DISJOINT AND COMPLETE UNIONS OF INCIDENCE STRUCTURES

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Abstract. Some decompositions of general incidence structures with regard to distinguished components (modular or simple) are considered and several structure theorems for them are deduced.

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Definition 1. Let G and M be non-empty sets and $I \subseteq G \times M$. Then the triple $\mathcal{J} = (G, M, I)$ is called an *incidence structure* (a *context*). If $A \subseteq G$, $B \subseteq M$ are non-empty sets, then denote

$$A^{\uparrow} := \{m \in M\,;\ gIm \quad \forall g \in A\},$$

$$B^{\downarrow} := \{g \in G\,;\ gIm \quad \forall m \in B\}.$$

Further notation: $\emptyset^{\uparrow} := M$, $\emptyset^{\downarrow} := G$,

$$g^{\uparrow} := \{g\}^{\uparrow} \text{ for all } g \in G,$$
$$m^{\downarrow} := \{m\}^{\downarrow} \text{ for all } m \in M,$$
$$A^{\uparrow\downarrow} := (A^{\uparrow})^{\downarrow} \text{ for all } A \subseteq G,$$
$$B^{\downarrow\uparrow} := (B^{\downarrow})^{\uparrow} \text{ for all } B \subseteq M.$$

(See [3]).

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Definition 2. Let $\mathcal{J} = (G, M, I)$ be an incidence structure. If $G_1 \subseteq G$, $M_1 \subseteq M$ are non-empty subsets and $I_1 = I \cap (G_1 \times M_1)$, then the incidence structure $\mathcal{J}_1 = (G_1, M_1, I_1)$ is called a *substructure* of $\mathcal{J}$.

Definition 3. Let $\mathcal{J} = (G, M, I)$ be an incidence structure. $\mathcal{J}$ is called

1. *complete* if $I = G \times M$,
2. *open* if $g^{\uparrow} \neq M$ for all $g \in G$ and $m^{\downarrow} \neq G$ for all $m \in M$,
3. *trivial* if $|G| = |M| = 1$,
4. *regular* if $g^{\uparrow} \neq \emptyset$ for all $g \in G$ and $m^{\downarrow} \neq \emptyset$ for all $m \in M$,
5. *simple* if $|\,g^{\uparrow}| = 1$ for all $g \in G$ and $|m^{\downarrow}| = 1$ for all $m \in M$.

Let $\mathcal{J} = (G, M, I)$ be a simple incidence structure. It will be useful to express G and M as indexed families $G = \{g_\nu\,;\, \nu \in T_1\}$, $M = \{m_\mu\,;\, \mu \in T_2\}$ where $g_{\nu_1} = g_{\nu_2}$ iff $\nu_1 = \nu_2$ and $m_{\mu_1} = m_{\mu_2}$ iff $\mu_1 = \mu_2$. By Definition 3, for every $g_i \in G$ there exists exactly one $m_j \in G$ such that $g_i I m_j$, and vice-versa. Hence the map $\alpha\colon T_1 \to T_2$, defined by $\alpha(i) = j$ iff $g_i I m_j$ for all $i \in T_1$, is injective. Assume that there exists an $l \in T_2$, $l \notin \alpha(T_1)$. Then there exists a $g_i \in G$ such that $g_i I m_l$. It follows that $\alpha(i) = l$, a contradiction. Thus $\alpha(T_1) = T_2$ and the map α is a one-to-one map of T_1 onto T_2 so that we can identify both sets of indices. If we denote $p_i := m_{\alpha(i)}$ for all $i \in T_1$, then we have $g_i I p_j \Leftrightarrow g_i I m_{\alpha(j)} \Leftrightarrow \alpha(i) = \alpha(j) \Leftrightarrow i = j$.

Let $\mathcal{J} = (G, M, I)$ be a simple incidence structure. Then T will serve as an index set for elements of G, M such that the relation I is defined by $g_i I m_j$ iff $i = j$. In what follows we will suppose that incidence relations in simple incidence structures are expressed like this.

Definition 4. An incidence structure $\mathcal{J} = (G, M, I)$ is said to be the *union of substructures* $\mathcal{J}_\nu = (G_\nu, M_\nu, I_\nu)$, $\nu \in T$, if $\{G_\nu\,;\, \nu \in T\}$ and $\{M_\nu\,;\, \nu \in T\}$ are decompositions of G and M. In this case we will write $\mathcal{J} = \bigcup\limits_{\nu \in T} \mathcal{J}_\nu$.

Remark 1. If a family $\{P_\nu\,;\, \nu \in T\}$ forms a decomposition of a non-empty set P, then we will write $P = \dot{\bigcup\limits_{\nu \in T}} P_\nu$.

Let $\mathcal{J} = (G, M, I)$ be an incidence structure and $G_\nu \subseteq G$, $M_\nu \subseteq M$ non-empty subsets for all $\nu \in T$. Then denote $\mathcal{J}_{ij} := (G_i, M_j, I_{ij})$ the substructure of $\mathcal{J}$, where $I_{ij} = I \cap (G_i \times M_j)$ for $i, j \in T$. Moreover, put $\mathcal{J}_{ii} = \mathcal{J}_i$ and $I_{ii} = I_i$ for all $i \in T$.

Theorem 1. *If $\mathcal{J} = \bigcup\limits_{\nu \in T} \mathcal{J}_\nu$ as in Definition 4, then $I = \dot{\bigcup\limits_{i,j \in T}} I_{ij}$.*

Proof. Consider the substructures $\mathcal{J}_{ij}$ of $\mathcal{J}$, $i, j \in T$. Then $\bigcup\limits_{i,j \in T} I_{ij} \subseteq I$. Let $(g, m) \in I$. Since $G = \dot{\bigcup\limits_{\nu \in T}} G_\nu$ and $M = \dot{\bigcup\limits_{\nu \in T}} M_\nu$, there exist $i, j \in T$ such that

$g \in G_i$ and $m \in M_j$. Then $(g,m) \in I_{ij}$, $I = \bigcup_{i,j\in T} I_{ij}$. If $(g,m) \in I_{i_1j_1} \cap I_{i_2j_2}$, then $(g,m) \in (G_{i_1} \times M_{j_1}) \cap (G_{i_2} \times M_{j_2})$ and $g \in G_{i_1} \cap G_{i_2}$, $m \in M_{j_1} \cap M_{j_2}$, a contradiction. Thus $I = \bigcup_{i,j\in T} I_{ij}$. □

Definition 5. Let an incidence structure $\mathcal{J} = (G, M, I)$ be the union of substructures $\mathcal{J}_\nu$, $\nu \in T$. This union is called *disjoint* if $I_{ij} = \emptyset$ for distinct $i, j \in T$, and will be denoted by $\mathcal{J} = \dot{\bigcup}_{\nu\in T} \mathcal{J}_\nu$. The union is called *complete* if $I_{ij} = G_i \times M_j$ for distinct $i, j \in T$, and will be denoted by $\mathcal{J} = \bar{\bigcup}_{\nu\in T} \mathcal{J}_\nu$.

Remark 2. 1. Let $\mathcal{J} = \bigcup_{\nu\in T} \mathcal{J}_\nu$. Then $\mathcal{J} = \dot{\bigcup}_{\nu\in T} \mathcal{J}_\nu$ iff $I = \dot{\bigcup}_{\nu\in T} I_\nu$ and $\mathcal{J} = \bar{\bigcup}_{\nu\in T} \mathcal{J}_\nu$ iff $I = (\bigcup_{\nu\in T} I_\nu) \cup (\bigcup_{i,j\in T} (G_i \times M_j))$ where $i \neq j$.

2. If $|T| = 1$, then $\mathcal{J} = \dot{\bigcup} \mathcal{J} = \bar{\bigcup} \mathcal{J}$. Let $\mathcal{J} = (G, M, I)$ be a simple incidence structure, where $G = \{g_\nu;\, \nu \in T\}$, $M = \{m_\nu;\, \nu \in T\}$ and $g_i I m_j$ iff $i = j$. If $\mathcal{J}_\nu = (\{g_\nu\}, \{m_\nu\}, I_\nu)$, $\nu \in T$, are substructures of $\mathcal{J}$ then $\mathcal{J}$ is the disjoint union of substructures $\mathcal{J}_\nu$, $\nu \in T$.

3. If $\mathcal{J}$ is a disjoint union of substructures $\mathcal{J}_\nu$, $\nu \in T$ then $\mathcal{J}$ is regular iff $\mathcal{J}_\nu$ are regular for all $\nu \in T$. If $\mathcal{J}$ is a complete union of substructures $\mathcal{J}_\nu$, $\nu \in T$, then $\mathcal{J}$ is open iff $\mathcal{J}_\nu$ are open for all $\nu \in T$.

Remark 3. If an incidence structure $\mathcal{J}$ is a union of substructures $\mathcal{J}_\nu$, $\nu \in T$ then write operators $\uparrow$, $\downarrow$ as right superscripts ($X^\uparrow$) for the incidence relation I in $\mathcal{J}$ and as left superscripts (${}^\uparrow X$) for incidence relations I_ν in substructures $\mathcal{J}_\nu$. Furthermore, write $G^\nu = G - G_\nu$ and $M^\nu = M - M_\nu$ for all $\nu \in T$.

Theorem 2. *Let $\mathcal{J} = (G, M, I)$ be the disjoint union of substructures $\mathcal{J}_\nu$, $\nu \in T$. If $A \subseteq G_i$, $A \neq \emptyset$ and $B \subseteq M_i$, $B \neq \emptyset$ for some $i \in T$ then $A^\uparrow = {}^\uparrow A$, $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow} A$ and $B^\downarrow = {}^\downarrow B$, $B^{\downarrow\uparrow} = {}^{\uparrow\downarrow} B$, respectively. If $a \in G_i$, $b \in G_j$ and $m \in M_i$, $n \in M_j$ for $i, j \in T$, $i \neq j$, then $\{a, b\}^\uparrow = \emptyset$ and $\{m, n\}^\downarrow = \emptyset$, respectively.*

Proof. Let $A \subseteq G_i$, $A \neq \emptyset$. Then $m \in A^\uparrow$ iff aIm for all $a \in A$. Since $I = \dot{\bigcup}_{\nu\in T} I_\nu$, we obtain $aI_i m$ for all $a \in A$, $A^\uparrow = {}^\uparrow A$ and $A^\uparrow \subseteq M_i$. Similarly we obtain $B^\downarrow = {}^\downarrow B$, $B^\downarrow \subseteq G_i$. This yields $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow} A$ and $B^{\downarrow\uparrow} = {}^{\uparrow\downarrow} B$.

Let $a \in G_i$, $b \in G_j$, $i \neq j$. If $m \in \{a, b\}^\uparrow$ then aIm and bIm, hence $m \in M_i \cap M_j$, which is a contradiction to $M_i \cap M_j = \emptyset$. Similarly we proceed when elements $m \in M_i$, $n \in M_j$ are under consideration. □

Theorem 3. *Let an incidence structure $\mathcal{J}$ be the complete union of substructures $\mathcal{J}_\nu$, $\nu \in T$.*

1. *If $A \subseteq G_i$ and $B \subseteq M_i$, $i \in T$, then $A^{\uparrow} = M^i \cup {}^{\uparrow}A$ and $B^{\downarrow} = G^i \cup {}^{\downarrow}B$. If the incidence structure $\mathcal{J}$ is open then $A^{\uparrow\downarrow} = {}^{\downarrow\uparrow}A$ and $B^{\downarrow\uparrow} = {}^{\uparrow\downarrow}B$.*
2. *Let $a \in G_i$ and $b \in G_j$ for distinct $i, j \in T$. Then $\{a, b\}^{\uparrow} = (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b$. If the incidence structure $\mathcal{J}$ is open then $\{a, b\}^{\uparrow\downarrow} = {}^{\downarrow\uparrow}a \cup {}^{\downarrow\uparrow}b$. Let $m \in M_i$, $n \in M_j$, $i \neq j$, $i, j \in T$. Then $\{m, n\}^{\uparrow} = (G^i \cap G^j) \cup {}^{\downarrow}m \cup {}^{\downarrow}n$. If $\mathcal{J}$ is open then $\{m, n\}^{\downarrow\uparrow} = {}^{\uparrow\downarrow}m \cup {}^{\uparrow\downarrow}n$.*

Proof. Let $g \in G$. Since $G = \dot{\bigcup\limits_{\nu \in T}} G_\nu$, there exists $l \in T$ such that $g \in G_l$. By Definition 1, $g^{\uparrow} = \{m \in M\,;\, gIm\}$ and from $I = (\bigcup\limits_{\nu \in T} I_\nu) \cup (\bigcup\limits_{i,j \in T} (G_i \times M_j))$ where $i \neq j$, we obtain $g^{\uparrow} = M^l \cup {}^{\uparrow}g$. Similarly, for $m \in M$ there exists $k \in T$ such that $m \in M_k$ and $m^{\downarrow} = G^k \cup {}^{\downarrow}m$.

1. Let $A \subseteq G_i$ and $A = \emptyset$. Then $A^{\uparrow} = M = M^i \cup M_i = \mathrm{M}^i \cup {}^{\uparrow}\emptyset = M^i \cup {}^{\uparrow}A$. If $A \neq \emptyset$ then $A^{\uparrow} = \bigcap\limits_{a \in A} a^{\uparrow} = \bigcap\limits_{a \in A} (M^i \cup {}^{\uparrow}a) = M^i \cup (\bigcap\limits_{a \in A} {}^{\uparrow}a) = M^i \cup {}^{\uparrow}A$.

Let $\mathcal{J}$ be an open incidence structure. Then $(M^i)^{\downarrow} = G_i$ for all $i \in T$. We obtain $A^{\uparrow\downarrow} = (A^{\uparrow})^{\downarrow} = (M^i \cup {}^{\uparrow}A)^{\downarrow} = (M^i)^{\downarrow} \cap ({}^{\uparrow}A)^{\downarrow}$. As ${}^{\uparrow}A \subseteq M_i$, we have $({}^{\uparrow}A)^{\downarrow} = G^i \cup {}^{\downarrow\uparrow}A$ and $A^{\uparrow\downarrow} = G_i \cap (G^i \cup {}^{\downarrow\uparrow}A) = (G_i \cap G^i) \cup (G_i \cap {}^{\downarrow\uparrow}A) = {}^{\downarrow\uparrow}A$.

If $B \subseteq M_i$ then the proof is similar.

2. Let $a \in G_i$, $b \in G_j$, $i \neq j$. Then $\{a, b\}^{\uparrow} = a^{\uparrow} \cap b^{\uparrow} = (M^i \cup {}^{\uparrow}a) \cap (M^j \cup {}^{\uparrow}b) = (M^i \cap M^j) \cup (M^j \cap {}^{\uparrow}a) \cup (M^i \cap {}^{\uparrow}b) \cup ({}^{\uparrow}a \cap {}^{\uparrow}b)$. Since $M^j \cap {}^{\uparrow}a = {}^{\uparrow}a$, $M^i \cap {}^{\uparrow}b = {}^{\uparrow}b$, ${}^{\uparrow}a \cap {}^{\uparrow}b = \emptyset$ we have $\{a, b\}^{\uparrow} = (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b$.

Let $\mathcal{J}$ be an open incidence structure. For every $i, j \in T$ we obtain $(M^i \cap M^j)^{\downarrow} = (\bigcup\limits_{l \neq i,j} M_l)^{\downarrow} = G_i \cup G_j$. Hence, $\{a, b\}^{\uparrow\downarrow} = (\{a, b\}^{\uparrow})^{\downarrow} = ((M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b)^{\downarrow} = (M^i \cap M^j)^{\downarrow} \cap ({}^{\uparrow}a)^{\downarrow} \cap ({}^{\uparrow}b)^{\downarrow} = (G_i \cup G_j) \cap (G^i \cup {}^{\downarrow\uparrow}a) \cap (G^j \cup {}^{\downarrow\uparrow}b) = [(G_i \cup G_j) \cap (G^i \cap G^j)] \cup [(G_i \cup G_j) \cap {}^{\downarrow\uparrow}a] \cup [(G_i \cup G_j) \cap {}^{\downarrow\uparrow}b]$. Now, $(G_i \cup G_j) \cap (G^i \cap G^j) = (G_i \cup G_j) \cap (\bigcup\limits_{l \neq i,j} G_l) = \emptyset$. By virtue of ${}^{\downarrow\uparrow}a \subseteq \mathrm{G}_i$, ${}^{\downarrow\uparrow}b \subseteq G_j$, it follows that $(G_i \cup G_j) \cap {}^{\downarrow\uparrow}a = {}^{\downarrow\uparrow}a$, $(G_i \cup G_j) \cap {}^{\downarrow\uparrow}b = {}^{\downarrow\uparrow}b$. Thus $\{a, b\}^{\uparrow\downarrow} = {}^{\downarrow\uparrow}a \cup {}^{\downarrow\uparrow}b$.

For $m \in M_i$ and $n \in M_j$ the proof is similar. □

Definition 6. Let $\mathcal{J} = (G, M, I)$, $\mathcal{J}_1 = (G_1, M_1, I_1)$ be incidence structures. A map $\varphi\colon G \cup M \to G_1 \cup M_1$ is called a *homomorphism* of $\mathcal{J}$ onto $\mathcal{J}_1$ if

1. $\varphi(G) := \{\varphi(g)\,;\, g \in G\} = G_1, \varphi(M) := \{\varphi(m)\,;\, m \in M\} = M_1$,
2. $aIm \Longrightarrow \varphi(a) I_1 \varphi(m)$,
3. for $a' I_1 m'$ there are elements $a \in G$, $m \in M$ such that aIm, $\varphi(a) = a'$ and $\varphi(m) = m'$.

Remark 4. 1. Let $\mathcal{J} = (G, M, I)$ be an incidence structure and let $\overline{G}, \overline{M}$ be decompositions of G, M. Put $\mathcal{R} = (\overline{G}, \overline{M})$ and consider the incidence structure

$\mathcal{J}_{\mathcal{R}} = (\overline{G}, \overline{M}, I_{\mathcal{R}})$ where $\bar{g} I_{\mathcal{R}} \overline{m}$ iff there is an $h \in \bar{g}$ with $n \in \overline{m}$, hIm for every $\bar{g} \in \overline{G}$, $\overline{m} \in \overline{M}$. The map $\varphi_{\mathcal{R}}$ defined by

$$\varphi_{\mathcal{R}}\colon \begin{cases} g \mapsto \bar{g} & \forall g \in G, \\ m \mapsto \overline{m} & \forall m \in M, \end{cases}$$

is a homomorphism of $\mathcal{J}$ onto $\mathcal{J}_{\mathcal{R}}$. (See [1], Theorem 1.)

2. Let φ be an incidence structure homomorphism of $\mathcal{J} = (G, M, I)$ onto $\mathcal{J}_1 = (G_1, M_1, I_1)$. If we put $\bar{g} = \{h \in G;\, \varphi(h) = \varphi(g)\}$, $\overline{m} = \{n \in M;\, \varphi(n) = \varphi(m)\}$ then $G_\varphi = \{\bar{g};\, g \in G\}$ is a decomposition of the set G and $M_\varphi = \{\overline{m};\, m \in M\}$ is a decomposition of the set M. If we denote $\mathcal{R}_\varphi = (G_\varphi, M_\varphi)$ then the map ξ defined by

$$\xi\colon \begin{cases} \bar{g} \mapsto \varphi(g) & \forall \bar{g} \in G_\varphi, \\ \overline{m} \mapsto \varphi(m) & \forall \overline{m} \in M_\varphi, \end{cases}$$

is an isomorphism (i.e., both sided homomorphism) between $\mathcal{J}_{\mathcal{R}_\varphi}$ and $\mathcal{J}_1$. (See [1], Theorem 1.)

Theorem 4. *Let $\mathcal{J} = (G, M, I)$ be an incidence structure. Then the following conditions are equivalent.*

1. *$\mathcal{J}$ is the disjoint union of substructures $\mathcal{J}_\nu = (G_\nu, M_\nu, I_\nu)$, $\nu \in T$, where $|T| \geqslant 2$ and $I_\nu \neq \emptyset$ for all $\nu \in T$.*
2. *There exists a homomorphism of $\mathcal{J}$ onto a simple non-trivial incidence structure.*

Proof. 1. $\Longrightarrow$ 2. Let the assumption 1 hold. Then the sets $\overline{G} = \{G_\nu;\, \nu \in T\}$, $\overline{M} = \{M_\nu;\, \nu \in T\}$ are decompositions of the sets G, M. Put $\mathcal{R} = (\overline{G}, \overline{M})$ and consider the incidence structure $\mathcal{J}_{\mathcal{R}} = (\overline{G}, \overline{M}, I_{\mathcal{R}})$ from Remark 4. We will prove that $\mathcal{J}_{\mathcal{R}}$ is a simple incidence structure. Let $G_i \in \overline{G}$. Then there exist $g \in G_i$ and $m \in M_i$ such that gI_im, because $I_i \neq \emptyset$. By Theorem 1, we have gIm and by Remark 4, we obtain $G_i I_{\mathcal{R}} M_i$ and $|G_i^{\uparrow}| \geqslant 1$. Similarly we get $|M_j^{\downarrow}| \geqslant 1$ for every $M_j \in \overline{M}$. Now suppose that $G_i I_{\mathcal{R}} M_j$ for $i, j \in T$. Then there exist $g \in G_i$ and $m \in M_j$ such that gIm, and according to Definition 5 and Remark 2 there exists an $l \in T$ such that $g \in G_l$, $m \in M_l$ and gI_lm. But $g \in G_i \cap G_l$ and $m \in M_j \cap M_l$, which means that $i = j = l$ so that $|G_i^{\uparrow}| = 1$. Similarly we obtain $|M_j^{\downarrow}| = 1$ for all $M_j \in \overline{M}$. Thus $\mathcal{J}_{\mathcal{R}}$ is simple. Because of $|T| \geqslant 2$, we have $|\overline{G}| \geqslant 2$, $|\overline{M}| \geqslant 2$ and $\mathcal{J}_{\mathcal{R}}$ is not trivial.

According to Remark 4 the map $\varphi_{\mathcal{R}}\colon \mathcal{J} \to \mathcal{J}_{\mathcal{R}}$ is a homomorphism of $\mathcal{J}$ onto $\mathcal{J}_{\mathcal{R}}$.

2. $\Longrightarrow$ 1. Let $\varphi\colon \mathcal{J} \to \mathcal{J}'$ be a homomorphism of $\mathcal{J}$ onto a simple incidence structure $\mathcal{J}' = (G', M', I')$. Suppose that $G' = \{g'_\nu;\, \nu \in T\}$, $M' = \{m'_\nu;\, \nu \in T\}$ and $g'_i I' m'_j$ iff $i = j$. Since $\mathcal{J}'$ is non-trivial, it follows that $|T| \geqslant 2$.

By Remark 4, we obtain the structure $\mathcal{J}_{\mathcal{R}_\varphi} = (G_\varphi, M_\varphi, I_{\mathcal{R}_\varphi})$, where $G_\varphi = \{\bar{g};\ g \in G\}$, $M_\varphi = \{\bar{m};\ m \in M\}$ and $\bar{g} I_{\mathcal{R}_\varphi} \bar{m}$ iff there are $h \in \bar{g}$, $n \in \bar{m}$ such that hIn. Furthermore, put $G_i := \bar{g}$ iff $\varphi(g) = g_i'$ and $M_i := \bar{m}$ iff $\varphi(m) = m_i'$ and consider substructures $\mathcal{J}_i = (G_i, M_i, I_i)$, where $I_i = I \cap (G_i \times M_i)$ for all $i \in T$. Then $\varphi(G_i) = g_i'$, $\varphi(M_i) = m_i'$ and $g_i' I' m_i'$. By Condition 3 from Definition 6 there exist $g \in G_i$, and $m \in M_i$ such that gIm. Then gI_im and hence $I_i \neq \emptyset$ for all $i \in T$.

We will prove that $\mathcal{J} = \dot{\bigcup\limits_{\nu \in T}} \mathcal{J}_\nu$. Since G_φ, M_φ are decompositions of G, M, the sets $\{G_\nu;\ \nu \in T\}$ and $\{M_\nu;\ \nu \in T\}$ are decompositions of G, M, too. Now the set $\{I_\nu;\ \nu \in T\}$ is a decomposition of the set I. We have gIm so that $\varphi(g) I' \varphi(m)$. If $\varphi(g) = g_i'$ then $\varphi(m) = m_i'$ and $(g, m) \in G_i \times M_i$. This yields $(g, m) \in I_i$ and $I_i \subseteq I$ for all $i \in T$. From $G_i \cap G_j = \emptyset$ and $M_i \cap M_j = \emptyset$ for $i \neq j$, we get $I = \bigcup\limits_{\nu \in T} I_\nu$. □

R e m a r k 5. There exists a homomorphism of an arbitrary incidence structure with non-empty incidence relation onto a trivial simple incidence structure.

Theorem 5. *Every regular incidence structure is a homomorphic image of a certain simple incidence structure.*

P r o o f. Let $\mathcal{J} = (G, M, I)$ be a regular incidence structure. Set $G = \{g_\nu;\ \nu \in P_1\}$, $M = \{m_\mu;\ \mu \in P_2\}$ and define the set $U \subseteq P_1 \times P_2$ by $(i, j) \in U$ iff $g_i I m_j$. Let $U = \{u_\xi;\ \xi \in T\}$. We consider the map $\alpha\colon U \to P_1$, given by $\alpha(i, j) = i$ for all $(i, j) \in U$. If $i \in P_1$, then $|g_i^\uparrow| \neq \emptyset$ because $\mathcal{J}$ is regular. Hence there exists $m_j \in M$ such that $g_i I m_j$. It follows that $(i, j) \in U$, $\alpha(i, j) = i$ and so α is a map onto P_1. For every $i \in P_1$, put $\alpha^{-1}(i) = U_i = \{u_\eta;\ \eta \in T_i\}$ where $T_i \subseteq T$. Similarly, define a map $\beta\colon U \to P_2$ such that $\beta(i, j) = j$. This map is onto. Denote $\beta^{-1}(j) = U^j = \{u_\kappa;\ \kappa \in T^j\}$ where $T^j \subseteq T$.

Now consider the simple incidence structure $\mathcal{J}_1 = (G_1, M_1, I_1)$ where $G_1 = \{b_\xi;\ \xi \in T\}$, $M_1 = \{p_\xi;\ \xi \in T\}$ and $b_i I_1 p_j$ iff $i = j$. Put $\bar{b}_i = \{b_\xi;\ \xi \in T_i\}$ for $i \in P_1$ and $\bar{p}_j = \{p_\xi;\ \xi \in T^j\}$ for $j \in P_2$.

The family $\{\bar{b}_i;\ i \in P_1\}$ forms a decomposition of G_1. If $b_l \in G_1$ then $l \in T$, and there exists a $u_l \in U$. We express it as $u_l = (p, q)$ so that $\alpha(u_l) = p$, $u_l \in U_p$ and consequently, $l \in T_p$, $b_l \in \bar{b}_p$, $G_1 = \bigcup\limits_{i \in T_1} \bar{b}_i$. If $b_l \in \bar{b}_{i_1} \cap \bar{b}_{i_2}$ then $l \in T_{i_1} \cap T_{i_2}$ and $u_l \in U_{i_1} \cap U_{i_2}$, which yields $i_1 = i_2$. Obviously, $\bar{b}_i \neq \emptyset$ for all $i \in P_1$. Similarly one can prove that the family $\{\bar{m}_j;\ j \in P_2\}$ forms a decomposition of M_1.

It is clear that

$$u_l = (i, j),\ l \in T \Leftrightarrow u_l \in U_i \cap U^j \Leftrightarrow l \in T_i \cap T^j \Leftrightarrow b_l \in \bar{b}_i,\ p_l \in \bar{p}_j.$$

Finally consider the map $\varphi\colon G_1 \cup M_1 \to G \cup M$ given by $\varphi(b_i) = g_j$ iff $b_i \in \bar{b}_j$ for all $b_i \in G_1$ and $\varphi(p_i) = m_j$ iff $p_i \in \bar{p}_j$ for all $p_i \in M_1$. We claim that φ

is a homomorphism of $\mathcal{J}_1$ onto $\mathcal{J}$: In deed, first it is obvious that $\varphi(G_1) = G$, $\varphi(M_1) = M$. If $b_l I_1 p_k$ then $l = k$. If $\varphi(b_l) = g_i$ then $b_l \in \bar{b}_i$ and similarly for $\varphi(p_l) = m_j$, $p_l \in \bar{p}_j$. This implies $u_l = (i,j) \in U$ and we obtain $g_i I m_j$, $\varphi(b_l) I \varphi(p_l)$.

If $g_i I m_j$ then there exists an $l \in T$ with $u_l = (i,j)$ and it follows that $b_l \in \bar{b}_i$, $p_l \in \bar{p}_j$. This yields $\varphi(b_l) = g_i$, $\varphi(p_l) = m_j$ and $b_l I_1 p_l$. □

Modular incidence structures have been defined in [2]:

Definition 7. An incidence structure $\mathcal{J} = (G, M, I)$ is said to be *modular* if it satisfies the following conditions:

(M1) $\{a,b\}^{\uparrow} \neq \emptyset \quad \forall a, b \in G,$

(M2) $\{m,n\}^{\downarrow} \neq \emptyset \quad \forall m, n \in M,$

(M3) $a, b \in G,\ x \in \{a,b\}^{\uparrow\downarrow},\ x \neq a \Longrightarrow \{a,x\}^{\uparrow} \subseteq \{a,b\}^{\uparrow},$

(M4) $m, n \in M,\ y \in \{m,n\}^{\downarrow\uparrow},\ y \neq m \Longrightarrow \{m,y\}^{\downarrow} \subseteq \{m,n\}^{\downarrow}.$

Theorem 6. *Let an incidence structure $\mathcal{J} = (G, M, I)$ be the complete union of incidence structures $\mathcal{J}_\nu = (G_\nu, M_\nu, I_\nu)$ where $\nu \in T$ and $|T| > 1$. Then the following two conditions are equivalent:*

1. *$\mathcal{J}$ is open modular.*
2. *$|G| \geqslant 3$ and each of $\mathcal{J}_\nu$ is either open modular, or simple non-trivial, or a trivial incidence structure with empty incidence relation.*

Proof. 1. $\Longrightarrow$ 2. As $\mathcal{J}$ is open, all substructures $\mathcal{J}_\nu$ are open by Remark 2. Since $|T| > 1$, we have $|G| \geqslant 2$ and $|M| \geqslant 2$. Suppose that $|G| = 2$, $G = \{a, b\}$. It follows that $\mathcal{J}_1 = (\{a\}, M_1, I_1)$, $\mathcal{J}_2 = (\{b\}, M_2, I_2)$ where $M = M_1 \dot{\cup} M_2$. Moreover, $\mathcal{J}_{12} = (\{a\}, M_2, I_{12})$, $\mathcal{J}_{21} = (\{b\}, M_1, I_{21})$ where $I_{12} = \{a\} \times M_2$, $I_{21} = \{b\} \times M_1$. Since $\mathcal{J}_1, \mathcal{J}_2$ are open, $I_1 = I_2 = \emptyset$ and $|m^{\downarrow}| = 1$ for all $m \in M$. But $\mathcal{J}$ is modular so that, according to Theorem 3 of [2], $\mathcal{J}$ is not open, which is a contradiction. Hence $|G| \geqslant 3$ and similarly, $|M| \geqslant 3$.

Let $\mathcal{J}_i = (G_i, M_i, I_i)$, $i \in T$, be substructures of $\mathcal{J}$.

(1) Let $|G_i| = 1$. Then $G_i = \{a\}$ for some $a \in G$. Furthermore, suppose that $I_i \neq \emptyset$. Then there exists an $m \in M_i$ such that $a I_i m$ and it follows that $\{a\} = {}^{\downarrow}m$. According to Theorem 3, $m^{\downarrow} = G^i \cup {}^{\downarrow}m = G^i \cup G_i = G$. We have obtained a contradiction to Condition 1. Therefore $I_i = \emptyset$.

Let m, n be distinct elements of M_i. Then ${}^{\downarrow}m = \emptyset = {}^{\downarrow}n$ and $m^{\downarrow} = n^{\downarrow} = G^i$, in contradiction to Theorem 4 of [2]. Thus $m = n$ and $|M_i| = 1$. Hence $\mathcal{J}_i$ is trivial and its incidence relation is empty. The case $|M_i| = 1$ can be considered analogously.

(2) Let $|G_i| > 1$. Then $|M_i| > 1$, too. Suppose that ${}^{\uparrow}a = \emptyset$ for some $a \in G_i$. By Theorem 3 we have $a^{\uparrow} = M^i \cup {}^{\uparrow}a = M^i$. Since $|G_i| > 1$, there exists a $b \in G_i$, $b \neq a$ and from $b^{\uparrow} = M^i \cup {}^{\uparrow}b$ we get $a^{\uparrow} \subseteq b^{\uparrow}$. But this is a contradiction to Theorem 4 of [2], so that $|{}^{\uparrow}a| \geqslant 1$. Similarly we prove $|{}^{\downarrow}m| \geqslant 1$.

(a) Suppose that $|{}^{\uparrow}a| = 1$ for some $a \in G_i$. Then there exists an $m \in M_i$ such that aI_im and ${}^{\uparrow}a = \{m\}$. Further suppose that there exists a $b \in G_i$, $b \neq a$ such that bI_im. Then $m \in {}^{\uparrow}b$ and ${}^{\uparrow}a \subseteq {}^{\uparrow}b$. Since $a^{\uparrow} = M^i \cup {}^{\uparrow}a$ and $b^{\uparrow} = M^i \cup {}^{\uparrow}b$, we have $a^{\uparrow} \subseteq b^{\uparrow}$, which is again a contradiction to Theorem 4 of [2]. This implies $|{}^{\downarrow}m| = 1$ and ${}^{\downarrow}m = \{a\}$.

Let n be an arbitrary element of M_i, $n \neq m$. Then $n \notin {}^{\uparrow}a$. Suppose there exist distinct $b, c \in G_i$, such that bI_in, cI_in. Clearly ${}^{\uparrow}\{a,b\} = \emptyset$ and by Theorem 3, $\{a,b\}^{\uparrow} = M^i$. Now ${}^{\downarrow\uparrow}\{a,b\} = G_i$ and $c \in {}^{\downarrow\uparrow}\{a,b\}$. By Theorem 3 it follows that ${}^{\downarrow\uparrow}\{a,b\} = \{a,b\}^{\uparrow\downarrow}$. Hence $c \in \{a,b\}^{\uparrow\downarrow}$ and from $n \notin M^i$, one gets $n \notin \{a,b\}^{\uparrow}$. Moreover, $n \in \{b,c\}^{\uparrow}$, hence $\{b,c\}^{\uparrow} \not\subseteq \{b,a\}^{\uparrow}$, which is a contradiction to (M3). From $|{}^{\downarrow}n| \geqslant 1$ we obtain $|{}^{\downarrow}n| = 1$.

Let b be an arbitrary element of G_i, $b \neq a$. Suppose there exist distinct $n, p \in M_i$ such that bI_in, bI_ip. Then ${}^{\downarrow}\{m,n\} = \emptyset$ and ${}^{\uparrow\downarrow}\{m,n\} = M_i$, and therefore $p \in {}^{\uparrow\downarrow}\{m,n\} = \{m,n\}^{\downarrow\uparrow}$. Moreover, $b \in \{n,p\}^{\downarrow}$ and $b \notin \{m,n\}^{\downarrow}$ so that $\{n,p\}^{\downarrow} \not\subseteq \{m,n\}^{\downarrow}$, in contradiction to (M4). Hence $|{}^{\uparrow}b| = 1$ and $\mathcal{J}_i$ is simple.

Similarly we prove that $|{}^{\downarrow}m| = 1$ implies that $\mathcal{J}_i$ is simple.

(b) Let us suppose that there exists $a \in G_i$ such that $|{}^{\uparrow}a| > 1$. Then by part (a) $|{}^{\uparrow}x| > 1$ for all $x \in G_i$ and $|{}^{\downarrow}m| > 1$ for all $m \in M_i$. We prove that every incidence structure $\mathcal{J}_i$ satisfies conditions (M1)–(M4).

To (M1): Let $a, b \in G_i$ such that ${}^{\uparrow}\{a,b\} = \emptyset$. Then ${}^{\downarrow\uparrow}\{a,b\} = \{a,b\}^{\uparrow\downarrow} = G_i$ and for arbitrary $x \in G_i$ we obtain $x \in \{a,b\}^{\uparrow\downarrow}$. As $\mathcal{J}$ is modular, (M3) implies $\{x,a\}^{\uparrow} \subseteq \{a,b\}^{\uparrow}$ whenever $x \neq a$, in other words $M^i \cup {}^{\uparrow}\{x,a\} \subseteq M^i \cup {}^{\uparrow}\{a,b\}$. As ${}^{\uparrow}\{a,b\} = \emptyset$, we obtain ${}^{\uparrow}\{x,a\} = \emptyset$. By $|{}^{\uparrow}a| > 1$, there exists an $m \in M_i$ such that aI_im. As $|{}^{\downarrow}m| > 1$, there exists a $c \in G_i$, $c \neq a$ such that cI_im. Hence $m \in {}^{\uparrow}\{c,a\}$, which is a contradiction. Then ${}^{\uparrow}\{a,b\} \neq \emptyset$.

Condition (M2) can be proved similarly as (M1).

To (M3): Let $a, b \in G_i$ and $c \in {}^{\downarrow\uparrow}\{a,b\}$, $c \neq a$. Then $c \in \{a,b\}^{\uparrow\downarrow}$. By (M3), $\{c,a\}^{\uparrow} \subseteq \{a,b\}^{\uparrow}$ i.e. $M^i \cup {}^{\uparrow}\{c,a\} \subseteq M^i \cup {}^{\uparrow}\{a,b\}$. If $x \in {}^{\uparrow}\{c,a\}$ then $x \in M^i \cup {}^{\uparrow}\{a,b\}$ and, regarding $x \notin M^i$, we obtain $x \in {}^{\uparrow}\{a,b\}$. It follows that ${}^{\uparrow}\{c,a\} \subseteq {}^{\uparrow}\{a,b\}$.

Condition (M4) can be proved similarly as (M3).

2. $\implies$ 1. Each of $\mathcal{J}_\nu$, $\nu \in T$ is an open and consequently $\mathcal{J}$ is open. We show that $\mathcal{J}$ satisfies conditions (M1)–(M4).

To (M1): Let a, b be elements of G such that $a, b \in G_i$ for some $i \in T$. By virtue of $|T| > 1$, it follows that $M^i \neq \emptyset$ and $\{a, b\}^{\uparrow} = M^i \cup {}^{\uparrow}\{a, b\} \neq \emptyset$.

Let $a \in G_i$, $b \in G_j$ where $i \neq j$ and let $|T| = 2$. Then $\mathcal{J} = \mathcal{J}_1 \dot{\cup} \mathcal{J}_2$. According to the hypothesis $|G| \geqslant 3$ both structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are non-trivial. Hence, for instance, $\mathcal{J}_1$ is simple non-trivial or modular and so regular. If $a \in G_1$ and $b \in G_2$ then ${}^{\uparrow}a \neq \emptyset$ and, by Theorem 3, $\{a, b\}^{\uparrow} = (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b = {}^{\uparrow}a \cup {}^{\uparrow}b \neq \emptyset$. If $|T| > 2$ then $M^i \cap M^j \neq \emptyset$ and again $\{a, b\}^{\uparrow} \neq \emptyset$.

The condition (M2) can be proved similarly as the condition (M1).

To (M3): Let a, b be elements of G and $c \in \{a, b\}^{\uparrow\downarrow}$, $c \neq a$. We have to prove that $\{a, c\}^{\uparrow} \subseteq \{a, b\}^{\uparrow}$.

(a) Let $a, b \in G_i$ for a certain $i \in T$. Then $\{a, b\}^{\uparrow\downarrow} = {}^{\downarrow\uparrow}\{a, b\}$. If $\mathcal{J}_i$ is trivial with $I_i = \emptyset$ then $G_i = \{a\}$, $c = a = b$ and ${}^{\uparrow}\{a, c\} = {}^{\uparrow}\{a, b\} = \emptyset$. Further, $\{a, c\}^{\uparrow} = M^i = \{a, b\}^{\uparrow}$. If $\mathcal{J}_i$ is simple then, because of $a \neq c$, it follows that ${}^{\uparrow}\{a, c\} = \emptyset$ and ${}^{\uparrow}\{a, c\} \subseteq {}^{\uparrow}\{a, b\}$. If $\mathcal{J}_i$ is modular then we obtain the same conclusion as a consequence of (M3). Hence $\{a, c\}^{\uparrow} = M^i \cup {}^{\uparrow}\{a, c\} \subseteq M^i \cup {}^{\uparrow}\{a, b\} = \{a, b\}^{\uparrow}$.

(b) Let $a \in G_i$, $b \in G_j$, $i \neq j$.

If $x, y \in G_l$ for an arbitrary $l \in T$ then ${}^{\uparrow}y \subseteq {}^{\uparrow}x$ iff $y = x$. If $\mathcal{J}_l$ is simple then ${}^{\uparrow}\{x, y\} = {}^{\uparrow}x \cap {}^{\uparrow}y = \emptyset$ for $x \neq y$ and (M3) is valid. If $\mathcal{J}_l$ is modular, then $\mathcal{J}_l$ is open and we obtain (M3) by Theorem 4 of [1].

By the hypothesis $c \in \{a, b\}^{\uparrow\downarrow}$. That means, by Theorem 3, $c \in {}^{\downarrow\uparrow}a \cup {}^{\downarrow\uparrow}b$. Since ${}^{\downarrow\uparrow}a \cap {}^{\downarrow\uparrow}b = \emptyset$, c belongs to exactly one of the sets ${}^{\downarrow\uparrow}a$ and ${}^{\downarrow\uparrow}b$. Let $c \in {}^{\downarrow\uparrow}a$. Hence ${}^{\uparrow}a \subseteq {}^{\uparrow}c$ and $a = c$. This yields $\{a, c\}^{\uparrow} = (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}c = (M^i \cap M^j) \cup {}^{\uparrow}a \subseteq (M^i \cap M^j) \cup {}^{\uparrow}a \cup {}^{\uparrow}b = \{a, b\}^{\uparrow}$.

Condition (M4) can be proved similarly as (M3). □

Remark 6. Let $\mathcal{J} = (G, M, I)$ be a simple incidence structure with $|G| \geqslant 3$. We put $G = \{g_\nu\,;\ \nu \in T\}$, $M = \{m_\nu\,;\ \nu \in T\}$, $g_i I m_j$ iff $i = j$. If $\mathcal{J}'$ is a complementary incidence structure on $\mathcal{J}$ (i.e. $\mathcal{J}' = (G, M, (G \times M) - I)$), then $\mathcal{J}$ is open modular.

Remark 7. According to Theorem 6, we can extend every open modular incidence structure with help of other open modular or non-trivial simple incidence structures or of trivial ones the incidence relations of which is empty, to a new incidence structure which is open modular, too.

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