n} \alpha_i v_{n,i}$. Let $i_0 = k_n - k + 1$ and $x_0 = \sum_{i=1}^{i_0 - 1} \alpha_i v_{n,i}$ (if $i_0 \le 1$ then $x_0 = 0$). For $j = 1, ..., m$ let

$$x_j = \sum_{i=i_0+(j-1)m}^{i_0+jm-1} \alpha_i v_{n,i}.$$

We have $T^k x_0 \in X_n$, and so $\Big(\sum_{j=0}^m Q_j\Big)T^k x_0 = 0$. For $j = 1, ..., m$, we have

$$\begin{aligned} T^k x_j &= \sum_{i=i_0+(j-1)m}^{i_0+jm-1} \alpha_i T^k v_{n,i} = \sum_i \alpha_i 2^{k_n - i} T^{k - k_n + i} v_{n,k_n} \\ &= \sum_i \alpha_i \frac{2^{k_n - i} n}{2^{k_n - 1}} T^{k - k_n + i - 1}(u_0 + w_n) = s_j + q_j, \end{aligned}$$

where

$$s_j = 2^{k - k_n} n \sum_{i=i_0+(j-1)m}^{i_0+jm-1} \alpha_i u_{k - k_n + i - 1}$$

and

$$q_j = \sum_{i=i_0+(j-1)m}^{i_0+jm-1} \alpha_i 2^{1-i} n T^{k - k_n + i - 1} w_n.$$

Note that

$$\|s_j\| = n 2^{k - k_n} \sum_{i=i_0+(j-1)m}^{i_0+jm-1} |\alpha_i| = n 2^{k - k_n} \|x_j\|$$

and

$$\|q_j\| \le \sum_{i=i_0+(j-1)m}^{i_0+jm-1} |\alpha_i| 2^{1-i} n 2^{k - k_n + i - 1} \|w_n\| \le \frac{1}{4}\|s_j\|.$$

Note also that

$$T^k x_j \in Y_{j-1} \vee Y_j \vee Y_{j+1}.$$

Write $t_j = Q_{j-1} q_j, t_j' = Q_j q_j$ and $t_j'' = Q_{j+1} q_j$. For $j = 1, \dots, m-1$, we have

$$\Big\|\Big(\sum_{i=0}^j Q_i\Big)(T^k x - v_{1,1})\Big\| = \|t_1 - v_{1,1}\| + \|s_1 + t_1' + t_2\| + \|s_2 + t_1'' + t_2' + t_3\| + \cdots$$

$$\begin{aligned}
&\cdots + \|s_{j-1} + t''_{j-2} + t'_{j-1} + t_j\| + \|s_j + t''_{j-1} + t'_j + t_{j+1}\| \\
\geq{}& 1 - \|t_1\| + \|s_1\| - \|t'_1\| - \|t_2\| + \|s_2\| - \|t''_1\| - \|t'_2\| - \|t_3\| + \cdots \\
&\cdots + \|s_j\| - \|t''_j\| - \|t'_j\| - \|t_{j+1}\| \\
\geq{}& 1 + \big(\|s_1\| - \|t_1\| - \|t'_1\| - \|t''_1\|\big) + \cdots \\
&\cdots + \big(\|s_{j-1}\| - \|t_{j-1}\| - \|t'_{j-1}\| - \|t''_{j-1}\|\big) + \big(\|s_j\| - \|t_j\| - \|t'_j\|\big) - \|t_{j+1}\| \\
\geq{}& 1 + \frac{3}{4}(\|s_1\| + \|s_2\| + \cdots + \|s_j\|) - \frac{\|s_{j+1}\|}{4}.
\end{aligned}$$

Since $\left\|\Big(\sum_{i=0}^{j} Q_i\Big)(T^k x - v_{1,1})\right\| \leq \frac{1}{2}$ by (2), we have
$\|s_{j+1}\| \geq 3(\|s_1\| + \|s_2\| + \cdots + \|s_j\|) \geq 3\|s_j\|$. So $\|x_{j+1}\| \geq 3\|x_j\|$.
By induction, $\|x_m\| \geq 3\|x_{m-1}\| \geq \cdots \geq 3^{m-1}\|x_1\|$. Since $\|x_m\| \leq \|x\| \leq 1$, we have $\|x_1\| \leq 3^{1-m}$. Hence

$$\|Q_0T^kx\| = \|Q_0T^kx_1\| = \|t_1\| \leq 2^{k-k_n} n\frac{\|x_1\|}{4} \leq 2^{k-k_n-2}n3^{1-m} \leq \frac{2^m n}{3^m} \leq \frac{1}{2},$$

which is a contradiction with the fact that

$$\|Q_0T^kx\| \geq \|Q_0v_{1,1}\| - \|Q_0(T^kx - v_{1,1})\| \geq 1 - \|T^kx - v_{1,1}\| \geq \frac{3}{4}.$$

Remark. The construction above can be modified easily so that we obtain an operator $V \in B(Y)$ and a non-zero vector $y \in Y$ such that $\operatorname{int} J_V(y) \neq \emptyset$ and $J_V(y) \neq Y$.

Let X and $T \in B(X)$ be as in the previous example. Let $Y = X \oplus \ell_1$ and let $V = T \oplus 2S$, where $S \in B(\ell_1)$ is the backward shift. Let $y \neq 0$ and $Sy = 0$. Then $V(0 \oplus y) = 0$. It is easy to see that $J_V(0 \oplus y) = J_V(0 \oplus 0)$. Clearly $J_V(0 \oplus 0) \subset J_T(0) \oplus J_{2S}(0)$. Furthermore, it is easy to see that for all $\varepsilon > 0$, $y' \in \ell_1$ and all n sufficiently large there exists $y_n \in \ell_1$ with $\|y_n\| < \varepsilon$ and $(2S)^n y_n = y'$. This implies that $J_V(0 \oplus 0) = J_T(0) \oplus \ell_1$.

Hence $\operatorname{int} J_V(0 \oplus y) \neq \emptyset$ and $J_V(0 \oplus y) \neq Y$.

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