Guaranteed estimates of the constant in Friedrichs' inequality

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June 3–8, PANM 16, Dolní Maxov

Outline

$$\|v\|_0 \le C_{\mathrm{F}} \|\nabla v\|_0 \quad \forall v \in H_0^1(\Omega)$$

- Motivation – complementary error estimates
- Relationship with eigenvalues
- Rayleigh–Ritz approximation
- Method of a priori–a posteriori inequalities
- Generalizations
- Examples

Motivation – complementary error estimates

Classical formulation:

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

Weak formulation: $V = H_0^1(\Omega)$

$$u \in V : \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in V$$

Error bound: $u_h \in V$

$$|||u - u_h||| \leq C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Notation:

- $L^2(\Omega)$ scalar product: $(\varphi, \psi) = \int_\Omega \varphi\psi \, \mathrm{d}x$
- $L^2(\Omega)$ norm: $\|v\|_0^2 = (v, v)$
- Energy norm: $|||e|||^2 = (\nabla e, \nabla e) = \|\nabla e\|_0^2$

Motivation – complementary error estimates

Friedrich's inequality: $\|v\|_0 \le C_{\mathrm{F}} \|\nabla v\|_0 \quad \forall v \in V$

Divergence thm: $(\operatorname{div} \mathbf{y}, v) + (\mathbf{y}, \nabla v) = 0 \quad \forall v \in V, \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$

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$$|\!|\!| u - u_h |\!|\!| \le C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof: $v = u - u_h \in V$

$$\begin{aligned}
(\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) \\
&= (f + \operatorname{div} \mathbf{y}, v) + (\mathbf{y} - \nabla u_h, \nabla v) \\
&\le \|f + \operatorname{div} \mathbf{y}\|_0 \|v\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \|\nabla v\|_0 \\
&\le \left(C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \right) |\!|\!| v |\!|\!| \qquad \square
\end{aligned}$$

Relation with eigenvalues

Friedrichs' inequality:

$$\|v\|_0 \leq C_{\mathrm{F}} \|\nabla v\|_0 \quad \forall v \in V \quad \Rightarrow \quad C_{\mathrm{F}} = \sup_{v \in V} \frac{\|v\|_0}{\|\nabla v\|_0}$$

Laplace eigenvalue problem

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega, \qquad u_i = 0 \text{ on } \partial\Omega, \quad i = 1, 2, \ldots$$

Theorem: $C_{\mathrm{F}}^2 = \frac{1}{\lambda_1}$ where $\lambda_1 = \min_i \lambda_i$.

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Theorem: $C_{\mathrm{F}}^2 = \dfrac{1}{\lambda_1}$ where $\lambda_1 = \min_i \lambda_i$.

Proof:
Weak formulation: $u_i \in V: \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V$

$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} \quad \Leftrightarrow \quad \frac{1}{\lambda_1} = \sup_{v \in V} \frac{\|v\|_0^2}{\|\nabla v\|_0^2}$$

$\square$

Rayleigh–Ritz approximation of λ_1

Weak formulation:

$$u_i \in V : \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V$$

Rayleigh–Ritz method: $V^h \subset V$, $\dim V^h < \infty$

$$u_i^h \in V^h : \quad (\nabla u_i^h, \nabla v^h) = \lambda_i^h (u_i^h, v^h) \quad \forall v^h \in V^h$$

Theorem: $\lambda_1 \leq \lambda_1^h$

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Theorem: $\lambda_1 \leq \lambda_1^h$

Proof:

$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} \leq \inf_{v^h \in V^h} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} = \lambda_1^h$$

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□

Corollary: $C_{\mathrm{F}}^h \leq C_{\mathrm{F}}$

Lower bound on λ_1

Method of *a priori-a posteriori inequalities*.

Theorem (Kuttler and Sigillito, 1978):

$$\Rightarrow \quad \min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|w\|_0}{\|u_*\|_0},$$

where

- $\lambda_* \in \mathbb{R}$ and $u_* \in V$ are arbitrary
- $w \in V: \quad (\nabla w, \nabla v) = (\nabla u_*, \nabla v) - \lambda_*(u_*, v) \quad \forall v \in V$

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Friedrich's inequality:

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|w\|_0}{\|u_*\|_0} \leq C_{\mathrm{F}} \frac{\|\nabla w\|_0}{\|u_*\|_0}$$

Algorithm

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq C_{\mathrm{F}} \frac{\|\nabla w\|_0}{\|u_*\|_0}$$

Theorem:

$$\|\nabla w\|_0 \leq \|\nabla u_* - \mathbf{q}\|_0 + C_{\mathrm{F}} \|\lambda_* u_* + \operatorname{div} \mathbf{q}\|_0 \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega).$$

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- λ_1^h is closest to λ_1 (i.e. $i = 1$)
- $(*) \quad \Leftrightarrow \quad 0 \le X^2 + \alpha X + \beta - \lambda_1^h, \quad$ where $X = \sqrt{\lambda_1}$

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- λ_1^h is closest to λ_1 (i.e. $i = 1$)
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- $\quad \Rightarrow \quad X_2^2 \le \lambda_1, \quad$ where $X_2 = \left(\sqrt{\alpha^2 + 4(\lambda_1^h - \beta)} - \alpha \right) / 2$

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- $\Rightarrow \quad X_2^2 \le \lambda_1, \quad$ where $X_2 = \left(\sqrt{\alpha^2 + 4(\lambda_1^h - \beta)} - \alpha \right) / 2$
- $\Rightarrow \quad C_{\mathrm{F}} \le 1/X_2$

Computing $\mathbf{q}_h \in \mathbf{H}(\mathrm{div}, \Omega)$

$$\left(\|\nabla u_* - \mathbf{q}\|_0 + C_{\mathrm{F}} \, \|\lambda_* u_* + \mathrm{div}\, \mathbf{q}\|_0 \right)^2$$

$$\approx \left(\|\nabla u_1^h - \mathbf{q}\|_0 + (\lambda_1^h)^{-1/2} \|\lambda_1^h u_1^h + \mathrm{div}\, \mathbf{q}\|_0 \right)^2$$

$$\leq (1 + \varrho^{-1}) \|\nabla u_1^h - \mathbf{q}\|_0^2 + \frac{1+\varrho}{\lambda_h} \|\lambda_1^h u_1^h - \mathrm{div}\, \mathbf{q}\|_0^2, \quad \forall \varrho > 0$$

Minimize over $W_h \subset \mathbf{H}(\mathrm{div}, \Omega)$:
Find $\mathbf{q}_h \in W_h$:

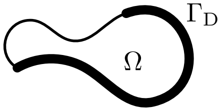
$$(\mathrm{div}\, \mathbf{q}_h, \mathrm{div}\, \psi_h) + \frac{\lambda_1^h}{\varrho} (\mathbf{q}_h, \psi_h) = \frac{\lambda_1^h}{\varrho} (\nabla u_1^h, \psi_h) - (\lambda_1^h u_1^h, \mathrm{div}\, \psi_h) \qquad \forall \psi_h \in W_h$$

Solve by standard Raviart-Thomas finite elements.

Generalizations

Linear elliptic problem:

$$-\operatorname{div}(\mathcal{A}\nabla u) + cu = f \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma_{\mathrm{D}}$$
$$n^T \mathcal{A}\nabla u = 0 \text{ on } \Gamma_{\mathrm{N}}$$



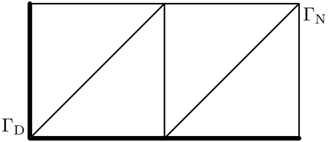
Friedrich's inequality:

$$\|v\|_0 \leq C_{\mathrm{F}} |\!|\!| v |\!|\!| \quad \forall v \in V$$

- $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{D}}\}$
- $|\!|\!| v |\!|\!|^2 = (\mathcal{A}\nabla u, \nabla u) + (cu, u)$

Example 1

$$-\Delta u = f \text{ in } (0,2)\times(0,1)$$
$$u = 0 \text{ on } \Gamma_{\mathrm{D}}$$
$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_{\mathrm{N}}$$



$$f = \frac{5\pi^2}{16} u$$
$$u = \sin\frac{\pi x_1}{4}\sin\frac{\pi x_2}{2}$$

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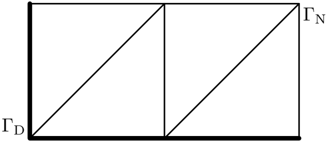
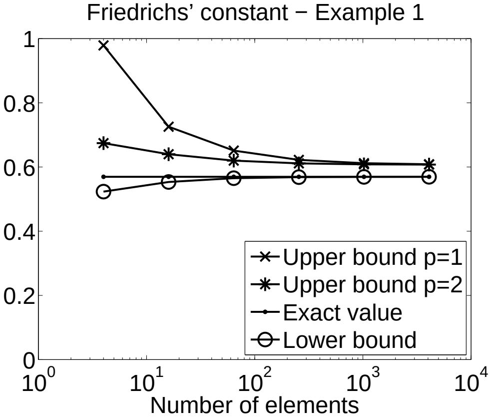


$$f = \frac{5\pi^2}{16} u$$
$$u = \sin\frac{\pi x_1}{4} \sin\frac{\pi x_2}{2}$$

$$C_{\mathrm{F}} = \frac{4}{\sqrt{5}\pi} \doteq 0.5694$$
$$C_{\mathrm{F}}^{\mathrm{low}} = 0.5693$$
$$C_{\mathrm{F}}^{\mathrm{up}} = 0.6004$$



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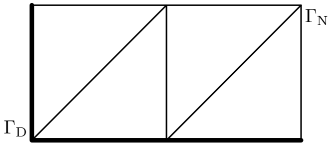
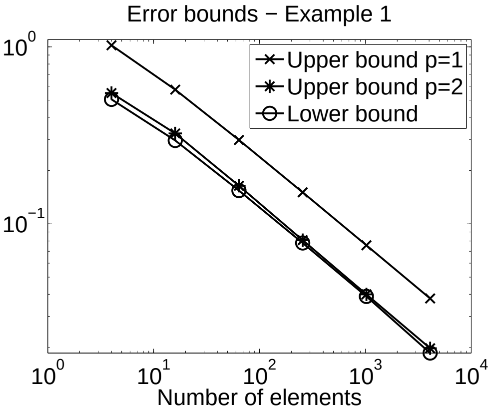


Lower bound:

$$|||u_h^{\mathrm{ref}} - u_h||| \le |||u - u_h|||$$

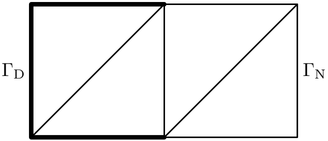
Upper bound:

$$|||u - u_h||| \le C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$



Example 2

$$-\Delta u = f \text{ in } (0,2)\times(0,1)$$
$$u = 0 \text{ on } \Gamma_{\mathrm{D}}$$
$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_{\mathrm{N}}$$



$$f = \frac{5\pi^2}{16}\sin\frac{\pi x_1}{4}\sin\frac{\pi x_2}{2}$$

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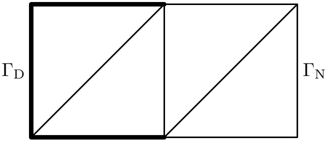
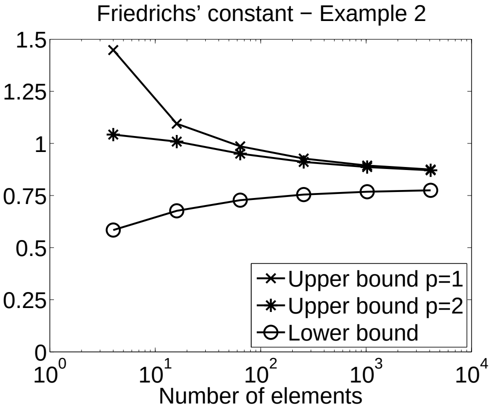


$$f = \frac{5\pi^2}{16} \sin \frac{\pi x_1}{4} \sin \frac{\pi x_2}{2}$$

$C_{\mathrm{F}} = ?$
$C_{\mathrm{F}}^{\mathrm{low}} = 0.7750$
$C_{\mathrm{F}}^{\mathrm{up}} = 0.8712$



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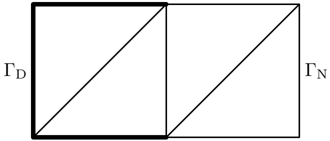
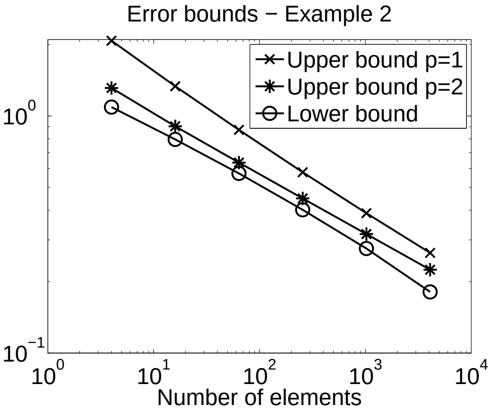


Lower bound:

$$|||u_h^{\mathrm{ref}} - u_h||| \le |||u - u_h|||$$

Upper bound:

$$|||u - u_h|||$$
$$\le C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0$$
$$+ \|\mathbf{y} - \nabla u_h\|_0$$
$$\forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$



Conclusions

- Practical method
- Guaranteed upper bound on Friedrichs' constant
- Easy to generalize to similar inequalities
- Computationally demanding
- Exact representation of the domain Ω

Thank you for your attention

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