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ON DETECTABLE COLORINGS OF GRAPHS

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Abstract. Let G be a connected graph of order $n \geqslant 3$ and let $c\colon E(G) \to \{1,2,\ldots,k\}$ be a coloring of the edges of G (where adjacent edges may be colored the same). For each vertex v of G, the color code of v with respect to c is the k-tuple $c(v) = (a_1, a_2, \ldots, a_k)$, where a_i is the number of edges incident with v that are colored i ($1 \leqslant i \leqslant k$). The coloring c is detectable if distinct vertices have distinct color codes. The detection number $\det(G)$ of G is the minimum positive integer k for which G has a detectable k-coloring. We establish a formula for the detection number of a path in terms of its order. For each integer $n \geqslant 3$, let $D_u(n)$ be the maximum detection number among all unicyclic graphs of order n and $d_u(n)$ the minimum detection number among all unicyclic graphs of order n. The numbers $D_u(n)$ and $d_u(n)$ are determined for all integers $n \geqslant 3$. Furthermore, it is shown that for integers $k \geqslant 2$ and $n \geqslant 3$, there exists a unicyclic graph G of order n having $\det(G) = k$ if and only if $d_u(n) \leqslant k \leqslant D_u(n)$.

Keywords: detectable coloring, detection number

MSC 2000: 05C15, 05C70

1. Introduction

Let G be a connected graph of order $n \geqslant 3$ and let $c\colon E(G) \to \{1,2,\ldots,k\}$ be a coloring of the edges of G for some positive integer k (where adjacent edges may be colored the same). The *color code* of a vertex v of G (with respect to c) is the ordered k-tuple

$$c(v) = (a_1, a_2, \ldots, a_k) \text{ (or simply, } c(v) = a_1a_2\ldots a_k\text{)},$$

where a_i is the number of edges incident with v that are colored i for $1 \leqslant i \leqslant k$. Therefore, $\sum\limits_{i=1}^{k} a_i = \deg_G v$. The coloring c is called *detectable* if distinct vertices have distinct color codes; that is, for every two vertices of G, there exists a color such that the number of incident edges with that color is different for these two vertices. The *detection number* $\det(G)$ of G is the minimum positive integer k for which G has a detectable k-coloring. Such a coloring is called a *minimum detectable coloring.* Since every nontrivial graph contains at least two vertices having the same degree, the vertices of a nontrivial connected graph cannot be distinguished by their degrees alone. Therefore, every connected graph of order 3 or more has detection number at least 2.

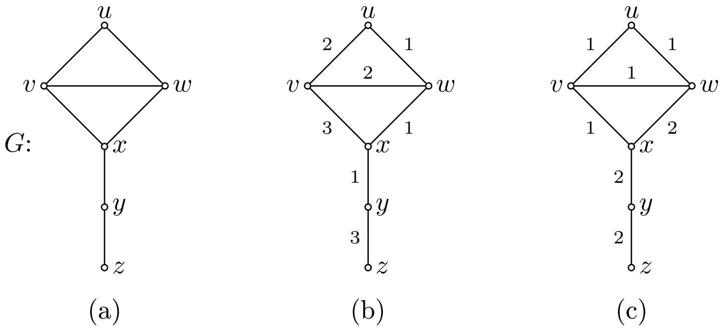
To illustrate these concepts, consider the graph G shown in Figure 1(a). A coloring of the edges of G is shown in Figure 1(b). For this 3-coloring c, the color codes of its vertices are

$$\begin{aligned} c(u) &= 110, \quad c(v) = 021, \quad c(w) = 210, \\ c(x) &= 201, \quad c(y) = 101, \quad c(z) = 001. \end{aligned}$$

Since the vertices of G have distinct color codes, c is a detectable coloring. Figure 1(c) shows yet another detectable coloring c' of the graph G of Figure 1(a). For this coloring,

$$c'(u) = 20,\ c'(v) = 30,\ c'(w) = 21,\ c'(x) = 12,\ c'(y) = 02,\ c'(z) = 01.$$

The coloring c' uses only two colors. Once a detectable 2-coloring for the graph G of Figure 1(c) was obtained, we can immediately conclude that $\det(G) = 2$ as every connected graph of order 3 or more has detection number at least 2.



(a) (b) (c)

Figure 1. A detectable coloring of a graph

The concept of detectable coloring was studied in [1], [2], [3], [4], [5], inspired by the basic problem in graph theory that concerns finding means to distinguish the vertices of a connected graph. The following results were stated in [2], [5].

Theorem A. *Let* c *be a* k*-coloring of the edges of a graph* G*. The maximum number of different color codes of the vertices of degree* r *in* G *is* $\binom{r+k-1}{r}$*.*

Theorem B. *If* c *is a detectable* k*-coloring of a connected graph* G *of order at least* 3*, then* G *contains at most* $\binom{r+k-1}{r}$ *vertices of degree* r*.*

Since vertices with distinct degrees in a connected graph always have distinct color codes, it is most challenging to find minimum detectable colorings of graphs having many vertices of the same degree. The detection numbers of complete graphs and complete bipartite graphs have been determined and detectable colorings of connected r-regular graphs and trees have been studied as well (see [2], [3], [4], [5]). The detection number of the cycle C_n of order n was established in [5].

Theorem C. *Let* $n \geqslant 3$ *be an integer and let* $l = \lceil \sqrt{n/2}\, \rceil$*. Then*

$$\det(C_n) = \begin{cases} 2l & \text{if } 2l^2 - l + 1 \leqslant n \leqslant 2l^2, \\ 2l - 1 & \text{if } 2(l-1)^2 + 1 \leqslant n \leqslant 2l^2 - l. \end{cases}$$

In this work, we first establish a formula for the detection number of paths in Section 2 and then study some extremal problems concerning detection numbers of unicyclic graphs in Section 3. We refer to the book [6] for graph theory notation and terminology not described in this paper.

2. DETECTABLE COLORING OF PATHS

In this section, we determine the detection numbers of all paths. In order to do this, we first present four results, the first of which is a consequence of Theorem B, the next two are well-known results in graph theory, and the fourth has a straightforward proof.

Corollary 2.1. *Let* $k \geqslant 2$ *be an integer. If* $n > \binom{k}{2} + 2$*, then* $\det(P_n) \geqslant k$*.*

Theorem D. *For each positive integer* k*, the complete graph* K_{2k} *can be factored into* $k-1$ *Hamiltonian cycles and a* 1*-factor.*

Theorem E. *For each positive integer* k*, the complete graph* K_{2k+1} *is Hamiltonian factorable* (*into* k *Hamiltonian cycles*)*.*

Lemma 2.2. *For each integer $n \geqslant 3$, there exists a unique positive integer l such that*

$$\binom{2l}{2} + 2 = 2l^2 - l + 2 \leqslant n \leqslant 2l^2 + 3l + 2 = \binom{2l+2}{2} + 1.$$

Furthermore, $l = \lceil \frac{1}{4}(-3 + \sqrt{8n-7}) \rceil$.

Theorem 2.3. *Let $n \geqslant 3$ and let $l = \lceil \frac{1}{4}(-3 + \sqrt{8n-7}) \rceil$. Then*

$$\det(P_n) = \begin{cases} 2l & \text{if } 2l^2 - l + 2 \leqslant n \leqslant 2l^2 + 3, \\ 2l + 1 & \text{if } 2l^2 + 4 \leqslant n \leqslant 2l^2 + 3l + 2. \end{cases}$$

Proof. Observe that $2l^2 - l + 2 \leqslant n \leqslant 2l^2 + 3l + 2$ by Lemma 2.2. It is easy to see that $\det(P_n) = 2$ for $3 \leqslant n \leqslant 5$ and so the result holds for $3 \leqslant n \leqslant 5$. Hence, we may restrict our attention to $n \geqslant 6$. We consider two cases, according to whether $2l^2 - l + 2 \leqslant n \leqslant 2l^2 + 3$ or $2l^2 + 4 \leqslant n \leqslant 2l^2 + 3l + 2$.

Case 1: $2l^2 - l + 2 \leqslant n \leqslant 2l^2 + 3$. By Corollary 2.1, if $n \geqslant \binom{2l}{2} + 3 = 2l^2 - l + 3$, then $\det(P_n) \geqslant 2l$. We now show that if $n = 2l^2 - l + 2$, then $\det(P_n) \geqslant 2l$. Since $n > \binom{2l-1}{2} + 2 = 2l^2 - 3l + 3$ for every $l \geqslant 1$, it follows that $\det(P_n) \geqslant 2l - 1$. Suppose that there exist a detectable $(2l-1)$-coloring c of P_n where $n = 2l^2 - l + 2$. Since the maximum number of vertices in a path with detection number $2l - 1$ is $\binom{(2l-1)+1}{2} + 2 = \binom{2l}{2} + 2 = 2l^2 - l + 2$, it follows that all possible color codes for the vertices of degree 2 are used in the coloring c. Observe that among the possible color codes for vertices of degree 2, there is a total of $2l - 2$ codes starting with 1. Indeed, among the codes containing exactly two 1's, there is a total of $2l - 2$ codes having 1 in the jth position for every $j = 1, 2, \ldots, 2l - 1$. Since the code of each end-vertex of $P_n = P_{2l^2-l+2}$ contains exactly one 1, it follows that in the corresponding detectable $(2l-1)$-tuple factorization of $P_n = P_{2l^2-l+2}$, two of the factors have an odd number of vertices of degree 1, which is not possible. Hence, $\det(P_n) = \det(P_{2l^2-l+2}) \neq 2l - 1$. Consequently, $\det(P_n) = \det(P_{2l^2-l+2}) \geqslant 2l$. This shows that $\det(P_n) \geqslant 2l$ if $2l^2 - l + 2 \leqslant n \leqslant 2l^2 + 3$.

We now show that $\det(P_n) \leqslant 2l$ if $2l^2 - l + 2 \leqslant n \leqslant 2l^2 + 3$ by considering two subcases, depending on whether $n = 2l^2 + 3$ or $2l^2 - l + 2 \leqslant n \leqslant l^2 + 2$.

Subcase 1.1: $n = 2l^2 + 3$. Let $V(K_{2l}) = \{1, 2, \ldots, 2l\}$. We now describe a method to assign a detectable coloring of the edges of P_{2l^2+3} with the elements of $V(K_{2l}) = \{1, 2, \ldots, 2l\}$. By Theorem D, there exists a factorization of K_{2l} into $l - 1$ Hamiltonian cycles

$$H_1, H_2, \ldots, H_{l-1}$$

and a 1-factor F. For each integer i with $1 \leqslant i \leqslant l - 1$, suppose that

$$H_i\colon\ 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2l}, 1,$$

where $a_{i,j}$ $(1 \leqslant j \leqslant 2l)$ is the jth vertex of H_i. We may assume, without loss of generality, that

$$H_1\colon\ 1, 2, \ldots, 2l, 1.$$

Therefore, $a_{1,j} = j$ for $1 \leqslant j \leqslant 2l$. Also, let b_1 be the neighbor of 1 in the 1-factor F of K_{2l}. Note that $b_1 \neq a_{i,2}$ and $b_1 \neq a_{i,2l}$ for every i with $1 \leqslant i \leqslant l-1$. Suppose that the edges of P_{2l^2+3} are encountered in the order

$$e_1, e_2, \ldots, e_{2l^2+2}$$

as we proceed along the path. For each integer k with $1 \leqslant k \leqslant 2l^2$, either $1 \leqslant k \leqslant 4l$ or $k = i(2l) + j$ for some integers i and j with $2 \leqslant i \leqslant l-1$ and $1 \leqslant j \leqslant 2l$. We now define a coloring $c\colon\ E(P_{2l^2+3}) \to V(K_{2l})$ of the edges of P_{2l^2+3} by

$$c(e_k) = \begin{cases} a_{1,\lceil k/2\rceil} = \lceil k/2\rceil & \text{if } 1 \leqslant k \leqslant 4l,\\ a_{i,j} & \text{if } k = i(2l)+j,\ 2 \leqslant i \leqslant l-1,\ 1 \leqslant j \leqslant 2l,\\ 1 & \text{if } k = 2l^2+1,\\ b_1 & \text{if } k = 2l^2+2. \end{cases}$$

In other words, we assign the color $\lceil k/2\rceil$ to the edge e_k for $1 \leqslant k \leqslant 4l$, color the next $2l$ edges $e_{2(2l)+j}$ $(1 \leqslant j \leqslant 2l)$ of P_{2l^2+3} by $a_{2,j}$, color the next $2l$ edges $e_{3(2l)+j}$ $(1 \leqslant j \leqslant 2l)$ by $a_{3,j}$ and so on. We continue this process until we have gone through all the Hamiltonian cycles $H_1, H_2, \ldots, H_{l-1}$. We have now assigned colors to the first $2l^2$ edges of P_{2l^2+3}. We assign the colors 1 and b_1 to the last two edges in that order. (Figure 2 illustrates a detectable $2l$-coloring for $P_{2l^2+3} = P_{21}$ for $l = 3$.) Since every vertex of degree 2 of P_{2l^2+3} is incident with two edges having a unique pair of colors and the edges incident with the end-vertices are colored 1 and $b_1 (\neq 1)$, c is a detectable $2l$-coloring of P_{2l^2+3} and so $\det(P_{2l^2+3}) \leqslant 2l$.

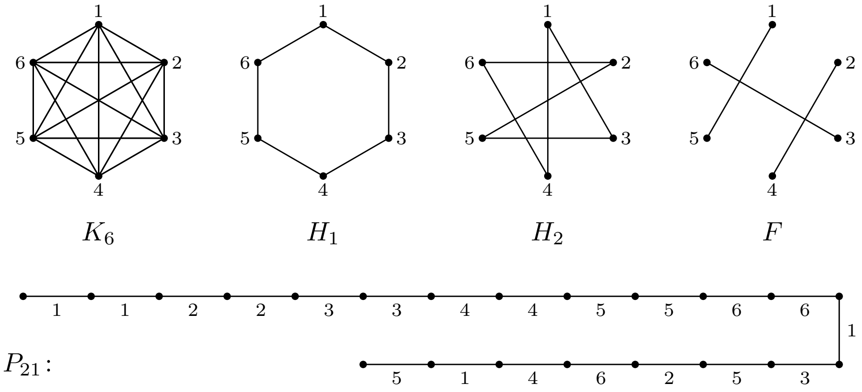


Figure 2. The detectable coloring of P_{21} in Subcase 1.1.

Subcase 1.2: *$n = 2l^2 + 3 - p$, for some integer p with $1 \leqslant p \leqslant l + 1$.* For each integer q with $1 \leqslant q \leqslant p$, let v_q be the vertex incident with e_{2q-1} and e_{2q} on P_{2l^2+3}. Suppressing the vertex v_q $(1 \leqslant q \leqslant p)$ so that e_{2q-1} and e_{2q} become the single edge f_q, we obtain a path P_{2l^2+3-p} of order $2l^2 + 3 - p$. Let c be the detectable $2l$-coloring of P_{2l^2+3} defined in Subcase 1.1. Define an edge coloring $c^*\colon\ E(P_{2l^2+3-p}) \to V(K_{2l})$ of P_{2l^2+3-p} by

$$c^*(e) = \begin{cases} c(e_{2q-1}) & \text{if } e = f_q \text{ for some } q \text{ with } 1 \leqslant q \leqslant p,\\ c(e) & \text{otherwise.}\end{cases}$$

The codes of the vertices of P_{2l^2+3-p} are all those of P_{2l^2+3} except those p $2l$-tuples for which 2 occurs in the qth coordinate for $1 \leqslant q \leqslant p$. This is a detectable $2l$-coloring of P_{2l^2+3-p} and so $\det(P_{2l^2+3-p}) \leqslant 2l$. Figure 3 illustrates a detectable $2l$-coloring of $P_{2l^2+3-p} = P_{17}$ for $l = 3$ and $p = 4 = l + 1$.

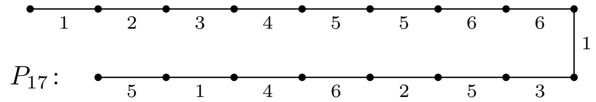


Figure 3. The detectable coloring of P_{17} in Subcase 1.2.

Case 2: $2l^2+4 \leqslant n \leqslant 2l^2+3l+2$. By Corollary 2.1, if $n > \binom{2l+1}{2}+2 = 2l^2+l+2$, then $\det(P_n) \geqslant 2l + 1$. Thus, if $n \geqslant 2l^2 + l + 3$, then $\det(P_n) \geqslant 2l + 1$. Now, let $n = 2l^2+l+2$. Since $n > \binom{2l}{2} = 2l^2-l+2$, it follows that $\det(P_n) = \det(P_{2l^2+l+2}) \geqslant 2l$. Suppose now that there exists a detectable $2l$-coloring c of $P_n = P_{2l^2+l+2}$. Because the largest possible number of vertices in a path with detection number $2l$ is $\binom{2l+1}{2} + 2 = 2l^2 + l + 2$, all possible color codes for the vertices of degree 2 are used in the coloring c. Observe that among the codes containing exactly two 1's, there is a total of $2l - 1$ codes having 1 in the jth position for every $j = 1, 2, \ldots, 2l$. The code of each end-vertex of $P_n = P_{2l^2+l+2}$ contains exactly one 1. This implies that in the corresponding detectable $2l$-tuple factorization of $P_n = P_{2l^2+l+2}$, all but two of the factors have an odd number of vertices of degree 1, which is not possible. Hence, $\det(P_n) = \det(P_{2l^2+l+2}) \geqslant 2l + 1$. Suppose now that $1 \leqslant p \leqslant l - 2$. Then $2l^2 + l + 2 - p \geqslant 2l^2 + l + 2 - (l - 2) = 2l^2 + 4$. But $2l^2 + 4 > 2l^2 - l + 2$. It follows that $\det(P_{2l^2+l+2-p}) \geqslant 2l$ for every $p = 1, 2, \ldots, l - 2$. If c is a detectable $2l$-coloring of $P_{2l^2+l+2-p}$, then in the corresponding $2l$-tuple factorization of $P_{2l^2+l+2-p}$, there would be at least $2l - (2 + 2(l - 2)) = 2$ factors having an odd number of vertices of degree 1 which is not possible. It follows then that $\det(P_n) \geqslant 2l + 1$ if $n \geqslant 2l^2 + 4$. Since $2l^2 + 3l + 2 > 2l^2 + 4$ for all $l \geqslant 1$, we have $\det(P_n) \geqslant 2l + 1$ if $2l^2 + 4 \leqslant n \leqslant 2l^2 + 3l + 2$.

It remains to show that $\det(P_n) \leqslant 2l+1$ whenever $2l^2+4 \leqslant n \leqslant 2l^2+3l+2$. This is accomplished by finding a detectable $(2l+1)$-coloring of P_n. We consider two subcases.

Subcase 2.1. $n = 2l^2+3l+2$. Let $V(K_{2l+1}) = \{1,2,\ldots,2l,2l+1\}$. We now describe a method to assign a detectable coloring of P_{2l^2+3l+2} with the elements of $V(K_{2l+1})$. By Theorem E, K_{2l+1} can be factored into l Hamiltonian cycles

$$H_1, H_2, \ldots, H_l.$$

For each integer i with $1 \leqslant i \leqslant l$, suppose that

$$H_i\colon\ 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2l+1}, 1,$$

where $a_{i,j}$ $(1 \leqslant j \leqslant 2l+1)$ is the jth vertex of H_i. We may assume, without loss of generality, that

$$H_1\colon\ 1,2,\ldots,2l+1,1.$$

Therefore, $a_{1,j} = j$ for $1 \leqslant j \leqslant 2l+1$. Suppose that the edges of P_{2l^2+3l+2} are encountered in the order

$$e_1, e_2, \ldots, e_{2l^2+3l+1}$$

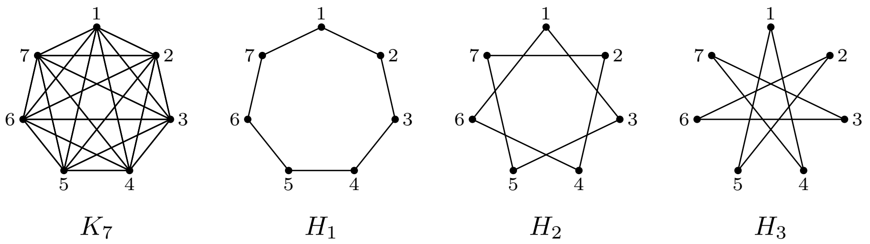
as we proceed along the path. We now define a coloring $c\colon\ E(P_{2l^2+3l+2}) \to V(K_{2l+1})$ of the edges of P_{2l^2+l+2} by

$$c(e_k) = \begin{cases} a_{1,\lceil k/2\rceil} = \lceil k/2\rceil & \text{if } 1 \leqslant k \leqslant 4l+2,\\ a_{i,j} & \text{if } k = i(2l+1)+j,\ 2 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant 2l+1. \end{cases}$$

That is, we color the first $4l+2$ edges e_k $(1 \leqslant k \leqslant 4l+2)$ of P_{2l^2+3l+2} by $\lceil k/2\rceil$, color the next $2l+1$ edges $e_{2(2l+1)+j}$ $(1 \leqslant j \leqslant 2l+1)$ of P_{2l^2+3l+2} by $a_{2,j}$, color the next $2l+1$ edges $e_{3(2l+1)+j}$ $(1 \leqslant j \leqslant 2l+1)$ by $a_{3,j}$ and so on. (Figure 4 illustrates the detectable $(2l+1)$-coloring of $P_{2l^2+3l+2} = P_{29}$ for $l=3$.) The last $2l+1$ edges $e_{l(2l+1)+j}$ $(1 \leqslant j \leqslant 2l+1)$ are then colored by $a_{l,j}$. Since every vertex of degree 2 of P_{2l^2+3l+2} is incident with two edges having a unique pair of colors and the edges incident with the two end-vertices are assigned 1 and $a_{l,2l+1} \neq 1$, it follows that c is a detectable $(2l+1)$-coloring of P_{2l^2+3l+2} and so $\det(P_{2l^2+3l+2}) \leqslant 2l+1$. Hence, $\det(P_{2l^2+3l+2}) = 2l+1$.

Subcase 2.2: $2l^2+4 \leqslant n \leqslant 2l^2+3l+1$. Let $n = (2l^2+3l+2)-p$, where $1 \leqslant p \leqslant 3l-2$. We consider two subcases, according to whether $1 \leqslant p \leqslant 2l+1$ or $2l+2 \leqslant p \leqslant 3l-2$.

Subcase 2.2.1: $1 \leqslant p \leqslant 2l+1$. For each integer q with $1 \leqslant q \leqslant p$, let v_q be the vertex incident with e_{2q-1} and e_{2q} on P_{2l^2+3l+2}. Suppressing the vertex v_q $(1 \leqslant q \leqslant$



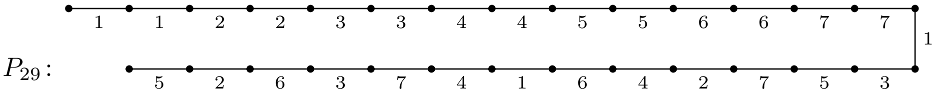


Figure 4. The detectable coloring of P_{29} in Subcase 2.1.

p) so that e_{2q-1} and e_{2q} become the single edge f_q, we obtain a path $P_{2l^2+3l+2-p}$ of order $2l^2+3l+2-p$. Let c be the detectable $(2l+1)$-coloring of P_{2l^2+3l+2} defined in Subcase 2.1. Define an edge coloring $c^*\colon\ E(P_{2l^2+3l+2-p}) \to V(K_{2l+1})$ of $P_{2l^2+3l+2-p}$ by

$$c^*(e) = \begin{cases} c(e_{2q-1}) & \text{if } e = f_q \text{ for some } q \text{ with } 1 \leqslant q \leqslant p,\\ c(e) & \text{otherwise.}\end{cases}$$

The codes of the vertices of $P_{2l^2+3l+2-p}$ are all those of P_{2l^2+3l+2} except those $(2l+1)$-tuples for which 2 occurs in the qth coordinate for $1 \leqslant q \leqslant p$. Since this is a detectable $(2l+1)$-coloring of $P_{2l^2+3l+2-p}$, it follows that $\det(P_{2l^2+3l+2-p}) \leqslant 2l+1$. Figure 5 illustrates the detectable $(2l+1)$-coloring of $P_{2l^2+3l+2-p} = P_{25}$ when $l=3$ and $p=4$.

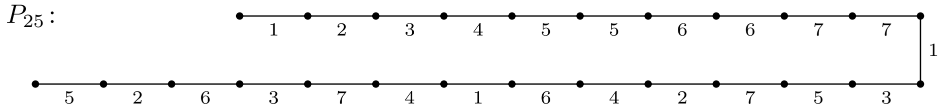


Figure 5. The detectable coloring of P_{25} in Subcase 2.2.1.

Subcase 2.2.2: $2l+2 \leqslant p \leqslant 3l-2$. Note that this subcase can only occur when $l \geqslant 4$. Let $p = (2l+1)+h$ where $1 \leqslant h \leqslant l-3$. Observe that $l-3 < 2l+1$ for all positive integers l. Recall that the edges of P_{2l^2+3l+2} are encountered in the order

$$e_1, e_2, \ldots, e_{2l^2+3l}, e_{2l^2+3l+1}$$

as we proceed along the path. Let v_i denote the vertex of P_{2l^2+3l+2} incident with e_i and e_{i+1} for $1 \leqslant i \leqslant 6l+3$. First, we construct a path $P_{(2l^2+3l+2)-(2l+1)}$ from P_{2l^2+3l+2} by

(1) deleting the vertices v_{4l+3}, v_{4l+4}, …, v_{6l+2} and therefore, deleting the $2l+1$ edges e_{4l+3}, e_{4l+4}, …, e_{6l+3} (which correspond to the Hamiltonian cycle H_2), and

(2) identifying the vertices v_{4l+2} and v_{6l+3}.

This produces a path $P_{(2l^2+3l+2)-(2l+1)}$ of order $(2l^2+3l+2)-(2l+1)$. Next, we suppress the vertex v_{2j-1} for $1 \leqslant j \leqslant h$, where the two edges e_{2j-1} and e_{2j} become the single edge f_j. This produces a path $P_{(2l^2+l+2)-(2l+1+h)} = P_n$. Let c be the detectable $(2l+1)$-coloring of P_{2l^2+3l+2} defined in Subcase 2.1. Define an edge coloring $c'\colon E(C_n) \to V(K_{2l+1})$ by

$$c'(e) = \begin{cases} c(e_{2j-1}) & \text{if } e = f_j \text{ for } 1 \leqslant j \leqslant h, \\ c(e) & \text{otherwise.} \end{cases}$$

Figure 6 illustrates a detectable $(2l+1)$-coloring of $P_{2l^2+3l+2} = P_{46}$ where $l = 4$ and a detectable $(2l+1)$-coloring of $P_{(2l^2+3l+2)-(2l+1+h)} = P_{36}$ (for $l = 4$ and $h = 1$) obtained from the coloring of P_{46}.

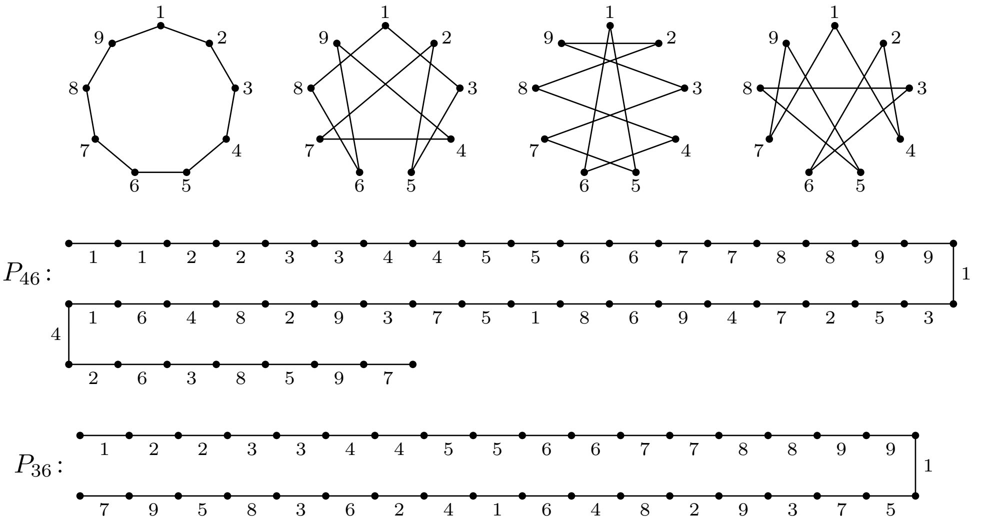


Figure 6. Detectable colorings of P_{46} and P_{36} in Subcase 2.2.2 in the proof of Theorem 2.3.

The codes of the vertices of P_n are all these of P_{2l^2+3l+2} except

(a) those $(2l+1)$-tuples for which 2 occurs in the jth coordinate for $1 \leqslant j \leqslant h$ (there are h such $(2l+1)$-tuples) and

(b) those $(2l+1)$-tuples that are produced from the Hamiltonian cycle H_2; that is, the codes of the vertices $v_{4l+3}, v_{4l+4}, \ldots, v_{6l+3}$ in the path P_{2l^2+3l+2} (there are $2l+1$ such $(2l+1)$-tuples).

Since c' is a detectable $(2l+1)$-coloring of P_n, it follows that $\det(P_n) \leqslant 2l+1$. □

3. Extremal problems on unicyclic graphs

A connected graph with exactly one cycle is called a *unicyclic graph.* A graph G of order n and size m is unicyclic if and only if G is connected and $m = n$. In this section, we study some extremal problems concerning detection numbers of unicyclic graphs, in particular, the problems of determining how large and how small the detection number of a unicyclic graph of a fixed order can be.

Observe that if n_i is the number of vertices of degree i in a unicyclic graph G with maximum degree Δ, then

$$n_1 = n_3 + 2n_4 + 3n_5 + \ldots + (\Delta - 2)n_\Delta. \tag{1}$$

For each integer $n \geqslant 3$, let $D_u(n)$ denote the maximum detection number among all unicyclic graphs of order n and $d_u(n)$ the minimum detection number among all unicyclic graphs of order n. That is, if $\mathcal{U}_n$ is the set of all unicyclic graphs of order n, then

$$\begin{aligned} D_u(n) &= \max\{\det(G)\colon\ G \in \mathcal{U}_n\} \\ d_u(n) &= \min\{\det(G)\colon\ G \in \mathcal{U}_n\}. \end{aligned}$$

Figure 7 shows all the unicyclic graphs of order n for $3 \leqslant n \leqslant 5$ together with a minimum detectable coloring for each. Hence $D_u(3) = d_u(3) = 3$, and $D_u(n) = 3$ and $d_u(n) = 2$ for $n = 4, 5$.

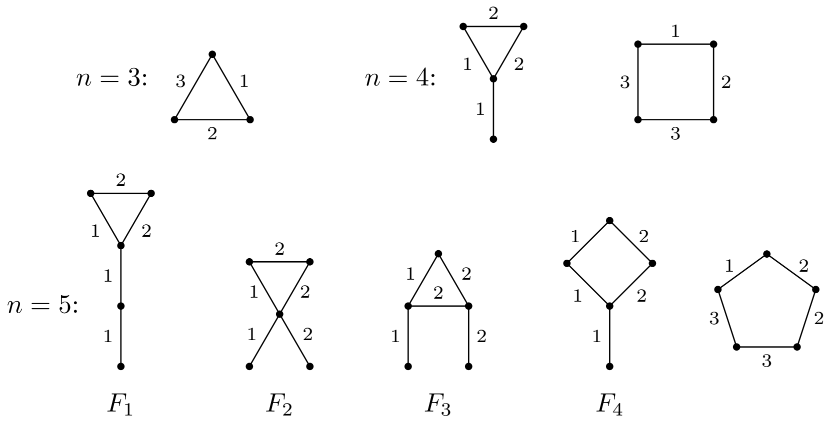


Figure 7. Minimum detectable colorings of unicyclic graphs of order $n = 3, 4, 5$.

In order to determine $D_u(n)$ for $n \geqslant 6$, we first present a lemma. For a graph F, let $m(F)$ denote the size of F.

Lemma 3.1. *Let G be connected graph of order $n \geqslant 3$. If H is a connected subgraph of G, then*

$$\det(G) - \det(H) \leqslant m(G) - m(H).$$

Proof. Color the $m(H)$ edges of H using $k = \det(H)$ colors and color the remaining $m(G)-m(H)$ edges of G using the colors $k+1, k+2, \ldots, k+(m(G)-m(H))$. This gives us a detectable $(m(G)-m(H)+k)$-coloring of G. It follows that $\det(G) \leqslant m(G) - m(H) + \det(H)$. □

The following is an immediate consequence of Lemma 3.1

Corollary 3.2. *Let G be a connected graph of order $n \geqslant 3$ and size m. If g is the girth of G, then*

$$\det(G) \leqslant m - g + \det(C_g).$$

Proposition 3.3. *For $n \geqslant 6$, $D_u(n) = n - 3$.*

Proof. It is easy to verify that $\det(K_{1,n-1} + e) = n - 3$ for $n \geqslant 6$ and so $D_u(n) \geqslant n - 3$ for $n \geqslant 6$. It remains to show that $D_u(n) \leqslant n - 3$ for $n \geqslant 6$. Let G be a unicyclic graph of order $n \geqslant 6$ and let g be the girth of G. If $3 \leqslant g \leqslant 5$, then G contains a subgraph F such that $F \in \{F_1, F_2, F_3, F_4, F_5\}$, where F_i $(1 \leqslant i \leqslant 4)$ is shown in Figure 7 and F_5 is the graph obtained from $C_5\colon v_1, v_2, v_3, v_4, v_5, v_1$ by adding a pendant edge vv_1. We have seen that $\det(F_i) = m(F_i) - 3$ for $1 \leqslant i \leqslant 4$. For the graph F_5, the 3-coloring c defined by $c(v_1v_2) = 1$, $c(v_2v_3) = c(v_3v_4) = 2$, and $c(v_4v_5) = c(v_1v_5) = c(vv_1) = 3$ is a minimum detectable coloring of F_5 and so $\det(F_5) = 3 = m(F_5) - 3$. Therefore, $\det(F) = m(F) - 3$ for each $F \in \{F_1, F_2, F_3, F_4, F_5\}$. It then follows by Lemma 3.1 that $\det(G) \leqslant m(G) + \det(F) - m(F) = n + (m(F) - 3) - m(F) = n - 3$ for $3 \leqslant g \leqslant 5$. If $g \geqslant 6$, then $\det(C_g) \leqslant g - 3$ by Theorem C. It then follows by Corollary 3.2 that $\det(G) \leqslant n - g + \det(C_g) \leqslant n - g + (g - 3) = n - 3$. Thus, $D_u(n) \leqslant n - 3$ for all $n \geqslant 6$. □

Next, we determine the minimum detection number among all unicyclic graphs of order n. According to Theorem B, every unicyclic graph of order $n \geqslant 3$ having detection number k contains at most k end-vertices and at most $\frac{1}{2}k(k+1)$ vertices of degree 2. It then follows by (1) that

$$n \leqslant k + \frac{k(k+1)}{2} + k = \frac{k^2 + 5k}{2}.$$

Furthermore, if G is a unicyclic graph of order $n = \frac{1}{2}(k^2 + 5k)$ with $\det(G) = k$, then G must contain exactly k end-vertices, exactly $\frac{1}{2}k(k+1)$ vertices of degree 2, and exactly k vertices of degree 3. We first determine $d_u(n)$ for the values of n mentioned above.

Theorem 3.4. *Let $k \geqslant 2$ be an integer. If $n = \frac{1}{2}(k^2+5k)$, then $d_u(n) = k$.*

Proof. First, we show that if $n = \frac{1}{2}(k^2+5k)$, then $d_u(n) \geqslant k$. Assume, to the contrary, that there exists a unicyclic graph G of order $\frac{1}{2}(k^2+5k)$ such that $\det(G) \leqslant k-1$. By Theorem B, G has at most $k-1$ end-vertices and at most $\frac{1}{2}k(k-1)$ vertices of degree 2. Therefore, G contains at least

$$\frac{k^2+5k}{2} - (k-1) - \frac{k(k-1)}{2} = 2k+1$$

vertices of degree 3 or more. It then follows by (1) that G contains at least $2k+1$ end-vertices, which is impossible. Thus, $d_u(n) \geqslant k$.

To show that $d_u(n) \leqslant k$, we construct a unicyclic graph G_k of order $n = \frac{1}{2}(k^2+5k)$ having detection number k such that G_k has exactly k end-vertices, exactly $\frac{1}{2}(k^2+k)$ vertices of degree 2, and exactly k vertices of degree 3. We consider two cases, according as to whether k is odd or even.

Case 1. *k is odd.* Then $k = 2l-1$ for some integer $l \geqslant 2$. We now construct G_k. Let

$$C_{2l^2-l}\colon\ v_1, v_2, \ldots, v_{2l^2-l}, v_1$$

be a cycle of length $2l^2-l$ and for $1 \leqslant i \leqslant k$, let Q_i be a copy of K_2 with $V(Q_i) = \{u_{i,1}, u_{i,2}\}$. Then the graph G_k is obtained from C_{2l^2-l} and Q_i $(1 \leqslant i \leqslant k)$ by adding the edges $v_{2i}u_{i,1}$ $(1 \leqslant i \leqslant k)$. Observe that G_k is a unicyclic graph of order $n = (2l^2-l) + 2(2l-1) = \frac{1}{2}(k^2+5k)$.

We now define a k-coloring c for the edges of G_k. First, we color the $2l^2-l$ edges of C_{2l^2-l} with the elements of $V(K_{2l-1}) = \{1, 2, \ldots, 2l, 2l-1\}$ as follows. Let $H_1, H_2, \ldots, H_{l-1}$ be $l-1$ pairwise edge-disjoint Hamiltonian cycles of K_{2l-1}. For each integer i with $1 \leqslant i \leqslant l-1$, suppose that $H_i\colon\ 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2l-1}, 1$, where $a_{i,j}$ $(1 \leqslant j \leqslant 2l-1)$ is the jth vertex of H_i and we assume that $H_1\colon\ 1, 2, \ldots, 2l-1, 1$. Therefore, $a_{1,j} = j$ for $1 \leqslant j \leqslant 2l-1$. Suppose that the edges of C_{2l^2-l} are encountered in the order

$$e_1, e_2, \ldots, e_{2l^2-l}, e_{2l^2-l+1} = e_1$$

as we proceed about the cycle in some direction. Then we define

$$c(e_k) = \begin{cases} a_{1,\lceil k/2\rceil} = \lceil k/2\rceil & \text{if } 1 \leqslant k \leqslant 4l-2,\\ a_{i,j} & \text{if } k = i(2l-1)+j,\ 2 \leqslant i \leqslant l-1, \text{ and } 1 \leqslant j \leqslant 2l-1. \end{cases}$$

It was shown in the proof of Theorem C that this coloring of the cycle C_{2l^2-l} is detectable. Furthermore, let $c(v_{2i}u_{i,1}) = c(u_{i,1}u_{i,2}) = i$ for $1 \leqslant i \leqslant k$. Thus c uses k colors. It remains to show that c is detectable. Note that the color codes of the vertices of G_k consist of all possible color codes for vertices of degrees 1 and 2 together with all the k-tuples whose only nonzero entry is 3 occurring in the ith coordinate for $1 \leqslant i \leqslant k$. Since each of the color codes described above occurs exactly once, c is a detectable k-coloring for G_k. Therefore, $\det(G_k) \leqslant k$ and consequently, $\det(G_k) = k$.

Case 2. *k is even.* Then $k = 2l$ for some positive integer l. If $k = 2$, then $n = 7$. Since the unicyclic graph G_2 of order 7 in Figure 8 has detection number 2 (as shown in that figure), the result holds for $k = 2$. Thus we may assume that $k \geqslant 4$ and so $l \geqslant 2$.

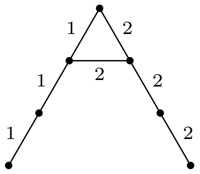


Figure 8. A detectable 2-coloring of G_2 in Case 2.

Let $C_{2l^2}\colon v_1, v_2, \ldots, v_{2l^2}, v_1$ be a cycle of length of $2l^2$. For $1 \leqslant i \leqslant l$, let Q_i be a copy of K_2 with $V(Q_i) = \{u_{i,1}, u_{i,2}\}$ and for $l+1 \leqslant i \leqslant 2l$, let $Q_i\colon u_{i,1}, u_{i,2}, u_{i,3}$ be a copy of a path of length 2. Then the graph G_k is obtained from C_{2l^2} and Q_i $(1 \leqslant i \leqslant k)$ by adding the edges $v_{2i}u_{i,1}$ $(1 \leqslant i \leqslant k)$. Observe that G_k is a unicyclic graph of order $n = 2l^2 + 2l + 3l = \frac{1}{2}(k^2 + 5k)$.

We now define a k-coloring c for the edges of G_k. First, we color the $2l^2$ edges of the cycle C_{2l^2} with the elements of $V(K_{2l}) = \{1, 2, \ldots, 2l\}$ as follows. Let $H_1, H_2, \ldots, H_{l-1}$ be $l-1$ pairwise edge-disjoint Hamiltonian cycles of K_{2l} and let F be the 1-factor of K_{2l} with $E(F) = \{x_iy_i\colon 1 \leqslant i \leqslant l\}$, where $x_l = 2l = k$. For each integer i with $1 \leqslant i \leqslant l-1$, suppose that $H_i\colon 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2l}, 1$, where $a_{i,j}$ $(1 \leqslant j \leqslant 2l)$ is the jth vertex of H_i and $H_1\colon 1, 2, \ldots, 2l, 1$, say. Therefore, $a_{1,j} = j$ for $1 \leqslant j \leqslant 2l$. Suppose that the edges of C_{2l^2} are encountered in the order

$$e_1, e_2, \ldots, e_{2l^2}, e_{2l^2+1} = e_1,$$

as we proceed about the cycle in some direction. For each integer k with $1 \leqslant k \leqslant 2l^2$, either $1 \leqslant k \leqslant 4l$ or $k = i(2l) + j$ for some integers i and j with $2 \leqslant i \leqslant l-1$ and $1 \leqslant j \leqslant 2l$. We now define

$$c(e_k) = \begin{cases} a_{1,\lceil k/2 \rceil} = \lceil k/2 \rceil & \text{if } 1 \leqslant k \leqslant 4l, \\ a_{i,j} & \text{if } k = i(2l) + j,\ 2 \leqslant i \leqslant l-1 \text{ and } 1 \leqslant j \leqslant 2l. \end{cases}$$

This coloring of C_{2l^2} is detectable, which was shown in the proof of Theorem C. Furthermore, for $1 \leqslant i \leqslant l$, let $c(v_{2i}u_{i,1}) = c(u_{i,1}u_{i,2}) = x_i$ and for $l+1 \leqslant i \leqslant 2l$, let $c(v_{2i}u_{i,1}) = x_{i-l}$ and let $c(u_{i,1}u_{i,2}) = c(u_{i,2}u_{i,3}) = y_{i-l}$. Thus c uses k colors. It remains to show that c is detectable. Note that in the coloring c all possible color codes for vertices of degrees 1 and 2 are used exactly once. The vertices of degree 3, namely v_{2i} $(i = 1, 2, \ldots, l)$, also have distinct color codes since v_{2i} is the only vertex whose code has an entry that is at least 2 in the ith position. Therefore, $\det(G_k) \leqslant k$ and consequently, $\det(G_k) = k$. □

With the aid of Theorem 3.4, we are now able to establish the following.

Theorem 3.5. *For each integer $k \geqslant 2$, if*

$$\frac{(k-1)^2 + 5(k-1)}{2} + 1 \leqslant n \leqslant \frac{k^2+5k}{2}$$

then $d_u(n) = k$.

Proof. First, we show that if $\frac{1}{2}(k^2+3k-2) = \frac{1}{2}((k-1)^2+5(k-1))+1 \leqslant n \leqslant \frac{1}{2}(k^2+5k)$, then $d_u(n) \geqslant k$. Assume, to the contrary, that there exists a unicyclic graph G of order $n \geqslant \frac{1}{2}(k^2+3k-2)$ such that $\det(G) \leqslant k-1$. By Theorem B, G has at most $k-1$ end-vertices and at most $\frac{1}{2}(k^2-k)$ vertices of degree 2. Therefore, G contains at least $n-(k-1)-\frac{1}{2}(k^2-k) \geqslant \frac{1}{2}(k^2+3k-2)-(k-1)-\frac{1}{2}(k^2-k) = k$ vertices of degree 3 or more. It then follows by (1) that G has at least k end-vertices which is impossible.

We next show that $d_u(n) \leqslant k$ if $\frac{1}{2}(k^2+3k-2) \leqslant n \leqslant \frac{1}{2}(k^2+5k)$. Theorem 3.4 shows that this is true when $n = \frac{1}{2}(k^2+5k)$. Assume therefore that $n = \frac{1}{2}(k^2+5k) - p$ where $1 \leqslant p \leqslant k+1$. We consider two cases, according to whether k is odd or even.

Case 1. *k is odd.* Then $k = 2l-1$ for some integer $l \geqslant 2$. There are two subcases, depending on whether $1 \leqslant p \leqslant k$ or $p = k+1$.

Subcase 1.1. $1 \leqslant p \leqslant k$. Construct the unicyclic graph G of order $\frac{1}{2}(k^2+5k)-p$ from the unicyclic graph G_k described in Theorem 3.4 by suppressing the vertices $u_{i,1}$ so that the edges $v_{2i}u_{i,1}$ and $u_{i,1}u_{i,2}$ become the single edge $v_{2i}u_{i,2}$ where $1 \leqslant i \leqslant p$. Define a k-coloring c^* of G by

$$c^*(e) = \begin{cases} c(v_{2i}u_{i,1}) & \text{if } e = v_{2i}u_{i,2} \text{ for some } i \text{ with } 1 \leqslant i \leqslant p,\\ c(e) & \text{otherwise.} \end{cases}$$

The color codes for the vertices of G are all those of G_k except those $(2l-1)$-tuples for which 2 occurs in the ith coordinate (and 0 occurs everywhere else) for $1 \leqslant i \leqslant p$.

Thus c^* is a detectable k-coloring of G. Since G has k end-vertices, it follows that $\det(G) = k$. Consequently, $d_u(n) \leqslant k$.

Subcase 1.2. $p = k + 1$ and so $n = \frac{1}{2}(k^2 + 3k - 2)$. Consider the unicyclic graph G of order $\frac{1}{2}(k^2 + 3k - 2) + 1 = \frac{1}{2}(k^2 + 3k)$ together with the edge coloring described in Subcase 1.1 (that is, when $p = k$ in Subcase 1.1.). We delete the edge $v_{2k}v_{2k-1}$, identify the vertices v_{2k} and v_{2k-1}, and label this new vertex by v. This gives us a unicyclic graph G' of order $n = \frac{1}{2}(k^2 + 3k - 2)$. Observe that the color codes of the vertices of G' are those of G except for those of v_{2k} and v_{2k-1}, and that v is the only vertex of G' of degree 3 whose color code has 2 as the kth coordinate. Hence, we have a detectable k-coloring of G'. Since G' has k end-vertices, it follows that $\det(G') \geqslant k$ and consequently, $\det(G') = k$. Therefore, $d_u(n) \leqslant k$.

Case 2. *k is even.* Then $k = 2l$ for some positive integer l. The result holds for $k = 2$ (that is, $l = 1$) as the graphs in Figure 9 show. For $k \geqslant 4$ (and so $l \geqslant 2$), we consider three subcases, according to whether $1 \leqslant p \leqslant k/2$, $k/2 + 1 \leqslant p \leqslant k$, or $p = k + 1$.

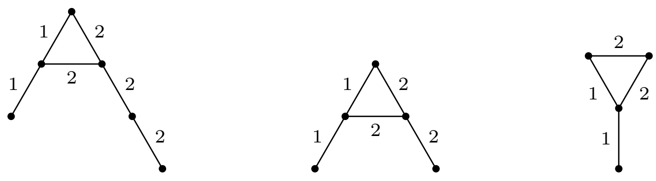


Figure 9. Detectable colorings when $k = 2$ in Case 2.

Subcase 2.1. $1 \leqslant p \leqslant k/2$. Construct the unicyclic graph G of order $\frac{1}{2}(k^2 + 5k) - p$ from the unicyclic graph G_k described in Theorem 3.4 by suppressing the vertices $u_{i,1}$ so that the edges $v_{2i}u_{i,1}$ and $u_{i,1}u_{i,2}$ become the single edge $v_{2i}u_{i,2}$ where $1 \leqslant i \leqslant p$. Define a k-coloring c^* of G by

$$c^*(e) = \begin{cases} c(v_{2i}u_{i,1}) & \text{if } e = v_{2i}u_{i,2} \text{ for some } i \text{ with } 1 \leqslant i \leqslant p, \\ c(e) & \text{otherwise.} \end{cases}$$

The color codes for the vertices of G are all those of G_k except those $(2l)$-tuples for which 2 occurs in the x_ith coordinate (and 0 occurs everywhere else) for $1 \leqslant i \leqslant p$. Thus c^* is a detectable k-coloring of G. Since G has k end-vertices, it follows that $\det(G) \leqslant k$. Consequently, $d_u(n) \leqslant k$.

Subcase 2.2. $k/2 + 1 \leqslant p \leqslant k$. Construct the unicyclic graph G' of order $\frac{1}{2}(k^2 + 5k) - p$ from the unicyclic graph G of order $\frac{1}{2}(k^2 + 5k) - k/2 = \frac{1}{2}(k^2 + 4k)$ described in Subcase 2.1 (that is, when $p = k/2$ in Subcase 2.1) by suppressing the vertices $u_{i,2}$ so that the edges $u_{i,1}u_{i,2}$ and $u_{i,2}u_{i,3}$ become the single edge $u_{i,1}u_{i,3}$

where $k/2+1 \leqslant i \leqslant p$. Define a k-coloring c' for G' by

$$c'(e) = \begin{cases} c^*(u_{i,2}u_{i,3}) & \text{if } e = u_{i,1}u_{i,3} \text{ for some } i \text{ with } k/2+1 \leqslant i \leqslant p, \\ c^*(e) & \text{otherwise.} \end{cases}$$

The color codes for the vertices of G' are all those of G except those $(2l)$-tuples for which 2 occurs in the $y_{i-k/2}$th coordinate (and 0 occurs everywhere else) for $k/2+1 \leqslant i \leqslant p$. Thus c' is a detectable k-coloring of G'. Since G' has k end-vertices, it follows that $\det(G') = k$. Consequently, $d_u(n) \leqslant k$.

Subcase 2.3. $p = k+1$. That is, $n = \frac{1}{2}(k^2+3k-2)$. Consider the unicyclic graph G' of order $\frac{1}{2}(k^2+3k-2)+1 = \frac{1}{2}(k^2+3k)$ together with the edge coloring described in Subcase 2.2 (that is, when $p = k$ in Subcase 2.2). We delete the edge $v_{2k}v_{2k-1}$, identify the vertices v_{2k} and v_{2k-1}, and label this new vertex by v. This gives us a unicyclic graph G'' of order $n = \frac{1}{2}(k^2+3k-2)$. Observe that the color codes of the vertices of G'' are those of G' except for those of v_{2k} and v_{2k-1}, and that v is the only vertex of G'' of degree 3 whose color code has 2 as the kth coordinate. Hence, we have a detectable k-coloring of G''. Since G'' has k end-vertices, it follows that $\det(G'') = k$. Consequently, $d_u(n) \leqslant k$. □

Solving for the smallest integer k for which $n \leqslant \frac{1}{2}(k^2+5k)$, we obtain the following.

Theorem 3.6. *For each integer* $n \geqslant 4$,

$$d_u(n) = \left\lceil \frac{-5+\sqrt{8n+25}}{2} \right\rceil.$$

By Theorem 3.6, $d_u(n) \approx \sqrt{2n}$ for large values of n. We now determine all pairs k, n of integers for which there exists a unicyclic graph of order n having detection number k.

Theorem 3.7. *Let* $k \geqslant 2$ *and* $n \geqslant 3$ *be integers. There exists a unicyclic graph* G *of order* n *such that* $\det(G) = k$ *if and only if* $d_u(n) \leqslant k \leqslant D_u(n)$.

Proof. By definition, if G is a unicyclic graph of order n such that $\det(G) = k$, then $d_u(n) \leqslant k \leqslant D_u(n)$. It remains to verify the converse. The result holds for $3 \leqslant n \leqslant 5$ as the graphs in Figure 7 show. Furthermore, the graphs in Figure 10 show that the result holds for $n = 6, 7$ as well.

We now assume that $n \geqslant 8$ and so $k \geqslant 3$. In this case, we show that if

$$d_u(n) = \left\lceil \frac{-5+\sqrt{8n+25}}{2} \right\rceil \leqslant k \leqslant n-3 = D_u(n),$$

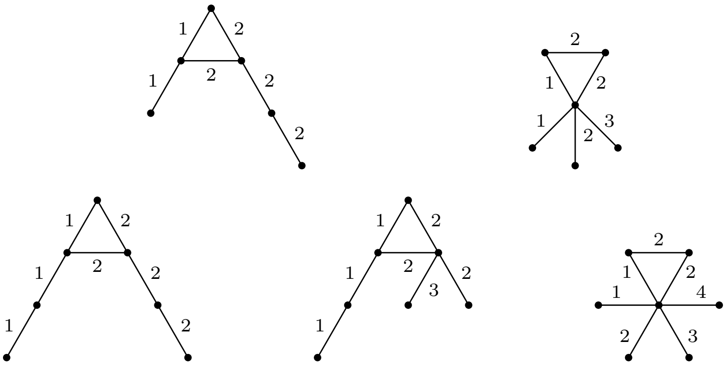


Figure 10. Minimum detectable colorings for graphs of order $n = 6, 7$.

then there is a unicyclic graph G of order n such that $\det(G) = k$. For each integer i with $i = 0, 1, \ldots, n - d_u(n) - 3$, we construct a unicyclic graph H_i such that H_i has order n and $\det(H_i) = d_u(n) + i$. Let H be the unicyclic graph of order n described in the proof of Theorem 3.5 and c the $d_u(n)$-coloring described in the proof of Theorem 3.5 as well. We first construct a unicyclic graph H_0 from H as follows.

(a) If vertex $u_{1,1} \in V(H)$, then delete the vertex $u_{1,2}$; while if $u_{1,1} \notin V(H)$, then delete the vertex v_3 and join the vertices v_2 and v_4.

(b) Delete the edge v_1v_2, add the vertex v, and join v to v_1 and v_2.

Then H_0 has exactly $d_u(n)$ end-vertices and so $\det(H_0) \geqslant d_u(n)$. Define the coloring $c_0\colon E(H_0) \to \{1, 2, \ldots, d_u(n)\}$ by

$$c_0(e) = \begin{cases} c(e) & \text{if} \quad e \in E(H), \\ 1 & \text{if} \quad e \notin E(H). \end{cases}$$

Then c_0 is a detectable $d_u(n)$-coloring of H_0. Thus $\det(H_0) \leqslant d_u(n)$ and so $\det(H_0) = d_u(n)$.

Observe that if $l = \lceil d_u(n)/2 \rceil$, then the girth of H_0 is $2l^2 - l$, $2l^2$, $2l^2 - l + 1$, or $2l^2 + 1$, depending on (1) the parity of $d_u(n)$ and (2) whether the vertex $u_{1,1}$ is in H or not. In each case, if we denote the girth of H_0 by $g(l)$, then $g(l) > 3$ and so $H_0 \neq K_{1,n-1} + e$. Note that the vertices v_1, v and v_2, in this order, are consecutive vertices in the cycle of H_0. For the purpose of notation, we relabel the vertex v_2 as w_0. Since $H_0 \neq K_{1,n-1} + e$ (as $g(l) > 3$), it follows that there exists a vertex x_0 in H_0 adjacent to w_0 such that $\deg_{H_0} x_0 \neq 1$ and $x_0 \notin \{v, v_1\}$.

We now construct a unicyclic graph H_1 from H_0 by deleting the edge w_0x_0, identifying the vertices w_0 and x_0, labeling the new vertex by w_1, introducing a new vertex y_1, and joining y_1 to w_1. We note that H_1 has order n and has $d_u(n) + 1$ end-vertices. Thus, $\det(H_1) \geqslant d_u(n) + 1$. To show that $\det(H_1) \leqslant d_u(n) + 1$, we provide a detectable $(d_u(n)+1)$-coloring of H_1. Define $c_1\colon E(H_1) \to \{1,2,\ldots,d_u(n)+1\}$ by

$$c_1(e) = \begin{cases} c_0(e) & \text{if} \quad e \in E(H_0), \\ d_u(n)+1 & \text{if} \quad e = w_1y_1. \end{cases}$$

Then c_1 is detectable $(d_u(n)+1)$-coloring of H_1. This implies that $\det(H_1) = d_u(n)+1$.

In general, we construct H_{i+1} from H_i and obtain the edge coloring c_{i+1} from c_i, where $0 \leqslant i \leqslant n - d_u(n) - 4$, as follows:

(1) Let x_i be a vertex in H_i that is adjacent to w_i such that $\deg_{H_i} x_i \neq 1$ and $x_i \notin \{v, v_1\}$.

(2) Construct H_{i+1} by deleting the edge w_ix_i, identifying the vertices w_i and x_i, labeling the new vertex by w_{i+1}, introducing a new vertex y_{i+1}, and joining y_{i+1} to w_{i+1}.

(3) Define $c_{i+1}\colon E(H_{i+1}) \to \{1,2,\ldots,d_u(n)+i+1\}$ by

$$c_{i+1}(e) = \begin{cases} c_i(e) & \text{if} \quad e \in E(H_i), \\ d_u(n)+i+1 & \text{if} \quad e = w_{i+1}y_{i+1}. \end{cases}$$

Observe that for every integer $i = 0,1,\ldots,n-d_u(n)-3$:

(i) H_i is a unicyclic graph of order n with $d_u(n)+i$ end-vertices;

(ii) c_i is a detectable $(d_u(n)+i)$-coloring of H_i;

(iii) Parts (i) and (ii) imply that $\det(H_i) = d_u(n)+i$.

Figure 11 illustrates how to construct the unicyclic graphs H_i $(0 \leqslant i \leqslant 6)$ for $n = 12$. In this case, $d_u(12) = 3$, $D_u(12) = 9$, and $\det(H_i) = d_u(n)+i = 3+i$ for $0 \leqslant i \leqslant 6$.

□

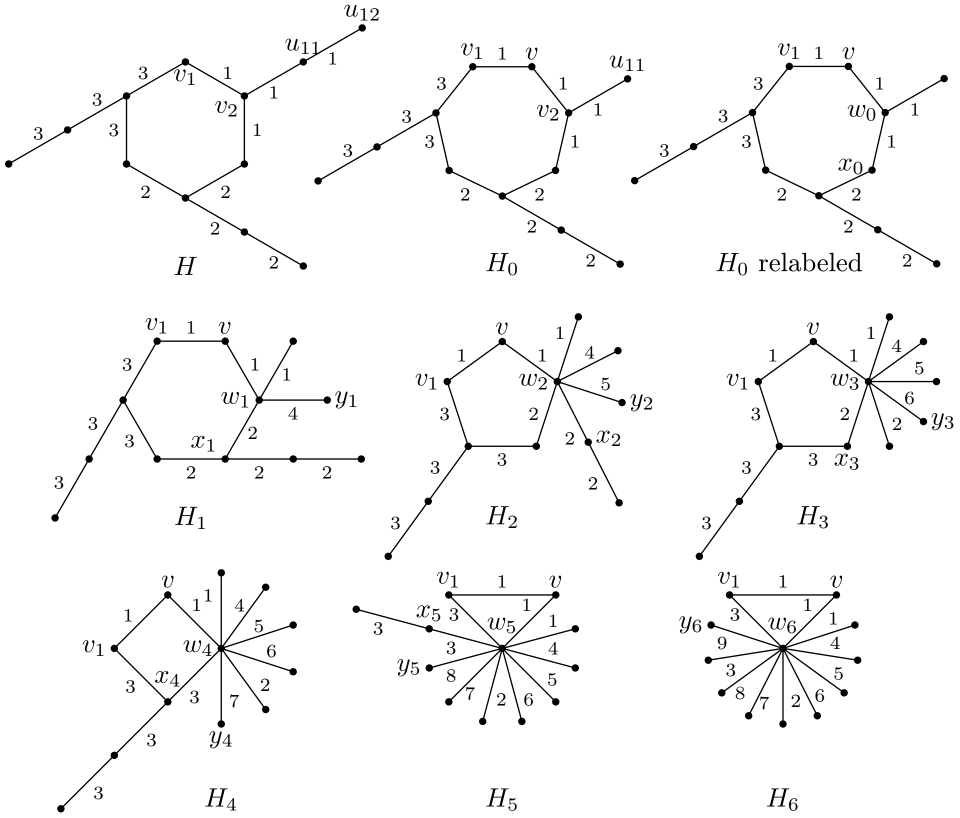


Figure 11. Constructing unicyclic graphs in the proof of Theorem 3.7 for $n = 12$.

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