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TRANSFINITE RANGES AND THE LOCAL SPECTRUM

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Abstract. Let T be a Banach space operator. For ordinal numbers α we define the α-ranges $R^\alpha(T)$ which generalize the ranges of powers $R(T^n)$. The intersection $\bigcap_\alpha R^\alpha(T)$ is the coeur algébrique of T. Moreover, the coeur algébrique (and more generally the α-ranges for limit ordinals α) have similar properties as the coeur analytique of T. So it is possible to introduce algebraic local spectra which have properties analogous to those of classical (analytic) local spectra studied in local spectral theory.

1. Introduction.

Let X be a Banach space. As usually we denote by $B(X)$ the set of all bounded linear operators acting on X. For $T \in B(X)$ let $R(T)$ and $N(T)$ denote the range $R(T) = TX$ and kernel $N(T) = \{x \in X : Tx = 0\}$, respectively.

If $T \in B(X)$ then the ranges $R^n(T) = T^nX$ form a decreasing sequence of linear manifolds

$$X = R^0(T) \supset R^1(T) \supset \cdots \supset R^n(T) \supset R^{n+1}(T) \cdots$$

There are two possibilities: either there is $n \in \mathbb{N}$ for which

$$R^n(T) = R^{n+1}(T) \tag{1}$$

or not: if (1) holds then also $R^m(T) = R^n(T)$ for all $m \geq n$. In this situation we say that the operator T has finite descent; the minimum $n \in \mathbb{N}$ for which (1) holds is called the descent of T.

It is possible to extend this construction to more general ordinals [S]. The α range of an operator $T \in B(X)$ is an extension to ordinal numbers α of the usual range $R^n(T) = T^nX$ associated with a natural number n. The intersection of all the α ranges coincides with the coeur algébrique, the largest linear manifold $Y \subset X$ for which $TY = Y$.

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2. Descent

Let X will be a (complex or real) Banach space and $T \in B(X)$. Then with

$$R^n(T) = T^n X \qquad (n \in \mathbb{N}),$$

there is inclusion

$$R^{n+1}(T) \subset R^n(T) \qquad (n \in \mathbb{N}).$$

Formally $R^n(T)$ is defined by induction: specifically

$$R^0(T) = X;\ R^{n+1}(T) = TR^n(T) \quad (n \geq 0). \tag{2}$$

The procedure (2) can be carried out for ordinals $\alpha \in \mathrm{Ord}$:

$$R^{\alpha+1}(T) = TR^\alpha(T) \quad (\alpha \in \mathrm{Ord}),$$

while for limit ordinals

$$R^\beta(T) = \bigcap_{\alpha<\beta} R^\alpha(T).$$

Let ω_0 be the first infinite ordinal. Note that $R^{\omega_0}(T) = \bigcap_{n=0}^\infty T^n X$ (usually denoted by $R^\infty(T)$) is called the hyperrange of T and used in operator theory frequently.

When in pursuit of $R^\alpha(T)$ we stray outside the natural numbers to more general ordinals we can no longer make sense of an operator

$$T^\alpha : X \to X.$$

It is easy to see that the α-ranges $R^\alpha(T)$ form a non-increasing "sequence" of linear manifolds, $R^\alpha(T) \subset R^\beta(T)$ if $\alpha \geq \beta$. Moreover, if $R^{\alpha+1}(T) = R^\alpha(T)$ for some α, then $R^\beta(T) = R^\alpha(T)$ for all $\beta > \alpha$. A standard cardinality argument shows that the sequence $R^\alpha(T)$ eventually stops: if $\alpha > \mathrm{card}\, X$ (more precisely if α is greater than the cardinality of a Hamel basis in X) then $R^{\alpha+1}(T) = R^\alpha(T)$.

Definition 1. Let $T \in B(X)$. The descent $\mathrm{dsc}\,(T)$ is the smallest ordinal number α for which $R^{\alpha+1}(T) = R^\alpha(T)$.

The coeur algébrique of T is defined by $\mathrm{co}\,(T) = \bigcap_\alpha R^\alpha(T) = R^{\mathrm{dsc}\,(T)}(T)$.

Remark 2. There is a simple characterization of the coeur algébrique of $T \in B(X)$. A vector $x_0 \in X$ belongs to $\mathrm{co}\,(T)$ if and only if there exist vectors $x_1, x_2, \ldots$ such that $Tx_i = x_{i-1}$ for all $i \geq 1$.

Indeed, it is easy to see that $T\mathrm{co}\,(T) = \mathrm{co}\,(T)$. If $x_0 \in \mathrm{co}\,(T)$ then we can find inductively vectors $x_1, x_2, \dots \in \mathrm{co}\,(T)$ such that $Tx_i = x_{i-1}$ for all $i \geq 1$.

Conversely, suppose that there are vectors x_i satisfying $Tx_i = x_{i-1} \quad (i \geq 1)$. Let M be the linear manifold generated by the vectors $x_i \quad (i \geq 0)$.

Clearly $TM = M$. It is easy to see that $M \subset R^\alpha(T)$ for all α, and so $M \subset \mathrm{co}\,(T)$. Hence $x_0 \in \mathrm{co}\,(T)$.

Thus $\mathrm{co}\,(T)$ is the union of all linear manifolds $M \subset X$ satisfying $TM = M$ and it is the largest linear manifold with this property.

Remark 3. All the previous definitions make sense for any set X and a mapping $f : X \to X$. Thus it is possible to define the α-ranges $R^\alpha(f)$ of f and the coeur $\mathrm{co}\,(f) = \bigcap_\alpha R^\alpha(f)$. The characterization of $\mathrm{co}\,(f)$ also remains true (of course the ranges $R^\alpha(f)$ and the coeur $\mathrm{co}\,(f)$ are now sets, not linear manifolds).

Proposition 4. For each ordinal number α there exists a Banach space X and an operator $T \in B(X)$ such that $\mathrm{dsc}\,(T) = \alpha$.

Proof. Let α be an ordinal number. Let X be the ℓ_1 space with a standard basis $e_{\alpha_1,\dots,\alpha_n}$, where $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n$ are ordinal numbers satisfying $\alpha > \alpha_1 > \dots > \alpha_n$. More precisely, the elements of X are the sums

$$x = \sum_{\alpha_1,\dots,\alpha_n} c_{\alpha_1,\dots,\alpha_n} e_{\alpha_1,\dots,\alpha_n}$$

with (real or complex) coefficients $c_{\alpha_1,\dots,\alpha_n}$ such that

$$\|x\| := \sum_{\alpha_1,\dots,\alpha_n} |c_{\alpha_1,\dots,\alpha_n}| < \infty.$$

The operator $T \in B(X)$ is defined by $Te_{\alpha_1,\dots,\alpha_n} = e_{\alpha_1,\dots,\alpha_{n-1}}$ if $n \geq 2$ and $Te_{\alpha_1} = 0$. By the transfinite induction we can prove that $R^\beta(T) = \bigvee\{e_{\alpha_1,\dots,\alpha_n} : \alpha_n \geq \beta\}$. Thus $R^\beta \neq \{0\}$ for $\beta < \alpha$ and $R^\alpha(T) = \{0\}$. So $\mathrm{dsc}\,(T) = \alpha$. □

Remark 5. It is interesting to note that the dual notion - ascent - behaves in a different way. If we define the transfinite kernels $N^\alpha(T)$ of an operator $T \in B(X)$ in a dual way by $N^0(T) = \{0\}$, $N^{\alpha+1}(T) = T^{-1}N^\alpha(T)$ and $N^\alpha(T) = \bigcup_{\beta<\alpha} N^\beta(T)$ for limit ordinals α, then this sequence stops at the latest at ω_0. We have $N^k(T) = N(T^k)$ for $k < \infty$ and $N^{\omega_0}(T) = \bigcup_{k=0}^\infty N(T^k)$ (which is usually denoted by $N^\infty(T)$). It is easy to see that $T^{-1}N^{\omega_0}(T) = N^{\omega_0}(T)$.

It is well known that if $\mathrm{asc}\,(T) < \infty$ and $\mathrm{dsc}\,(T) < \infty$ then $\mathrm{asc}\,(T) = \mathrm{dsc}\,(T)$. It may happen that $\mathrm{asc}\,(T) < \infty$ and $\mathrm{dsc}\,(T)$ is infinite, however, in this case $\mathrm{dsc}\,(T) = \omega_0$.

Proposition 6. Let $T \in B(X)$ and $\mathrm{asc}\,(T) < \infty$. Then $\mathrm{dsc}\,(T) \leq \omega_0$.

Proof. Let $\mathrm{asc}\, T = p < \infty$. We show that $TR^{\omega_0}(T) = R^{\omega_0}(T)$. Let $x \in R^{\omega_0}(T)$. Then there exists a vector $u \in X$ such that $T^{p+1}u = x$. Let $v = T^p u$. So $Tv = x$. We show that $v \in R^{\omega_0}(T)$.

Let $n \in \mathbb{N}$, $n > p$. Since $x \in R(T^n)$, there exist $y \in X$ with $T^n y = x$. So $T^{p+1}(u - T^{n-p-1}y) = 0$. Since $\mathrm{asc}\,(T) = p$, we have $v - T^{n-1}y =$

$T^p(u - T^{n-p-1}y) = 0$. So $v = T^{n-1}y \in R(T^{n-1})$. Since n was arbitrary, we have $v \in R^{\omega_0}(T)$ and $TR^{\omega_0}(T) = R^{\omega_0}(T)$. Hence $\mathrm{co}\,(T) = R^{\omega_0}(T)$ and $\mathrm{dsc}\,(T) \le \omega_0$. □

Proposition 7. Let $A, B \in B(X)$, $AB = BA$, let α be an ordinal number. Then
(i) $BR^\alpha(A) \subset R^\alpha(A)$;
(ii) $R^\alpha(AB) \subset R^\alpha(A)$;
(iii) if α is a limit ordinal and $n \in \mathbb{N}$ then $R^\alpha(A^n) = R^\alpha(A)$.

In particular, $B\mathrm{co}\,(A) \subset \mathrm{co}\,(A)$, $\mathrm{co}\,(AB) \subset \mathrm{co}\,(A)$ and $\mathrm{co}\,(A^n) = \mathrm{co}\,(A)$ for each $n \in \mathbb{N}$.

Proof. (i) By the transfinite induction. We have $BR^0(A) = BX \subset X = R^0(A)$. If $BR^\alpha(A) \subset R^\alpha(A)$, then $BR^{\alpha+1}(A) = BAR^\alpha(A) = ABR^\alpha(A) \subset AR^\alpha(A) = R^{\alpha+1}(A)$. If α is a limit ordinal and $BR^\beta(A) \subset R^\beta(A)$ for all $\beta < \alpha$, then

$$BR^\alpha(A) = B \bigcap_{\beta<\alpha} R_\beta(A) \subset \bigcap_{\beta<\alpha} BR^\beta(A) \subset \bigcap_{\beta<\alpha} R^\beta(A) = R_\alpha(A).$$

(ii) Again by transfinite induction. The statement is clear for $\alpha = 0$. If $R^\alpha(AB) \subset R^\alpha(A)$, then $R^{\alpha+1}(AB) = ABR^\alpha(AB) \subset R^{\alpha+1}(A)$. If α is a limit ordinal and $R^\beta(AB) \subset R^\beta(A)$ for all $\beta < \alpha$, then $R^\alpha(AB) = \bigcap_{\beta<\alpha} R^\beta(AB) \subset \bigcap_{\beta<\alpha} R^\beta(A) = R^\alpha(A)$.

(iii) Let α be a limit ordinal number. If $\alpha = \omega_0$ then we have $R^{\omega_0}(A) = \bigcap_{k=0}^\infty R(A^k) = \bigcap_{k=0}^\infty R(A^{nk}) = R^{\omega_0}(A^n)$.

Suppose that (iii) is not true and let α be the smallest limit ordinal for which this is not true. Then either $\alpha = \beta + \omega_0$ for some limit ordinal β or $\alpha = \sup\{\beta < \alpha : \beta \text{ limit ordinal}\}$.

If $\alpha = \beta + \omega_0$ for some limit ordinal β then

$$R^\alpha(A) = \bigcap_{k=0}^\infty A^k R^\beta(A) = \bigcap_{k=0}^\infty A^k R^\beta(A^n) = R^\alpha(A^n).$$

If $\alpha = \sup\{\beta < \alpha : \beta \text{ limit ordinal}\}$ then

$$R^\alpha(A) = \bigcap_{\beta<\alpha, \beta \text{ limit}} R^\beta(A) = \bigcap_{\beta<\alpha, \beta \text{ limit}} R^\beta(A^n) = R^\alpha(A^n).$$

□

3. Local spectra

In this section X will be a complex Banach space.

Recall that the coeur analytique $K(T)$ is defined as the set of all vectors $x_0 \in X$ for which there exist vectors $x_1, x_2, \dots \in X$ such that $Tx_i = x_{i-1}$ $(i \ge 1)$ and $\sup_n \|x_n\|^{1/n} < \infty$, see [M]. Equivalently, $x_0 \in K(T)$ if there exists an analytic function $f : U \to X$ defined on a neighborhood of 0 such that $(T-z)f(z)=x_0 \quad (z\in U)$ (f is defined by $f(z)=\sum_{n=0}^\infty x_n z^n$). Clearly $K(T)\subset \mathrm{co}\,(T)$.

The coeur analytique plays an important role in the local spectral theory. We show that the coeur algébrique co$\,(\cdot)$, and more generally the transfinite ranges R_α have similar properties and it is possible to construct parallel local spectra.

Recall [KM], [MM] that a non-empty subset $\mathcal{R}\subset B(X)$ is called a regularity if it satisfies the following two conditions:
(i) Let $T\in B(X)$ and $n\in\mathbb{N}$. Then $T\in\mathcal{R}\Leftrightarrow T^n\in\mathcal{R}$;
(ii) Let $A,B,C,D\in B(X)$ be mutually commuting operators satisfying $AC+BD=I$. Then

$$AB\in\mathcal{R}\Leftrightarrow A\in\mathcal{R} \text{ and } B\in\mathcal{R}.$$

Any regularity gives rise to an abstract spectrum $\sigma_{\mathcal{R}}$. For $T\in B(X)$ we define $\sigma_{\mathcal{R}}(T)=\{\lambda\in\mathbb{C}: T-\lambda\notin\mathcal{R}\}$.

The spectrum $\sigma_{\mathcal{R}}$ defined as above exhibits nice properties, especially it satisfies the spectral mapping property: $\sigma_{\mathcal{R}}(f(T))=f(\sigma_{\mathcal{R}}(T))$ for each $T\in B(X)$ and each locally non-constant function f analytic on a neighborhood of $\sigma(T)$.

The abstract spectra $\sigma_{\mathcal{R}}$ include most of the natural spectra considered in operator theory. For example, the local spectrum can be defined in the following way:

For $x\in X$ let $\mathcal{R}_{x,K}=\{T\in B(X): x\in K(T)\}$. Then $\mathcal{R}_{x,K}$ is a regularity and the local spectrum at x can be defined by

$$\sigma_x(T)=\{\lambda\in\mathbb{C}: T-\lambda\notin\mathcal{R}_{x,K}\}=\{\lambda\in\mathbb{C}: x\notin K(T-\lambda)\}$$

(the usual equivalent definition of the local spectrum is $\lambda\notin\sigma_x(T)\Leftrightarrow$ there exists a function $f:U\to X$ analytic on a neighbourhood U of λ such that $(T-z)f(z)=x \quad (z\in U)$; note that the traditional notation of the local spectrum is rather illogically $\sigma_T(x)$). This implies the spectral mapping property for the local spectrum: $\sigma_x(f(T))=f(\sigma_x(T))$ for all $x\in X$, $T\in B(X)$ and each locally non-constant function f analytic on a neighborhood of $\sigma(T)$.

We show that the coeur algébrique and the transfinite ranges give also rise to regularities, and so it is possible to define the corresponding spectra in a similar way as in the local spectral theory.

Definition 8. Let $x\in X$ and let α be a limit ordinal number. Write $\mathcal{R}_{x,\alpha}=\{T\in B(X): x\in R^\alpha(T)\}$. Write further $\mathcal{R}_{x,\mathrm{co}}=\{T\in B(X): x\in R_{\mathrm{co}}\,(T)\}$.

For each $x\in X$ we have clearly $\mathcal{R}_{x,\alpha}\supset\mathcal{R}_{x,\beta}$ whenever $\alpha\le\beta$ and $\mathcal{R}_{x,\mathrm{co}}=\bigcap_\alpha\mathcal{R}_{x,\alpha}$.

Lemma 9. *Let $A,B,C,D\in B(X)$ be mutually commuting operators satisfying $AC+BD=I$. Then $N(A)\subset\mathrm{co}\,(B)$ and $N(B)\subset\mathrm{co}\,(A)$.*

Moreover, $N^\infty(A) \subset \mathrm{co}\,(B)$ and $N^\infty(B) \subset \mathrm{co}\,(A)$.

Proof. Let $x_0 \in N(A)$. Then $BDx_0 = x_0$. For $j \in \mathbb{N}$ set $x_j = D^j x_0$. Then for $j \geq 1$ we have

$$Bx_j = BD^j x_0 = D^{j-1}x_0 = x_{j-1}.$$

So $x_0 \in \mathrm{co}\,(B)$. The inclusion $N(B) \subset \mathrm{co}\,(A)$ follows from symmetry.

Let $n \in \mathbb{N}$. Since $AC + BD = I$ implies $A^n C_n + B^n D_n = I$ for some $B_n, D_n \in B(X)$ commuting with each other and with A^n, B^n, see [KM], we have $N(A^n) \subset \mathrm{co}\,(B^n) = \mathrm{co}\,(B)$. Thus $N^\infty(A) \subset \mathrm{co}\,(B)$ and similarly $N^\infty(B) \subset \mathrm{co}\,(A)$. □

Lemma 10. Let $A, B, C, D \in B(X)$ be mutually commuting operators satisfying $AC + BD = I$. Then $R^\alpha(AB) = R^\alpha(A) \cap R^\alpha(B)$ for each ordinal number α. In particular, $\mathrm{co}\,(AB) = \mathrm{co}\,(A) \cap \mathrm{co}\,(B)$.

Proof. Clearly $R^\alpha(AB) \subset R^\alpha(A) \cap R^\alpha(B)$ by Proposition 7 (ii). We prove the second inclusion by the transfinite induction.

Suppose that $R^\alpha(AB) = R^\alpha(A) \cap R^\alpha(B)$ and $x \in R^{\alpha+1}(A) \cap R^{\alpha+1}(B)$. Then $x = Au = Bv$ for some $u \in R^\alpha(A)$ and $v \in R^\alpha(B)$. So $x = Au \in R^\alpha(A)$ and $x = Bv \in R^\alpha(B)$. By the induction hypothesis $x \in R^\alpha(AB)$.

Let $\beta < \alpha$. Then $ABR^\beta(AB) = R^{\beta+1}(AB) \supset R^\alpha(AB)$, so there exists $w \in R^\beta(AB)$ with $ABw = x$. We have $u - Bw \in N(A) \subset \mathrm{co}\,(B) \subset R^{\beta+1}(B)$ and $Bw \in BR^\beta(AB) \subset BR^\beta(B) = R^{\beta+1}(B)$. Thus $u \in R^{\beta+1}(B)$. Hence $u \in \bigcap_{\beta<\alpha} R^{\beta+1}(B) = R^\alpha(B)$. Thus $u \in R^\alpha(A) \cap R^\alpha(B) = R^\alpha(AB)$. In a similar way we can prove $v \in R^\alpha(AB)$.

Set $y = Du + Cv \in R^\alpha(AB)$. Then

$$ABy = ABDu + ABCv = BDAu + ACBv = BDx + ACx = x$$

and $x \in ABR^\alpha(AB) = R^{\alpha+1}(AB)$.

If α is a limit ordinal and $R^\beta(AB) = R^\beta(A) \cap R^\beta(B)$ for all $\beta < \alpha$, then

$$R^\alpha(A) \cap R^\alpha(B) = \bigcap_{\beta<\alpha} R^\beta(A) \cap \bigcap_{\beta<\alpha} R^\beta(B)$$

$$= \bigcap_{\beta<\alpha} (R^\beta(A) \cap R^\beta(B)) = \bigcap_{\beta<\alpha} R^\beta(AB) = R^\alpha(AB).$$

□

Corollary 11. Let α be a limit ordinal number and $x \in X$. Then $\mathcal{R}_{x,\alpha}$ is a regularity. In particular, $\mathcal{R}_{x,\mathrm{co}}$ is a regularity.

Proof. Let $T \in B(X)$ and $n \in \mathbb{N}$. We have

$$T \in \mathcal{R}_{x,\alpha} \Leftrightarrow x \in R^\alpha(T) \Leftrightarrow x \in R^\alpha(T^n) \Leftrightarrow T^n \in \mathcal{R}_{x,\alpha}.$$

Let A, B, C, D, be mutually commuting operators satisfying $AC + BD = I$. Then

$$AB \in \mathcal{R}_{x,\alpha} \Leftrightarrow x \in R^\alpha(AB) \Leftrightarrow x \in R^\alpha(A) \cap R^\alpha(B) \Leftrightarrow A \in \mathcal{R}_{x,\alpha} \text{ and } B \in \mathcal{R}_{x,\alpha}.$$

So $\mathcal{R}_{x,\alpha}$ is a regularity.

Since $\operatorname{co}(T) = R^\alpha(T)$ for all $T \in B(X)$ for any limit ordinal $\alpha > \operatorname{card} X$, we have that $\mathcal{R}_{x,\mathrm{co}}$ is a regularity. □

Definition 12. Let $x \in X$ and let α be a limit ordinal. For $T \in B(X)$ write $\sigma_{x,\alpha}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{R}_{x,\alpha}\}$. Write further $\sigma_{x,\mathrm{co}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{R}_{x,\mathrm{co}}\}$.

Clearly $\sigma_{x,\omega_0}(T) = \{\lambda \in \mathbb{C} : x \notin R^{\omega_0}(T - \lambda)\}$ and and $\sigma_{x,co}(T) = \{\lambda \in \mathbb{C} : x \notin \operatorname{co}(T - \lambda)\}$. Clearly for all $x \in X$ and $T \in B(X)$ we have

$$\sigma_{x,\omega_0}(T) \subset \sigma_{x,2\omega_0}(T) \subset \cdots \subset \sigma_{x,\mathrm{co}}(T) \subset \sigma_x(T) \subset \sigma_{\mathrm{sur}}(T),$$

where $\sigma_x(T)$ denotes the classical local spectrum defined above and $\sigma_{\mathrm{sur}}(T) = \{\lambda \in \mathbb{C} : (T - \lambda)X \neq X\}$ is the surjective spectrum.

The spectrum $\sigma_{x,\mathrm{co}}$ was implicitly considered for example in [JS], [L], [LV], [MMN], [PV]. In these papers there were considered algebraic spectral spaces $E_T(F)$ for any subset $F \subset \mathbb{C}$. In our terminology $E_T(F) = \{x \in X : \sigma_{x,\mathrm{co}}(T) \subset F\}$. For a survey of results concerning the algebraic spectral subspaces see [LN], p.48.

Proposition 13. Let $T \in B(X)$ and let α be a limit ordinal. Then

$$\bigcup_{x \in X} \sigma_{x,\alpha}(T) = \sigma_{\mathrm{sur}}(T).$$

Corollary 14. Let $T \in B(X)$, $x \in X$ and let f be a function analytic on a neighborhood of $\sigma(T)$. Then

$$\sigma_{x,\alpha}(f(T)) = f(\sigma_{x,\alpha}(T))$$

for each limit ordinal α. In particular,

$$\sigma_{x,\mathrm{co}}(f(T)) = f(\sigma_{x,\mathrm{co}}(T)).$$

In general the spectra $\sigma_{x,\mathrm{co}}$ and $\sigma_{x,\alpha}$ are not closed even for normal operator on a Hilbert space.

Example 15. Let H be a separable infinite-dimensional Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Let $T \in B(H)$ be defined by $Te_n = n^{-1}e_n$. For $k = 0, 1, \ldots$ let $x_k = \sum_{n=1}^\infty n^{k-n} e_n \in H$. Then $Tx_k = x_{k-1}$ for all $k \in \mathbb{N}$. So $x_0 \in \operatorname{co}(T)$ and $0 \notin \sigma_{x_0,\mathrm{co}}(T)$.

On the other hand, for each $n \in \mathbb{N}$, $x_0 \notin R(T - n^{-1})$, and so $n^{-1} \in \sigma_{x,\mathrm{co}}(T)$. In fact for each limit ordinal α we have $n^{-1} \in \sigma_{x_0,\alpha}(T)$ and $0 \notin \sigma_{x_0,\alpha}(T)$.

The previous example shows also that in general $\sigma_{x_0,\mathrm{co}}(T) \neq \sigma_{x_0}(T)$ since $\sigma_{x_0}(T)$ is always closed.

Another important notion studied in local spectral theory is that of analytic residuum. Let $T \in B(X)$. Denote by S_T the set of all complex numbers λ such that there exists a nonzero function $f : U \to X$ analytic on a neighbourhood of λ such that $(T - z)f(z) = 0 \quad (z \in U)$. The analytic residuum of T is the closure $\overline{S_T}$.

The analytic residuum can be also introduced by means of regularities. Let

$$\mathcal{S}(X) = \{T \in B(X) : K(T) \cap N(T) = \{0\}\}.$$

Then $\mathcal{S}(X)$ is a regularity and the corresponding spectrum $\sigma_{\mathcal{S}}(T) = S_T$, see [KM]. In particular, $S_{f(T)} = f(S_T)$ and $\overline{S_{f(T)}} = f(\overline{S_T})$ for each locally non-constant function f analytic on a neighbourhood of $\sigma(T)$.

Again we can define a parallel algebraic notion.

Definition 16. Let $\mathcal{S}^{alg}(X) = \{T \in B(X) : \mathrm{co}\,(T) \cap N(T) = \{0\}\}$.

Theorem 17. $\mathcal{S}^{alg}(X)$ is a regularity.

Proof. Let $A, B \in B(X), AB = BA \notin \mathcal{S}^{alg}(X)$. We prove that either $A \notin \mathcal{S}^{alg}(X)$ or $B \notin \mathcal{S}^{alg}(X)$. Let $x_i \in X$ satisfy $ABx_i = x_{i-1} \quad (i = 1, 2, \dots)$, where $x_0 = 0$ and $x_1 \neq 0$. Set $u_i = B^i x_i \quad (i = 0, 1, \dots)$. Then $u_0 = 0$ and $Au_i = u_{i-1} \quad (i = 1, 2, \dots)$. If $u_1 \neq 0$ then $A \notin \mathcal{S}^{alg}(X)$.

Suppose on the contrary $u_1 = Bx_1 = 0$. Set $v_0 = 0, v_i = A^{i-1}x_i \quad (i = 1, 2, \dots)$. Then $Bv_i = v_{i-1} \quad (i = 1, 2, \dots)$ and $v_1 = x_1 \neq 0$. Thus $B \notin \mathcal{S}^{alg}(X)$. Hence $A, B \in \mathcal{S}^{alg}(X), AB = BA$ implies $AB \in \mathcal{S}^{alg}(X)$.

In particular $A \in \mathcal{S}^{alg}(X) \Rightarrow A^n \in \mathcal{S}^{alg}(X) \quad (n = 1, 2, \dots)$.

Let $A \notin \mathcal{S}^{alg}(X)$ and let $x_i \in X$ satisfy $x_0 = 0$, $x_1 \neq 0$ and $Ax_i = x_{i-1} \quad (i \geq 1)$. Then $y_i = x_{ni}$ satisfy the same conditions for A^n , so that $A^n \notin \mathcal{S}^{alg}(X)$. Hence $A \in \mathcal{S}^{alg}(X) \Leftrightarrow A^n \in \mathcal{S}^{alg}(X)$.

Suppose that A, B, C, D are mutually commuting operators satisfying $AC + BD = I$ and $A \notin \mathcal{S}^{alg}(X)$. Let $x_i \in X$ satisfy $Ax_i = x_{i-1} \quad (i = 1, 2, \dots), x_0 = 0$ and $x_1 \neq 0$. Set $x_{i,0} = x_i \quad (i \geq 0)$ and $x_{0,i} = 0 \quad (i \geq 1)$.

Define inductively $x_{i,j} = Cx_{i-1,j} + Dx_{i,j-1} \quad (i, j \geq 1)$.

We show by induction

$$Ax_{i,j} = x_{i-1,j} \qquad (i \geq 1, j \geq 0) \tag{3}$$

and

$$Bx_{i,j} = x_{i,j-1} \qquad (i \geq 0, j \geq 1). \tag{4}$$

This is clear for $i = 0$ or $j = 0$. Let $i, j \geq 1$ and suppose that (3) and (4) is true for all $i' \leq i, j' \leq j, (i', j') \neq (i, j)$. Then

$$Ax_{i,j} = ACx_{i-1,j} + ADx_{i,j-1} = (I - BD)x_{i-1,j} + ADx_{i,j-1}$$
$$= x_{i-1,j} - Dx_{i-1,j-1} + Dx_{i-1,j-1} = x_{i-1,j}.$$

Similarly,

$$Bx_{i,j} = BCx_{i-1,j} + BDx_{i,j-1} = BCx_{i-1,j} + (I - AC)x_{i,j-1} = x_{i,j-1}.$$

Set $y_i = x_{i,i} \quad (i \geq 0)$. Then $ABy_i = y_{i-1} \quad (i \geq 1)$, $y_0 = 0$ and $y_1 \neq 0$. Thus $AB \notin \mathcal{S}^{alg}(X)$, so that $AB \in \mathcal{S}^{alg}(X) \Rightarrow A, B \in \mathcal{S}^{alg}(X)$. □

For $T \in B(X)$ define $S_T^{alg} = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{S}^{alg}(X)\}$.

Corollary 18. Let $T \in B(X)$ and let f be a locally non-constant function analytic on a neighbourhood of $\sigma(T)$. Then

$$f(S_T^{alg}) = S_{f(T)}^{alg} \quad \text{and} \quad f(\overline{S_T^{alg}}) = \overline{S_{f(T)}^{alg}}.$$

References

[JS] B.E. Johnson, A.M. Sinclair: Continuity of derivatives and a problem of Kaplansky, Amer. J. Math. 90 (1968), 1067–73.

[KM] V. Kordula, V. Müller: On the axiomatic theory of spectrum, Studia Math. 119 (1996), 109-128.

[M] M. Mbekhta: Généralisation de la décomposition de Kato aux opérateurs paranormaux et spetraux, Glasgow Math. J. 29 (1987), 159–175.

[MM] M. Mbekhta, V. Müller: On the axiomatic theory of spectrum II., Studia Math. 119 (1996), 129-147.

[L] K.B. Laursen: Algebraic spectral subspaces and automatic continuity, Czechoslovak Math. J. 38 (1988), 157–172.

[LN] K.B. Laursen, M.M. Neumann: An Introduction to Local Spectral Theory, London Math. Soc. Monographs New Series 20, Oxford Science Publication, Clarence Press, Oxford, 2000.

[LV] K.B. Laursen, P. Vrbova: Some remarks on the surjectivity spectrum of linear operators, Czechoslovak Math. J. 39 (1989), 730–739.

[MMN] T.L. Miller, V.G. Miller, M.M. Neumann: Spectral subspaces of subscalar and related operators, Proc. Amer. Math. Soc. 132 (2004), no. 5, 1483-1493.

[PV] V. Ptak, P. Vrbova: Algebraic spectral subspaces, Czechoslovak Math. J. 38 (1988), 342–350.

[S] P. Saphar: Contribution à l'étude des applications linéaires dans un espace de Banach, Bull. Soc. Math. France 92 (1964), 363–384.

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