A MAXIMUM MODULUS ESTIMATE FOR THE STOKES EQUATIONS

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The classical maximum principle states that a smooth non-constant function u being harmonic in a bounded domain $\Omega \subset \mathbb{R}^n$ takes its maximum value on the boundary $\partial\Omega$, i. e.

$$\max_{x \in \overline{\Omega}} u(x) \leq \max_{y \in \partial\Omega} u(y)\,.$$

This principle extends also to solutions of more general elliptic equations with regular coefficients. It is not true, however, for solutions of higher order equations or of elliptic systems. In the present paper, instead of the Laplace equation $-\Delta u = 0$, we consider the Stokes system

$$-\Delta v + \operatorname{grad} p = 0 \text{ in } \Omega\,, \qquad \operatorname{div} v = 0 \text{ in } \Omega\,, \qquad v = b \text{ on } \partial\Omega,$$

well-known from hydrodynamics: Here v is the velocity field and p some kinematic pressure function defined in a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The function $b \in C^0(\partial\Omega)$ is a given boundary value satisfying the necessary compatibility condition

$$\int_{\partial\Omega} b(y) \cdot N(y) \mathrm{d}o = 0, \tag{1}$$

where N denotes the exterior unit normal vector on $\partial\Omega$. In this case, for the function $|v|$ there is no maximum principle: Consider in the unit ball $\Omega := B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\} \subset \mathbb{R}^n$ the functions

$$v(x) := (2x_1^2 - |x|^2 + n(1 - |x|^2), 2x_1x_2, 2x_1x_3, \ldots, 2x_1x_n), \quad p(x) := -2(n^2 + n - 2)x_1.$$

These functions represent a solution of the Stokes system, and we find $|v(x)| = 1$ on $\partial\Omega$, but $|v(0)| = n$. Nevertheless, a so-called maximum modulus estimate might be valid in this case. This means to prove the existence of some constant $c = c_\Omega > 0$ depending only on Ω with

$$\max_{x \in \overline{\Omega}} |v(x)| \;\leq\; c \;\max_{y \in \partial\Omega} |v(y)|. \tag{2}$$

Our main result concerning the Stokes equations is now stated in

Theorem 1: *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain with a compact boundary $\partial\Omega \in C^{1,\alpha}$ ($\alpha > 0$). Let $b \in C^0(\partial\Omega)$ be a given boundary value satisfying (1). Then there is a constant $c = c_\Omega > 0$ such that the solution v of the Stokes system satisfies a maximum modulus estimate of the type (2)*

We can prove this theorem with the method of boundary integral equations as follows. First we choose for the solution v a representation in form of a suitable layer potential with an unknown source density ψ:

$$v(x) := \; D_n\psi(x) \; := \int_{\partial\Omega} D_n(x, y)\psi(y) do_y \qquad x \in \Omega\,.$$

Approaching the boundary, this representation leads to a system

$$b(x) = A_n \psi(x) \qquad x \in \partial\Omega$$

of boundary integral equations, where $A_n : C^0(\partial\Omega) \to C^0(\partial\Omega)$ is a certain boundary integral operator. Now we have to prove two conditions:

(i) A_n is bounded and bijective.

(ii) $\int\limits_{\partial\Omega} |D_n(x,y)| do_y \leq \; c_2 \quad$ for all $\; x \in \mathbb{R}^n$.

Since (i) implies that the inverse operator A_n^{-1} is bounded, i.e. $\|A_n^{-1}\|_\infty \leq \; c_1$, the asserted maximum modulus estimate follows from

$$\begin{aligned} \max_{x\in\overline{\Omega}} |v(x)| \; &= \; \max_{x\in\overline{\Omega}} \left| \int\limits_{\partial\Omega} D_n(x,y) A_n^{-1} b(y) do_y \right| \\ &\leq \; \max_{x\in\overline{\Omega}} \int\limits_{\partial\Omega} |D_n(x,y)| \, ||A_n^{-1}||_\infty |b(y)| do_y \; \leq \; c_1 c_2 \max_{y\in\partial\Omega} |b(y)| \, . \end{aligned}$$

Let us first represent the solution v by a pure hydrodynamical double layer potential

$$v(x) := D_n\psi(x) := \int\limits_{\partial\Omega} D_n(x,y)\psi(y) do_y, \qquad x \in \Omega,$$

where the $n \times n$ double layer kernel matrix $D_n(x,y) = (D_n^{i,j}(x,y))_{i,j=1,\ldots,n}$ is defined by

$$D_n^{i,j}(x,y) := \frac{-2}{\omega_n} \frac{z_i z_j z \cdot N}{|z|^{n+2}} \qquad z := x - y, \qquad N := N(y).$$

The resulting integral operator $A_n := \frac{1}{2} I_n + D_n$ (I_n denotes the identity matrix) is not bijektive. Therefore, instead of A_n, we use the operator $\tilde{A}_n$, defined by

$$\tilde{A}_n := A_n - \beta P_n, \qquad 0 \neq \beta \in \mathbb{R}, \qquad P_n\psi(x) := \int\limits_{\partial\Omega} \psi \cdot N \,\mathrm{do}\; N(x)\,.$$

For $\tilde{A}_n$ we get the following result.

Theorem 2: *Let $2 \leq n \in \mathbb{N}$ and $0 \neq \beta \in \mathbb{R}$. Then the operator $\tilde{A}_n = \frac{1}{2} I_n + D_n - \beta P_n$: $C^0(\partial\Omega) \longrightarrow C^0(\partial\Omega)$ is bounded and bijective. Let $b \in C^0(\partial\Omega)$ be given with (1). Then the function $\tilde{\psi} := \tilde{A}_n^{-1} b \in C^0(\partial\Omega)$ is also a solution of $A_n\psi = b$ on $\partial\Omega$.*

The above mentioned condition (ii) can also been proved, which implies the assertion of Theorem 1. A maximum modulus estimate for the Stokes equations in exterior domains does also hold true. In this case a representation of the solution in form of a single and a double layer potential has to be chosen.