ON PERFECT AND UNIQUE MAXIMUM INDEPENDENT SETS IN GRAPHS

Lutz Volkmann, Aachen

(Received September 24, 2003)

Abstract. A perfect independent set I of a graph G is defined to be an independent set with the property that any vertex not in I has at least two neighbors in I. For a nonnegative integer k, a subset I of the vertex set $V(G)$ of a graph G is said to be k-independent, if I is independent and every independent subset I' of G with $|I'| \geqslant |I| - (k-1)$ is a subset of I. A set I of vertices of G is a super k-independent set of G if I is k-independent in the graph $G[I, V(G) - I]$, where $G[I, V(G) - I]$ is the bipartite graph obtained from G by deleting all edges which are not incident with vertices of I. It is easy to see that a set I is 0-independent if and only if it is a maximum independent set and 1-independent if and only if it is a unique maximum independent set of G.

In this paper we mainly investigate connections between perfect independent sets and k-independent as well as super k-independent sets for $k = 0$ and $k = 1$.

Keywords: independent sets, perfect independent sets, unique independent sets, strong unique independent sets, super unique independent sets

MSC 2000: 05C70

1. Terminology and introduction

We will assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [2] or Lovász and Plummer [11]). In this paper, all graphs are finite, undirected, and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The *neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent to x, and the number $d_G(x) = |N_G(x)|$ is the *degree* of x. If $S \subseteq V(G)$, then we define the *neighborhood* of S by $N_G(S) = \bigcup_{x \in S} N_G(x)$. If S and T are two disjoint subsets of $V(G)$, then let $G[S, T]$ be the bipartite graph consisting of the partite sets S and T and all edges of G with one end in S and the other one in T, and we define $e_G(S, T) = |E(G[S, T])|$. A graph without any cycle is called a *forest*.

A set I of vertices is *independent* if no two vertices of I are adjacent. The *independence number* $\alpha(G)$ of a graph G is the maximum cardinality among the independent sets of vertices of G. Croitoru and Suditu [3] call an independent set I of a graph G a *perfect independent set* if any vertex not in I has at least two neighbors in I.

For a nonnegative integer k, by Siemes, Topp, Volkmann [12], an independent set I of the vertex set $V(G)$ of a graph G is said to be *k-independent*, if every independent subset I' of G with $|I'| \geqslant |I| - (k-1)$ is a subset of I. Furthermore, a set I of vertices of G is *super k-independent* if I is k-independent in the bipartite graph $G[I, V(G) - I]$. Obviously, a set I is 0-independent if and only if it is maximum independent and 1-independent if and only if it is a unique maximum independent set of G. In this paper we mainly deal with super k-independent sets for $k = 0, 1$. We call a super 0-independent and super 1-independent set also a *super independent* and *super unique independent* set, respectively.

If a bipartite graph G has partite sets A and B such that B is a unique maximum independent set of G, then Hopkins and Staton [5] speak of a *strong unique independence graph*. If a bipartite graph G has partite sets A and B such that B is a maximum independent set of G, then G will be called a *strong maximum independence graph*.

A *vertex cover* in G is a set of vertices that are incident with all edges of G. The minimum cardinality of a vertex cover in a graph G is called the *covering number* and is denoted by $\tau(G)$. A set of edges in a graph is called a *matching* if no two edges are incident. The size of any largest matching in G is called the *matching number* of G and is denoted by $\nu(G)$. It is easy to see and well-known that $\nu(G) \leqslant \tau(G)$ and $\alpha(G) + \tau(G) = |V(G)|$ for any graph G.

A *block* of a graph is a maximal connected subgraph having no cut-vertex. A *block-cactus* graph is a graph whose blocks are either complete graphs or cycles.

In this paper we investigate connections between perfect independent sets and k-independent as well as super k-independent sets for $k = 0$ and $k = 1$. In addition, we present various families of graphs with a strong unique (or maximum) independence spanning forest.

2. Preliminary results

In [1], p. 272, Berge proved that an independent set I in a graph G is 0-independent if and only if $|N_G(J) \cap I| \geqslant |J|$ for every independent subset J of $V(G) - I$. In [12], the authors presented the following extensions of Berge's result.

Theorem 2.1 (Siemes, Topp, Volkmann [12] 1994)**.** *For a nonnegative integer k, an independent set I of vertices of a graph G is a k-independent set in G if and only* *if*

$$|N_G(J) \cap I| \geqslant |J| + k$$

for every independent subset J of $V(G) - I$ with $J \neq \emptyset$ when $k \geqslant 1$.

Corollary 2.2. *For a nonnegative integer k, an independent set I of vertices of a graph G is a super k-independent set in G if and only if*

$$|N_G(J) \cap I| \geqslant |J| + k$$

for every subset J of $V(G) - I$ with $J \neq \emptyset$ when $k \geqslant 1$.

Proof. In view of the definition, I is a super k-independent set in G if and only if I is k-independent in the bipartite graph $G^* = G[I, V(G) - I]$. According to Theorem 2.1, this is equivalent to

$$|N_{G^*}(J) \cap I| \geqslant |J| + k$$

for every independent subset J of $V(G^*) - I$ with $J \neq \emptyset$ when $k \geqslant 1$. However, this is equivalent to

$$|N_G(J) \cap I| \geqslant |J| + k$$

for every subset J of $V(G) - I$ with $J \neq \emptyset$ when $k \geqslant 1$, and the proof is complete. □

Theorem 2.1 as well as Corollary 2.2 play an important role in our investigations.

Observation 2.3. *If G is a claw-free graph, then every perfect independent set is also a maximum independent set.*

Proof. If $I \subseteq V(G)$ is a perfect independent set and $J \subseteq V(G) - I$ an independent set, then $e_G(J, I) \geqslant 2|J|$. Since G is claw-free, we observe that

$$2|J| \leqslant e_G(J, I) = e_G(J, I \cap N_G(J)) \leqslant 2|I \cap N_G(J)|$$

and hence $|J| \leqslant |I \cap N_G(J)|$. Theorem 2.1 with $k = 0$ yields the desired result. □

Theorem 2.4 (Listing [9] 1862, König [8] 1936)**.** *A graph G is a forest if and only if $|E(G)| - |V(G)| + \sigma(G) = 0$, where $\sigma(G)$ denotes the number of components of G.*

Theorem 2.5 (König [6] 1916). *A graph is bipartite if and only if it contains no cycle of odd length.*

3. Perfect and super unique independent sets

Clearly, a super unique independent set is a unique maximum independent set, and a unique maximum independent set is a perfect independent set. In this section we will present some classes of graphs with the property that each perfect independent set is also a super unique independent set.

Proposition 3.1. *Let G be a graph with a perfect independent set I. If I is not a super unique independent set, then the bipartite graph $G[I, V(G) - I]$ contains a cycle.*

Proof. Since I is not a super unique independent set, there exists, in view of Corollary 2.2 with $k = 1$, a set $\emptyset \neq J \subseteq V(G) - I$ such that $|N_G(J) \cap I| \leqslant |J|$. Let $H = G[N_G(J) \cap I, J]$ be the induced bipartite subgraph of $G[I, V(G) - I]$. Since I is a perfect independent set, it follows that $|E(H)| \geqslant 2|J|$, and this leads to

$$|V(H)| = |N_G(J) \cap I| + |J| \leqslant 2|J| \leqslant |E(H)|.$$

Therefore, Theorem 2.4 implies that the graph H and hence also the bipartite graph $G[I, V(G) - I]$ contains a cycle. □

Proposition 3.1 and Theorem 2.5 immediately yield the following corollary.

Corollary 3.2. *Let G be a graph without any even cycle, and let I be an independent set. Then I is a perfect independent set if and only if I is a super unique independent set.*

Theorem 3.3. *If G is a graph, then every even cycle of G induces a complete subgraph of G if and only if the bipartite graph $G[I, V(G) - I]$ is a forest for each independent set $I \subseteq V(G)$.*

Proof. Assume that every even cycle of G induces a complete graph. Suppose that there exists an independent set $I \subseteq V(G)$ such that $G[I, V(G) - I]$ contains a cycle C. This implies $|I \cap V(C)| \geqslant 2$. Since C induces a complete graph, we arrive at the contradiction that I is an independent set.

Conversely, let $G[I, V(G) - I]$ be a forest for each independent set $I \subseteq V(G)$. Let $C = v_1 v_2 \ldots v_p v_1$ be an even cycle of length $p \geqslant 4$. We will prove by induction on p that C induces a complete subgraph. Let $A = \{v_1, v_3, \ldots, v_{p-1}\}$ and $B =$

$\{v_2, v_4, \ldots, v_p\}$. Neither $G[A, V(G) - A]$ nor $G[B, V(G) - B]$ is a forest and thus, neither A nor B is an independent set in G. Hence, there exist odd integers $1 \leqslant i < j \leqslant p - 1$ and even integers $2 \leqslant k < l \leqslant p$ such that v_i and v_j as well as v_k and v_l are adjacent. In the case that $p = 4$, it follows that C induces a complete graph. Let now $p \geqslant 6$ and assume, without loss of generality, that $i < k$. Then there are the two possibilities, namely $1 \leqslant i < k < l < j \leqslant p - 1$ or $1 \leqslant i < k < j < l \leqslant p$. In both cases we will show that C has a chord uw with $u \in A$ and $w \in B$.

If $1 \leqslant i < k < l < j \leqslant p - 1$, then

$$C_0 = v_i v_{i+1} \ldots v_k v_l v_{l+1} \ldots v_j v_i$$

is an even cycle with $|V(C_0)| < |V(C)|$. Therefore, by the induction hypothesis, C_0 induces a complete graph. In particular, $v_i v_l$ is a chord of C.

If $1 \leqslant i < k < j < l \leqslant p$, then

$$C_1 = v_i v_{i+1} \ldots v_k v_l v_{l-1} \ldots v_{j+1} v_j v_i,$$
$$C_2 = v_i v_j v_{j-1} \ldots v_{k+1} v_k v_l v_{l+1} \ldots v_i$$

are even cycles such that $|V(C_1)| + |V(C_2)| = |V(C)| + 4$ and hence $|V(C_1)| = |V(C_2)| = |V(C)|$ if and only if $|V(C)| = 4$. Since $|V(C)| \geqslant 6$, we conclude that $|V(C_1)| < |V(C)|$ or $|V(C_2)| < |V(C)|$. According to the induction hypothesis, the cycle C_1 or C_2 induces a complete graph. In particular, $v_i v_k, v_k v_j, v_j v_l, v_l v_i \in E(G)$. Since $|V(C)| \geqslant 6$, at least one of these four edges is a chord of C.

If C has a chord uw with $u \in A$ and $w \in B$, then we will finally show that C induces a complete graph. Let, without loss of generality, $u = v_1$ and $w = v_q$ with an even integer $4 \leqslant q \leqslant p - 2$. The cycles

$$C_3 = v_1 v_2 \ldots v_{q-1} v_q v_1, \quad C_4 = v_1 v_q v_{q+1} \ldots v_{p-1} v_p v_1$$

are even and such that $|V(C_3)|$, $|V(C_4)| < |V(C)|$. By the induction hypothesis, the cycles C_3 and C_4 induce complete graphs. Now let x and y be two arbitrary vertices in $V(C)$. If $x, y \in V(C_3)$ or $x, y \in V(C_4)$, then they are adjacent. If not, then $v_1 x v_q y v_1$ is a cycle of length four, and by the induction hypothesis, the vertices x and y are adjacent. Consequently, C induces a complete subgraph, and the proof is complete. □

Proposition 3.1 and Theorem 3.3 immediately lead to the following results.

Corollary 3.4. *Let G be a graph with the property that every even cycle induces a complete subgraph, and let I be an independent set. Then I is a perfect independent set if and only if I is a super unique independent set.*

Corollary 3.5. *Let G be a block-cactus graph such that every even block is a complete subgraph, and let I be an independent set. Then I is a perfect independent set if and only if I is a super unique independent set.*

Theorem 3.6. *Let G be a bipartite graph, and let $I \subseteq V(G)$ be an independent set. Then I is a unique maximum independent set if and only if I is a super unique independent set.*

Proof. Let I be a unique maximum independent set. Theorem 2.1 implies that $|N_G(J) \cap I| > |J|$ for all independent sets $\emptyset \neq J \subseteq V(G) - I$. Let A and B be the partite sets of G and let $L \neq \emptyset$ be an arbitrary subset of $V(G) - I$. It follows that $L \cap A$ and $L \cap B$ are independent sets such that, without loss of generality, $L \cap A \neq \emptyset$. We deduce from Theorem 2.1 that

$$|N_G(L \cap A) \cap I| > |L \cap A|, \quad |N_G(L \cap B) \cap I| \geqslant |L \cap B|.$$

Therefore, we obtain

$$|N_G(L) \cap I| = |N_G(L \cap A) \cap I| + |N_G(L \cap B) \cap I| > |L \cap A| + |L \cap B| = |L|.$$

Thus, with respect to Corollary 2.2, I is a super unique independent set, and the proof is complete. □

4. Perfect and unique independent sets

Proposition 4.1. *Let G be a graph with a perfect independent set I. If I is not a unique maximum independent set, then there exists an induced bipartite subgraph of G which is not a forest.*

Proof. Since I is not a unique maximum independent set, there exists, in view of Theorem 2.1 with $k = 1$, an independent set $\emptyset \neq J \subseteq V(G) - I$ such that $|N_G(J) \cap I| \leqslant |J|$. If we define the induced bipartite graph $H = G[N_G(J) \cap I, J]$, then, since I is a perfect independent set, it follows that $|E(H)| \geqslant 2|J|$. This yields

$$|V(H)| = |N_G(J) \cap I| + |J| \leqslant 2|J| \leqslant |E(H)|.$$

Therefore, Theorem 2.4 implies that the induced bipartite subgraph H is not a forest. □

Observation 4.2. *If G is a graph, then every even cycle of G contains a chord if and only if every induced bipartite subgraph of G is a forest.*

Proof. Assume that every even cycle contains a chord. Suppose that there exists an induced bipartite subgraph H with a cycle. Let C be a shortest cycle in H. Since C has a chord in G, this chord also belongs to H, a contradiction to the minimum length of C.

Conversely, assume that every induced bipartite subgraph of G is a forest. Let C be an even cycle in G. Suppose that C has no chord. Then C is an induced bipartite subgraph of G but no forest. This contradiction completes the proof. □

Proposition 4.1 and Observation 4.1 immediately lead to the next result.

Corollary 4.3. *Let G be a graph with the property that every even cycle contains a chord, and let I be an independent set. Then I is a perfect independent set if and only if I is a unique maximum independent set.*

5. Strong (unique) maximum independence spanning forests

In view of Theorem 2.1, we establish easily the following facts.

Corollary 5.1. *Let G be a bipartite graph.*

The graph G is a strong maximum independence graph if and only if there exist partite sets A and B such that $|N_G(S)| \geqslant |S|$ for all $S \subseteq A$.

The graph G is a strong unique independence graph if and only if there exist partite sets A and B such that $|N_G(S)| > |S|$ for all $\emptyset \neq S \subseteq A$.

Theorem 5.2 (König [7] 1931). *If G is a bipartite graph, then*

$$\tau(G) = \nu(G).$$

Theorem 5.3 (König-Hall, König [7] 1931, Hall [4] 1935). *Let G be a bipartite graph with partite sets A and B. Then G contains a matching M with the property that every vertex in A is incident with an edge in M if and only if $|N_G(S)| \geqslant |S|$ for all $S \subseteq A$.*

Theorem 5.4 (Lovász [10] 1970). *Let G be a bipartite graph with partite sets A and B. Then G contains a spanning forest F such that $d_F(v) = 2$ for all $v \in A$ if and only if $|N_G(S)| > |S|$ for all $\emptyset \neq S \subseteq A$.*

A proof of Theorem 5.4 can also be find in [11] on p. 20. Corollary 5.1 shows that Theorem 5.3 and Theorem 5.4 characterize the strong maximum and the strong unique independence graphs, respectively.

Theorem 5.5. *If G is a graph, then the following statements are equivalent.*

(a) $\nu(G) = \tau(G)$.

(b) *There exists a super independent set in G.*

(c) *Every maximum independent set in G is a super independent set.*

P r o o f. (a) $\Rightarrow$ (c): Let I be a maximum independent set, and let M be a maximum matching in G. This leads to

$$|V(G) - I| = \tau(G) = \nu(G) = |M|.$$

This implies that M is a matching in the bipartite graph $G[I, V(G) - I]$ with the property that every vertex in $V(G) - I$ is incident with an edge in M. It follows that $|N_G(S) \cap I| \geqslant |S|$ for all $S \subseteq V(G) - I$. Hence, by Corollary 2.2, I is a super independent set in G.

(b) $\Rightarrow$ (a): Let I be a super independent set in G. As a consequence of Corollary 2.2 we obtain $|N_G(S) \cap I| \geqslant |S|$ for all $S \subseteq V(G) - I$. Hence, by Theorem 5.3, there exists a matching M in the bipartite graph $G[I, V(G) - I]$ with the property that every vertex in $V(G) - I$ is incident with an edge in M. It follows that $\tau(G) = |V(G) - I| = |M| \leqslant \nu(G)$. Because of $\nu(G) \leqslant \tau(G)$, we deduce that $\nu(G) = \tau(G)$.

Since (c) $\Rightarrow$ (b) is immediate, the proof is complete. □

For reason of completeness, we will give a short proof of the next theorem by Hopkins and Staton [5].

Theorem 5.6 (Hopkins, Staton [5] 1985)**.** *Let G be a connected bipartite graph. The graph G is a strong unique independence graph if and only if G has a strong unique independence spanning tree T. In addition, the unique maximum independent sets of G and T coincide.*

P r o o f. Assume that G is a strong unique independence graph. Let A and B be the partite sets such that B is a unique maximum independent set of G. Combining Corollary 5.1 and Theorem 5.4, we find that G contains a spanning forest F such that $d_F(v) = 2$ for all $v \in A$. We now extend F to a spanning tree T of G by adding as many edges as necessary. This yields $d_T(v) \geqslant 2$ for all $v \in A$. Hence, B is a perfect independent set in T, and Corollary 3.2 implies that B is a unique independent set in T.

Conversely, assume that G has a strong unique independence spanning tree T with the partite sets A and B such that B is the unique maximum independent set of T. It follows easily from Theorem 2.5 that A and B are also independent sets in G. Obviously, B is also a unique maximum independent set in G. □

Using Theorem 5.3 instead of Theorem 5.4, one can prove the next result similar to Theorem 5.6. Its proof is therefore omitted.

Theorem 5.7 (Volkmann [13] 1988). *Let G be a connected bipartite graph. The graph G is a strong maximum independence graph if and only if G has a strong maximum independence spanning tree T. In addition, the maximum independent sets of G and T coincide.*

Theorem 5.8. *If G is a graph, then the following statements are valid.*

(a) *If G has a super unique independent set, then G has a strong unique independence spanning forest T with $\alpha(T) = \alpha(G)$.*

(b) *If G is a bipartite graph with a unique maximum independent set, then G has a strong unique independence spanning forest T with $\alpha(T) = \alpha(G)$.*

(c) *If $\nu(G) = \tau(G)$, then G has a strong maximum independence spanning forest T with $\alpha(T) = \alpha(G)$.*

(d) *If G is a bipartite graph, then G has a strong maximum independence spanning forest T with $\alpha(T) = \alpha(G)$.*

Proof. (a) Let I be a super unique independent set in G. This means that I is a unique maximum independent set in the bipartite graph $H = G[I, V(G) - I]$, and thus H is a strong unique independence graph. If $H_1, H_2, \ldots, H_p$ are the components of H, then $I \cap V(H_i)$ are strong unique independent sets in H_i for $i = 1, 2, \ldots, p$. In view of Theorem 5.6, each component H_i has a strong maximum independence spanning tree T_i with a unique maximum independent set $I \cap V(H_i)$ for $i = 1, 2, \ldots, p$. Obviously, $T = \bigcup_{i=1}^{p} T_i$ is a strong maximum independence spanning forest of G with $\alpha(T) = \alpha(G) = |I|$.

(b) Let I be a unique maximum independent set in the bipartite graph G. According to Theorem 3.6, I is a super unique independent set in G and (a) yields the desired result.

(c) Let $\nu(G) = \tau(G)$. In view of Theorem 5.5, G has a super independent set. Using Theorem 5.7 instead of Theorem 5.6, the proof is analogous to the proof of (a) and is therefore omitted.

(d) If G is bipartite, then Theorem 5.2 yields $\nu(G) = \tau(G)$. Now (c) leads to the desired result. □

Theorem 5.9. *Let G be a block-cactus graph such that every even block is a complete subgraph. If $I \subseteq V(G)$ is a perfect independent set, then $F = G[I, V(G)-I]$ is a strong unique independence spanning forest of G.*

Proof. In view of Theorem 3.3, F is a spanning forest of G. According to Corollary 3.5, I is a super unique independent set in G. Altogether, we see that F is a strong unique independence spanning forest of G with the unique maximum independent set I. □

Theorem 5.8 (b) and Theorem 5.9 are generalizations of the following result by Hopkins and Staton [5].

Corollary 5.10 (Hopkins, Staton [5] 1985)**.** *A tree T has a unique maximum independent set I if and only if T has a spanning forest F such that each component of F is a strong unique independence tree and each edge in $T - E(F)$ joins two vertices not in I.*

References

[1] *C. Berge*: Graphs. Second revised edition, North-Holland, 1985.
[2] *G. Chartrand, L. Lesniak*: Graphs and Digraphs. Third edition, Chapman and Hall, London, 1996.
[3] *C. Croitoru, E. Suditu*: Perfect stables in graphs. Inf. Process. Lett. *17* (1983), 53–56.
[4] *P. Hall*: On representatives of subsets. J. London Math. Soc. *10* (1935), 26–30.
[5] *G. Hopkins, W. Staton*: Graphs with unique maximum independent sets. Discrete Math. *57* (1985), 245–251.
[6] *D. König*: Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. Math. Ann. *77* (1916), 453–465.
[7] *D. König*: Graphs and matrices. Math. Fiz. Lapok *38* (1931), 116–119. (In Hungarian.)
[8] *D. König*: Theorie der endlichen und unendlichen Graphen. Akademische Verlagsgesellschaft, Leipzig, 1936; reprinted: Teubner-Archiv zur Mathematik, Band 6, Leipzig, 1986.
[9] *J. B. Listing*: Der Census räumlicher Complexe oder Verallgemeinerungen des Eulerschen Satzes von den Polyedern. Göttinger Abhandlungen *10* (1862).
[10] *L. Lovász*: A generalization of König's theorem. Acta Math. Acad. Sci. Hung. *21* (1970), 443–446.
[11] *L. Lovász, M. D. Plummer*: Matching Theory. Ann. Discrete Math. 29, North-Holland, 1986.
[12] *W. Siemes, J. Topp, L. Volkmann*: On unique independent sets in graphs. Discrete Math. *131* (1994), 279–285.
[13] *L. Volkmann*: Minimale und unabhängige minimale Überdeckungen. An. Univ. Bucur. Mat. *37* (1988), 85–90.

Author's address: *Lutz Volkmann*, Lehrstuhl II für Mathematik, RWTH Aachen, 52056 Aachen, Germany, e-mail: volkm@math2.rwth-aachen.de.