EXAMPLES FROM THE CALCULUS OF VARIATIONS
I. NONDEGENERATE PROBLEMS

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(Received October 23, 1997)

Abstract. The criteria of extremality for classical variational integrals depending on several functions of one independent variable and their derivatives of arbitrary orders for constrained, isoperimetrical, degenerate, degenerate constrained, and so on, cases are investigated by means of adapted Poincaré-Cartan forms. Without ambitions on a noble generalizing theory, the main part of the article consists of simple illustrative examples within a somewhat naive point of view in order to obtain results resembling the common Euler-Lagrange, Legendre, Jacobi, and Hilbert-Weierstrass conditions whenever possible and to discuss some modifications necessary in the degenerate case. The inverse and the realization problems are mentioned, too.

Keywords: variational integral, critical curve, adjoint module, diffiety, initial form, Poincaré-Cartan form, Lagrange problem, Mayer field, Weierstrass function

MSC 2000: 49-01, 49K15, 58A10, 49N45

We will deal with variational integrals depending on several functions of one independent variable subjected to a system of ordinary differential equations (the *Lagrange problem*), and the article is devoted to some improvements of the most fundamental classical aspects of the calculus of variations which are discussed in all textbooks, namely to the Euler-Lagrange, Legendre, Jacobi, and Hilbert-Weierstrass conditions. It is well-known that the common approach based on admissible variations causes many difficulties and fails in the degenerate case. Moreover, the final results involving uncertain coefficients (the *Lagrange multipliers*) and derived under vague limitations seem to be not quite useful in practice. To cope with such troubles, appropriately adapted *Poincaré-Cartan forms* naturally appearing in the theory of infinitely prolonged differential equations (*diffieties*) and the tool of "economical variables" (the *adjoint modules*) are called for help. To manifest the method as simply as possible, the topic is explained within a survey of examples. This approach intentionally results in a somewhat old-fashioned article which however permits to accentuate the crucial ideas without unnecessary formalism and as elementarily as possible. We do not suppose preliminary acquaintance with any advanced results and, for better convenience of the reader, the parts enclosed with asterisks $^{*}\dots{}_{*}$ may be skipped without loss of the main context.

Introduction

1. Notation. We will use a space $\mathbf{M}$ with (local) coordinates $h^1, h^2, \dots$ and the ring $\mathcal{F}(\mathbf{M})$ of (C^∞ smooth) real-valued functions $f = f(h^1, \dots, h^m)$ depending on a finite number $m = m(f)$ of variables. The space $\mathbf{M}$ is supplied with the $\mathcal{F}(\mathbf{M})$-module $\Phi(\mathbf{M})$ of differential 1-forms $\varphi = \sum f^i \, \mathrm{d}g^i$ ($f^i, g^i \in \mathcal{F}(\mathbf{M})$; finite sum), and with the $\mathcal{F}(\mathbf{M})$-module $\mathcal{T}(\mathbf{M})$ of vector fields $Z = \sum^{\infty} z^i \frac{\partial}{\partial h^i}$ ($z^i \in \mathcal{F}(\mathbf{M})$; infinite sum). The Lie derivatives

$$\begin{aligned} &\mathfrak{L}_Z f = Zf \in \mathcal{F}(\mathbf{M}), \\ &\mathfrak{L}_Z \varphi \in \Phi(\mathbf{M}), \\ &\mathfrak{L}_Z X = [Z, X] \in \mathcal{T}(\mathbf{M}) \end{aligned}$$

(where $X \in \mathcal{T}(\mathbf{M})$), the interior products like

$$\begin{aligned} &Z \rfloor \varphi = \varphi(Z) \in \mathcal{F}(\mathbf{M}), \\ &Z \rfloor \, \mathrm{d}f = Zf \in \mathcal{F}(\mathbf{M}), \\ &Z \rfloor \psi = \psi(Z, .) \in \Phi(\mathbf{M}) \end{aligned}$$

(where ψ is a 2-form), and the exterior differentials satisfying

$$\begin{aligned} &\mathfrak{L}_Z \varphi = Z \rfloor \, \mathrm{d}\varphi + \mathrm{d}Z \rfloor \varphi, \\ &\mathrm{d}\varphi(Z, X) = Z\varphi(X) - X\varphi(Z) - \varphi([Z, X]) \end{aligned}$$

will frequently occur. By means of invertible substitutions

$$(1) \qquad \begin{aligned} &h^i \equiv H^i(k^1, \dots, k^{m(i)}), \\ &k^j \equiv K^j(h^1, \dots, h^{n(j)}), \end{aligned}$$

other coordinates $k^1, k^2, \dots$ can be introduced. In more generality, if $\mathbf{N}$ is a space with coordinates $k^1, k^2, \dots$, then (1_1) determines a mapping $\mathbf{n} \colon \mathbf{N} \to \mathbf{M}$ where $\mathbf{n}^* h^i \equiv H^i \in \mathcal{F}(\mathbf{N})$. Such *injective mappings* (*inclusions*) will be denoted as $\mathbf{n} \colon \mathbf{N} \subset$

$\mathbf{M}$ and we will speak of a *subspace* ($\mathbf{N}$ of $\mathbf{M}$) and of a *restriction* $\mathbf{n}^*$ (of $\mathcal{F}(\mathbf{M})$ or $\Phi(\mathbf{M})$ to $\mathbf{N}$). Analogously, *surjective mappings* $\mathbf{n}$ are called *fibrations* (of $\mathbf{N}$ with the *factorspace* $\mathbf{M}$) and then $\mathbf{n}^*$ acts injectively and produces certain submodules.

Finitely-dimensional spaces will be occuring, too. In particular, one-dimensional spaces (subspaces) are *curves.* Concerning such fundamental concepts, we rely on a kind tolerance of the reader.

2. A sole variational integral. Let $\alpha \in \Phi(\mathbf{M})$ be a given form, the *density.* A curve (consisting of points) $P(t) \in \mathbf{M}$ $(0 \leqslant t \leqslant 1)$ is called *stationary* (for α) if the condition

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_Q \alpha|_{\lambda=0} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^1 Q(.,\lambda)^*\alpha|_{\lambda=0} = 0 \tag{2}$$

is satisfied for all families of curves $Q(t,\lambda) \in \mathbf{M}$ $(0 \leqslant t \leqslant 1)$ such that

$$Q(t,0) \equiv P(t),\ Q(0,\lambda) \equiv P(0),\ Q(1,\lambda) \equiv P(1) \tag{3}$$

where λ $(-\varepsilon < \lambda < \varepsilon, \varepsilon > 0)$ is a parameter. The stationary curves can be "in general" easily identified. To this aim, let

$$\mathrm{Adj}\ \mathrm{d}\alpha = \{Z\rfloor\,\mathrm{d}\alpha\colon\ \text{all } Z \in \mathcal{T}(\mathbf{M})\} \subset \Phi(\mathbf{M}) \tag{4}$$

be the *adjoint submodule* (generated by all the forms $Z\rfloor\,\mathrm{d}\alpha \in \Phi(\mathbf{M})$ mentioned) *to the form* $\mathrm{d}\alpha$. Assuming *genericity* (i.e., we suppose that the *dimension* denoted by $\ell(\mathrm{Adj}\ \mathrm{d}\alpha)$ of the module Adj $\mathrm{d}\alpha$ is a constant independent of the choice of the place), this adjoint submodule is *completely integrable* (i.e., locally has a basis consisting of total differentials) and moreover (owing to the *Pfaff-Darboux theorem*)

$$\mathrm{Adj}\ \mathrm{d}\alpha = \{\,\mathrm{d}u^i,\, \mathrm{d}v^i\colon\ i = 1,\dots,c\},\ \mathrm{d}\alpha = \sum \mathrm{d}u^i \wedge \mathrm{d}v^i \tag{5}$$

for appropriate $u^i, v^i \in \mathcal{F}(\mathbf{M})$ and $c = \ell(\mathrm{Adj}\ \mathrm{d}\alpha)/2$. (All composed functions $F(u^1,\dots,u^c,v^1,\dots,v^c)$ will be called *adjoint to* $\mathrm{d}\alpha$.) Using this result, we have

$$\int_Q \alpha - \int_P \alpha = \oint \alpha = \iint \mathrm{d}\alpha = \sum \oint u^i\,\mathrm{d}v^i = \sum \iint \mathrm{d}u^i \wedge \mathrm{d}v^i \tag{6}$$

with the curvilinear integral over the closed loop consisting of the variable arc $Q(t,\lambda)$ and reversely oriented arc $P(t)$, and the relevant double integral appearing by the *Green formula.* One can then find that (2) is satisfied if and only if all functions $u^i(P(t))$, $v^i(P(t))$ are constants (expressively: $P(t)$ reduces to a point in the space of adjoint variables). In other terms, *stationary curves are identical with solutions of the Pfaffian system*

$$(7) \qquad P^*\varphi \equiv 0 \quad (\varphi \in \mathrm{Adj}\ \mathrm{d}\alpha).$$

Assuming $c \neq 0$, the value of the difference $\int_Q \alpha - \int_P \alpha$ can be nevertheless made quite arbitrary, by an appropriate choice of Q.

We shall discuss additional requirements for the curves to ensure the constant sign of the difference. The requirements will be realized by differential equations. Roughly saying, such equations mean some relations among fractions $\mathrm{d}h^i/\mathrm{d}h^j$, that is, they may be expressed by Pfaffian equations $\mathrm{d}h^i = h^i_j\,\mathrm{d}h^j$ (h^i_j are certain functions depending on parameters). We shall however need a little more: all possible consequences of such equations ought be available without much effort. This demand is satisfied in the realm of *diffieties* (to follow).

3. Digression. Let $\Omega \subset \Phi(\mathbf{M})$ be a submodule of codimension one, hence the submodule

$$\mathcal{H}(\Omega) = \{Z\colon\ \omega(Z) \equiv 0 \text{ for all } \omega \in \Omega\} \subset \mathcal{T}(\mathbf{M})$$

is generated by a single vector field. Clearly $\mathfrak{L}_Z\omega = Z\rfloor\,\mathrm{d}\omega \in \Omega$ if $Z \in \mathcal{H}(\Omega)$ and $\omega \in \Omega$. We speak of a *diffiety* Ω if there exist $\omega^1, \dots, \omega^c \in \Omega$ such that

$$\Omega = \{\mathfrak{L}^k_Z\omega^i\colon\ Z \in \mathcal{H}(\Omega); i = 1, \dots, c;\ k = 0, 1, \dots\},$$

that is, the operator $\mathfrak{L}_Z$ (a single $Z \neq 0$, $Z \in \mathcal{H}(\Omega)$ is enough) repeatedly applied to all ω^i produces forms which altogether generate Ω.

More explicitly, choose $x \in \mathcal{F}(\mathbf{M})$ with $\mathrm{d}x \notin \Omega$, the so called *independent variable.* Then $\mathrm{d}h^i - g^i\,\mathrm{d}x \in \Omega$ for appropriate $g^i \in \mathcal{F}(\mathbf{M})$. If $X \in \mathcal{H}(\Omega)$ is defined by $Xh^i \equiv g^i$, then $Xx = 1$ whence

$$\mathrm{d}f - Xf\,\mathrm{d}x \in \Omega, \quad \mathfrak{L}_X(\mathrm{d}f - g\,\mathrm{d}x) = \mathrm{d}Xf - Xg\,\mathrm{d}x$$

for all $f, g \in \mathcal{F}(\mathbf{M})$. It follows easily that all forms

$$\mathfrak{L}^k_X(\mathrm{d}h^i - Xh^i\,\mathrm{d}x) = \mathrm{d}X^kh^i - X^{k+1}h^i\,\mathrm{d}x \quad (i = 1, \dots, c;\ k = 0, 1, \dots)$$

generate Ω provided c is fixed and large enough. Generators of this kind will be frequently employed, and the primary notation of coordinates will be appropriately adapted.

In particular, the *contact diffieties* provide a link to the classical theory. Let $\mathbf{M}(m)$ be the space with the so called *jet coordinates* x, w^i_r ($i = 1, \dots, m;\ r =$

$0, 1, \ldots$) and let $\Omega(m) \subset \Phi(\mathbf{M}(m))$ be the submodule generated by *contact forms* $\omega_r^i \equiv \mathrm{d}w_r^i - w_{r+1}^i\,\mathrm{d}x$. This is a diffiety since

$$\mathfrak{L}_X \omega_r^i \equiv \omega_{r+1}^i \quad \Big(X = \frac{\partial}{\partial x} + \sum^{\infty} w_{r+1}^i \frac{\partial}{\partial w_r^i} \in \mathcal{H}(\Omega(m))\Big). \tag{8}$$

In dealing with this diffiety, we will occasionally abbreviate by

$$f_r^i \equiv \frac{\partial f}{\partial w_r^i},\ f_{rs}^{ij} \equiv \frac{\partial^2 f}{\partial w_r^i \partial w_s^j}, \ldots, f_{rx}^i \equiv \frac{\partial^2 f}{\partial w_r^i \partial x}, \ldots$$

various derivatives of functions (and apologize for possible abuse).

A little surprisingly, formulae (8) can be carried over to arbitrary diffiety but we shall restrict ourselves to a certain subcase quite sufficient for our future needs. Denote by $\mathcal{R}(\Omega) \subset \Omega$ the largest completely integrable submodule of a diffiety Ω. Obviously $\mathfrak{L}_Z \mathcal{R}(\Omega) \subset \mathcal{R}(\Omega)$ for any $Z \in \mathcal{H}(\Omega)$ and it may be proved that $\mathcal{R}(\Omega)$ is the largest finite-dimensional submodule of Ω with this property. Moreover, assuming $\mathcal{R}(\Omega) = 0$, there exist so called *initial forms* $\pi^1, \ldots, \pi^\mu \in \Omega$ such that the forms

$$\pi_r^i \equiv \mathfrak{L}_Z^r \pi^i \ (i = 1, \ldots, \mu;\ r = 0, 1, \ldots;\ \text{fixed } Z \in \mathcal{H}(\Omega),\ Z \neq 0) \tag{9}$$

constitute a basis of Ω. (Obviously $\pi_0^i \equiv \pi^i$.) If $Z = X$ with $Xx = 1$ as above, then (9) reads $\mathrm{d}\pi_r^i \cong \mathrm{d}x \wedge \pi_{r+1}^i \pmod{\Omega \wedge \Omega}$. The initial forms are not uniquely determined (unlike the constant $\mu = \mu(\Omega)$) and will be explicitly stated in all examples. (Clearly $\mu = m$, $\pi_r^i \equiv \omega_r^i$ for the contact diffiety $\Omega = \Omega(m)$.)

Finally, let $\mathbf{n}\colon \mathbf{N} \subset \mathbf{M}$ be a subspace such that all *vectors* $Z \in \mathcal{H}(\Omega)$ *are tangent to* $\mathbf{N}$ (i.e., $\mathbf{n}^* f = 0$ implies $\mathbf{n}^* Z f = 0$). Such a subspace will appear if certain $f^1, \ldots, f^c \in \mathcal{F}(\mathbf{M})$ are chosen and we take all points of $\mathbf{M}$ satisfying the equations $Z^k f^i \equiv 0$ ($k = 0, 1, \ldots;\ Z \in \mathcal{H}(\Omega)$). The restriction $\mathbf{n}^*\Omega \subset \Phi(\mathbf{N})$ of a diffiety $\Omega \subset \Phi(\mathbf{M})$ is again a diffiety (on $\mathbf{N}$) called a *subdiffiety of* Ω. Concerning the notation, we follow the common practice: $Z \in \mathcal{H}(\Omega)$ is identified with its restriction $Z \in \mathcal{H}(\mathbf{n}^*\Omega)$ to $\mathbf{N}$, and analogous abbreviations

$$f = \mathbf{n}^* f \in \mathcal{F}(\mathbf{N}),$$
$$\varphi = \mathbf{n}^* \varphi \in \Phi(\mathbf{N}),$$
$$\omega = \mathbf{n}^* \omega \in \mathbf{n}^* \Omega$$

for $f \in \mathcal{F}(\mathbf{M})$, $\varphi \in \Phi(\mathbf{M})$, $\omega \in \Omega$ do not make much confusion. (Warning: since the initial forms change for the subdiffiety, abbreviations $\pi_r^i \equiv \mathbf{n}^* \pi_r^i$ should be forbidden.)

4. The critical curves. Let a density $\alpha \in \Phi(\mathbf{M})$ and a diffiety $\Omega \subset \Phi(\mathbf{M})$ be given. An *admissible* ($\mathcal{A}$) *curve* $P(t) \in \mathbf{M}$ ($0 \leqslant t \leqslant 1$) *to the diffiety* Ω (i.e., satisfying $P^*\omega \equiv 0$ for all $\omega \in \Omega$) is called *stationary* (for the density α, *with the constraint* Ω) if (2) is true for all $\mathcal{A}$-curves $Q(x,\lambda) \in \mathbf{M}$ ($0 \leqslant t \leqslant 1$, parameter λ) satisfying (3).

If an $\mathcal{A}$-curve is stationary for a density α, it is stationary for this α with the constraint Ω (triviality). In more generality, if an $\mathcal{A}$-curve is stationary for a density $\alpha + \omega$ ($\omega \in \Omega$ fixed), it is stationary for the original α with constraint Ω (triviality). Such a curve will be called a *critical* ($\mathcal{C}$) one. In other terms, *$\mathcal{C}$-curves* $P(t) \in \mathbf{M}$ ($0 \leqslant t \leqslant 1$) are solutions of the system

$$\text{(10)} \qquad \begin{aligned} &P^*\omega \equiv 0 \quad (\text{all } \omega \in \Omega),\\ &P^*\varphi \equiv 0 \quad (\varphi \in \operatorname{Adj} \mathrm{d}(\alpha+\omega),\ \text{appropriate } \omega \in \Omega), \end{aligned}$$

by definition.

As yet no surprise has appeared. However, assuming $\mathcal{R}(\Omega) = 0$, there exists a certain so called *Poincaré-Cartan* ($\mathcal{PC}$) *form* denoted $\breve{\alpha} = \alpha + \breve{\omega}$ ($\breve{\omega} \in \Omega$) such that all $\mathcal{C}$-curves satisfy (10_2) with this *universal choice* $\omega = \breve{\omega}$. The explicit construction will follow.

5. On the Poincaré-Cartan forms.

(i) The existence. We shall employ forms (9) where $Z = X$ together with the relevant form $\mathrm{d}x \notin \Omega$ as a basis of the module $\Phi(\mathbf{M})$. Let a $\mathcal{C}$-curve satisfy (10) where

$$\text{(11)} \qquad \mathrm{d}(\alpha+\omega) \cong \sum e^i_r \pi^i_r \wedge \mathrm{d}x \quad (\operatorname{mod} \Omega \wedge \Omega).$$

Clearly $e^i_r\,\mathrm{d}x \in \operatorname{Adj} \mathrm{d}(\alpha+\omega)$ (mod Ω) whence $P^* e^i_r \equiv 0$. This implies $P^*Xe^i_r \equiv 0$ (and even $P^*X^k e^i_r \equiv 0$; use $P^*\,\mathrm{d}e^i_r = \mathrm{d}P^*e^i_r \equiv 0$ where $\mathrm{d}e^i_r \cong Xe^i_r\,\mathrm{d}x$ (mod Ω) with induction on k). However, assuming $r \geqslant 1$ we have

$$\mathrm{d}(e^i_r \pi^i_{r-1}) \cong Xe^i_r\,\mathrm{d}x \wedge \pi^i_{r-1} + e^i_r\,\mathrm{d}x \wedge \pi^i_r \ (\operatorname{mod} \Omega \wedge \Omega),$$

and it follows that the form $\alpha + \omega$ in (10_2) can be replaced by $\alpha + \omega + e^i_r\pi^i_{r-1}$ with the result that the relevant summand $e^i_r\pi^i_r \wedge \mathrm{d}x$ in (11) disappears. Beginning with the highest possible indices $r \geqslant 1$ and repeatedly applying this reduction, only the initial summands can survive in (11), that is, we terminate with a certain

$$\text{(12)} \qquad \mathrm{d}(\alpha+\breve{\omega}) \cong \sum e^i \pi^i \wedge \mathrm{d}x \ (\operatorname{mod} \Omega \wedge \Omega)$$

instead of (11). One can then observe that such a form $\breve{\omega} \in \Omega$ with the property (12) is unique. So the special choice of the primary $\mathcal{C}$-curve does not matter and we have proved even more: *any* $\alpha + \breve{\omega}$ *with property* (12) *is a* $\mathcal{PC}$ *form.*

*(ii) The Lagrange multipliers. Let a subdiffiety $\mathbf{n}^*\Omega \subset \Phi(\mathbf{N})$ on the subspace $\mathbf{n}\colon \mathbf{N} \subset \mathbf{M}$ defined by certain equations $Z^k f^i \equiv 0$ as above be given. Then a $\mathcal{C}$-curve $P(t) \in \mathbf{M}$ $(0 \leqslant t \leqslant 1)$ for α and Ω satisfying moreover $P(t) \in \mathbf{N}$ $(0 \leqslant t \leqslant 1)$ may be regarded as a $\mathcal{C}$-curve for the restrictions $\mathbf{n}^*\alpha$ and $\mathbf{n}^*\Omega$ (triviality). One can observe that this α may be replaced by any density $\bar{\alpha} \in \Phi(\mathbf{M})$ of the kind

$$\bar{\alpha} = \alpha + \sum \lambda_k^i X^k f^i \,\mathrm{d}x \quad \big(\lambda_k^i \in \mathcal{F}(\mathbf{M})\big). \tag{13}$$

In other terms, any *$\mathcal{C}$-curve* (in $\mathbf{M}$) *for this $\bar{\alpha}$ and Ω which is moreover lying in* $\mathbf{N}$ *can be regarded as a $\mathcal{C}$-curve for the restrictions* $\mathbf{n}^*\alpha$ *and* $\mathbf{n}^*\Omega$. We shall see in a moment that the converse is valid as well, but let us continue with two adjustments of the result at this moment. First: *all summands in* (13) *with $k > 0$ can be deleted* by using analogous reduction as in (i). Second: *we may assume $\lambda_k^i \equiv \lambda_k^i(x)$ without any loss of generality* since (roughly speaking) only the values along the $\mathcal{C}$-curve under consideration are important. Summarizing the achievement, *a curve $P(t) \in \mathbf{N} \subset \mathbf{M}$ $(0 \leqslant t \leqslant 1)$ is critical for $\mathbf{n}^*\alpha$ and $\mathbf{n}^*\Omega$ if and only if it is critical for some density $\alpha + \sum \lambda^i(x) f^i \,\mathrm{d}x$.* In the particular case $\mathbf{M} = \mathbf{M}(m)$ and $\Omega = \Omega(m)$, it is easy to verify that our $\mathcal{C}$-curves are identical with the familiar extremals of the classical calculus of variations, the subdiffieties realize the familiar *Lagrange problem*, and our result reduces to the well-known *Lagrange multipliers rule*. (In our opinion, this is however a misleading way.)

(iii) Continuation. We have a $\mathcal{C}$-curve $P(t) \in \mathbf{N}$ $(0 \leqslant t \leqslant 1)$ for $\mathbf{n}^*\alpha$ and $\mathbf{n}^*\Omega$ and our aim is to prove that this is a $\mathcal{C}$-curve for an appropriate density (13) and the primary constraint Ω. Passing to the proof, our assumptions read

$$\begin{aligned} &P^*\mathbf{n}^*\omega \equiv 0 \quad (\text{all } \omega \in \Omega),\\ &P^*\varphi \equiv 0 \quad (\varphi \in \operatorname{Adj}\,\mathrm{d}\mathbf{n}^*(\alpha+\omega),\ \text{some } \omega \in \Omega). \end{aligned} \tag{14}$$

If the curve is regarded as lying in $\mathbf{M}$, then (14_1) is equivalent to (10_1). Concerning (14_2), there does exist a form $\beta \in \Phi(\mathbf{M})$ such that

$$\begin{aligned} &\mathbf{n}^*\beta = \mathbf{n}^*(\alpha+\omega),\\ &\mathbf{n}^*\operatorname{Adj}\,\mathrm{d}\beta = \operatorname{Adj}\,\mathrm{d}\mathbf{n}^*(\alpha+\omega). \end{aligned}$$

The first equation can be expressed by

$$\beta = \alpha + \omega + \sum a_k^{ij} X^k f^i \cdot \varphi^j + \sum b_k^i \,\mathrm{d}X^k f^i$$

for appropriate $a_k^{ij}, b_k^i \in \mathcal{F}(\mathbf{M})$ and $\varphi^j \in \Phi(\mathbf{M})$. This implies $\beta \cong \alpha + \omega + \sum \lambda_k^i X^k f^i \,\mathrm{d}x$ (mod Ω) for certain multipliers $\lambda_k^i \in \mathcal{F}(\mathbf{M})$, hence

$$\operatorname{Adj}\,\mathrm{d}\mathbf{n}^*(\alpha+\omega) = \mathbf{n}^*\operatorname{Adj}\Big(\alpha + \sum \lambda_k^i X^k f^i \cdot \mathrm{d}x + \omega\Big)$$

where the right hand pull-back $\mathbf{n}^*$ may be omitted (by abbreviation) and then (14_2) turns into (10_2).

Such uncertain Lagrange multipliers will not be mentioned any more.$_*$

(iv) A few definitions. We introduce the *Euler-Lagrange* ($\mathcal{EL}$) subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$ of all points satisfying the so called $\mathcal{EL}$ *conditions*

$$Z^k e^i \equiv 0 \quad (Z \in \mathcal{H}(\Omega);\ i = 1, \dots, \mu;\ k = 0, 1, \dots) \tag{15}$$

(where $e^i \in \mathcal{F}(\mathbf{M})$ are $\mathcal{EL}$ *coefficients* for our choice of the $\mathcal{PC}$ form) which is equipped with the important $\mathcal{EL}$ *subdiffiety* $\mathbf{e}^*\Omega \subset \Phi(\mathbf{E})$ of diffiety Ω. The properties (10) of our $\mathcal{C}$-curve can be expressed as

$$\begin{aligned} &P^*\omega = 0 \quad (\text{all } \omega \in \Omega),\\ &P(t) \in \mathbf{E} \quad (0 \leqslant t \leqslant 1), \end{aligned} \tag{16}$$

by direct verification using (12). It follows that the $\mathcal{EL}$ subspace does not depend on the choice of the initial forms (hence of the $\mathcal{PC}$ form). In most examples to appear, it will be of a finite dimension, hence $\mathbf{e}^*\Omega = \mathcal{R}(\mathbf{e}^*\Omega) \subset \Phi(\mathbf{E})$ will be a completely integrable submodule of codimension one. It follows that (16_1) regarded as a Pfaffian system on $\mathbf{E}$ can be identified with the familiar $\mathcal{EL}$ *differential equations.* In our approach, they do not involve any uncertain multipliers.

*(v) An overview of $\mathcal{PC}$ forms. All possible $\mathcal{PC}$ forms $\tilde\alpha$ are

$$\tilde\alpha = \breve\alpha + \sum f^j\omega^j \quad (f^j \in \mathcal{F}(\mathbf{M}) \text{ with } \mathbf{e}^* f^j \equiv 0,\ \omega^j \in \Omega) \tag{17}$$

where $\breve\alpha$ is a given $\mathcal{PC}$ form. Indeed, every such $\tilde\alpha$ is a $\mathcal{PC}$ form by direct verification. Conversely, let $\bar\alpha = \breve\alpha + \omega$ ($\omega \in \Omega$) be a certain $\mathcal{PC}$ form and assume $\mathrm{d}\bar\alpha \cong \sum f_{ir}\pi^i_r \wedge \mathrm{d}x \pmod{\Omega \wedge \Omega}$. Clearly $\mathbf{e}^* f_{ir} \equiv 0$. Then, applying an analogous reduction as in (i) to this $\bar\alpha$, one arrives just at the (unique) form $\breve\alpha$ satisfying (12) by certain adjustments like $f_{ir}\pi^i_{r-1}$ which provide the second summand in (17), hence the form of the kind $\tilde\alpha$ as above.

One can observe that $\mathbf{e}^*\tilde\alpha = \mathbf{e}^*\breve\alpha$ therefore $\mathbf{e}^*\,\mathrm{d}\tilde\alpha = \mathbf{e}^*\,\mathrm{d}\breve\alpha$, that means, after the restriction to $\mathbf{E}$, it does not matter which $\mathcal{PC}$ form is employed.

(vi) The trivial case. Assuming the development

$$\mathrm{d}\breve\alpha = \sum e^i\pi^i \wedge \mathrm{d}x + \sum a^{ij}_{rs}\pi^i_r \wedge \pi^j_s \quad (i < j,\ \text{or } i = j \text{ and } r < s) \tag{18}$$

instead of congruence (12), the property $\mathbf{E} = \mathbf{M}$ (hence $e^i \equiv 0$) easily implies $a^{ij}_{rs} \equiv 0$ by virtue the identity $\mathrm{d}^2\breve\alpha = 0$. It follows that $\breve\alpha = \mathrm{d}a$ (hence $\alpha \cong Xa\,\mathrm{d}x \pmod{\Omega}$) for an appropriate $a \in \mathcal{F}(\mathbf{M})$ in this case. (We neglect the topological obstructions.)$_*$

(vii) Important note. In our theory of $\mathcal{C}$-curves, the rather strong requirements $(3_{2,3})$ at the end points can be replaced by less restrictive ones if a certain $\mathcal{PC}$ form $\tilde{\alpha}$ is kept fixed. For instance, assuming $\tilde{\alpha} \in \{\mathrm{d}v^1, \ldots, \mathrm{d}v^c\}$ for appropriate functions $v^i \in \mathcal{F}(\mathbf{M})$, it is clearly sufficient to suppose only the equalities $v^i(P(0)) \equiv v^i(Q(0,\lambda))$, $v^i(P(1)) \equiv v^i(Q(1,\lambda))$. Such a weaker *fixed end points assumption* admits many far going modifications: the reasonings can be adapted for the case when the curves $Q(0,\lambda), Q(1,\lambda) \in \mathbf{M}$ $(-\varepsilon < \lambda < \varepsilon)$ with variable λ satisfy the Pfaffian equation $\tilde{\alpha} = 0$. This approach yields the *moving end points* theory. All improvements mentioned will be passed over in silence here.

(viii) On the degeneracy concept. We will see that there is a large amount of variational problems which admit a rather uniform approach resembling the classical theory, these are the *nondegenerate problems.* The remaining "exceptional" *degenerate* cases will cause difficulties for various reasons which cannot be easily predicted in advance. In this sense, it is better to made this concept more precise only case by case. Let us turn to examples.

Nondegenerate problems

6. An opening example. Let us deal with the subdiffiety $\Omega = \mathbf{m}^*\Omega(2)$ on the subspace $\mathbf{m}\colon \mathbf{M} \subset \mathbf{M}(2)$ consisting of all points which satisfy a certain equation $w_1^2 = g(x, w_0^1, w_0^2, w_1^1)$, hence all equations

$$X^r(w_1^2 - g) = w_{r+1}^2 - X^r g = w_{r+1}^2 - w_{r+1}^1 g_1^1 + \ldots \equiv 0 \quad (r = 0, 1, \ldots).$$

Functions x, w_r^1, w_0^2 $(r = 0, 1, \ldots)$ can be taken for coordinates on $\mathbf{M}$, (restrictions of) forms ω_r^1, ω_0^2 $(r = 0, 1, \ldots)$ yield a basis of Ω, and the vector field

$$X = \frac{\partial}{\partial x} + \sum^{\infty} w_{r+1}^1 \frac{\partial}{\partial w_r^1} + g \frac{\partial}{\partial w_0^2} \tag{19}$$

(the restriction of (8_2) where $i = 1, 2$) generates the module $\mathcal{H}(\Omega)$). Clearly

$$\begin{aligned} \mathfrak{L}_X \omega_r^1 &\equiv \omega_{r+1}^1, \\ \mathfrak{L}_X \omega_0^2 &= \omega_1^2 = g_0^1 \omega_0^1 + g_0^2 \omega_0^2 + g_1^1 \omega_1^1 \end{aligned} \tag{20}$$

and it follows that the form $\pi = \omega_0^2 - g_1^1 \omega_0^1$ satisfies

$$\mathfrak{L}_X \pi = (g_0^1 - X g_1^1)\omega_0^1 + g_0^2 \omega_0^2 = g_0^2 \pi + a\omega_0^1 \tag{21}$$

where $a = g_0^1 + g_0^2 g_1^1 - X g_1^1$. Assuming $a \neq 0$, this π can be taken for the (single) initial form (explicitly $\mu = 1$, $\pi^1 = \pi$). We will not discuss the easier case when $a = 0$

(one can observe that $\mathcal{R}(\Omega) = \{\pi\} \neq 0$ then). Let moreover $\alpha = f(x, w_0^1, w_0^2, w_1^1)\,\mathrm{d}x$ be a given density. Obviously

$$\mathrm{d}\alpha = (f_0^2\pi + (f_0^1 + f_0^2 g_1^1)\omega_0^1 + f_1^1\omega_1^1) \wedge \mathrm{d}x$$

and one can identify the $\mathcal{PC}$ form and the $\mathcal{EL}$ condition:

$$\breve{\alpha} = f\,\mathrm{d}x + f_1^1\omega_0^1 + \frac{b}{a}\pi, \qquad (e^1 =)e = f_0^2 - g_0^2\frac{b}{a} - X\frac{b}{a} = 0 \tag{22}$$

where $b = f_0^1 + f_0^2 g_1^1 - X f_1^1$. Indeed, $\mathrm{d}\breve{\alpha} \cong e\pi \wedge \mathrm{d}x \pmod{\Omega \wedge \Omega}$ easily follows from (20, 21). The top order term

$$e = -X\frac{b}{a} + \ldots = w_3^1 a^{-2}(af_{11}^{11} - bg_{11}^{11}) + \ldots$$

indicates that the condition $af_{11}^{11} \neq bg_{11}^{11}$ should mean the *nondegeneracy*. In this case the functions $x, w_0^1, w_1^1, w_2^1, w_0^2$ can be used for coordinates on the $\mathcal{EL}$ subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$ and the forms $\breve{\alpha}$, $\mathrm{d}\breve{\alpha}$ are expressed in terms of them so that we may identify $\breve{\alpha} = \mathbf{e}^*\breve{\alpha}$, $\mathrm{d}\breve{\alpha} = \mathbf{e}^*\,\mathrm{d}\breve{\alpha}$.

We pass to the extremality conditions, and let us deal with the case of the *minimum.*

In order to simulate the well-known Weierstrass-Hilbert method, it is necessary to discuss the differential $\mathrm{d}\breve{\alpha}$ in more detail. For our aims, the formula

$$\mathrm{d}\breve{\alpha} = \pi \wedge \xi + \Big(f_{11}^{11} - \frac{b}{a}g_{11}^{11}\Big)\omega_1^1 \wedge \omega_0^1 \tag{23}$$

where $\xi \cong e\,\mathrm{d}x + a^{-1}(f_{11}^{11} - \frac{b}{a}g_{11}^{11})\omega_2^1 \pmod{\omega_0^1, \omega_0^2, \omega_1^1}$ is sufficient. It follows that $\operatorname{Adj}\,\mathrm{d}\breve{\alpha} = \{\pi, \xi, \omega_1^1, \omega_0^1\}$ in terms of generators of the module $\operatorname{Adj}\,\mathrm{d}\breve{\alpha}$, whence

$$\operatorname{Adj}\,\mathrm{d}\breve{\alpha} = \{\,\mathrm{d}u^1, \mathrm{d}u^2, \mathrm{d}v^1, \mathrm{d}v^2\}, \qquad \mathrm{d}\breve{\alpha} = \mathrm{d}u^1 \wedge \mathrm{d}v^1 + \mathrm{d}u^2 \wedge \mathrm{d}v^2 \tag{24}$$

in terms of appropriate adjoint functions. It is well-known that there exist *Lagrangean subspaces* $\mathbf{l}\colon \mathbf{L} \subset \mathbf{E}$ to the form $\mathrm{d}\breve{\alpha}$, i.e., maximal subspaces satisfying the *Hamilton-Jacobi equation* $\mathrm{d}\breve{\alpha} = 0$. In our case, they are three-dimensional (two dimensional in the space of the adjoint functions) and are fibered by the curves $u^1 = \text{const.}, \ldots, v^2 = \text{const.}$ These are however just the $\mathcal{C}$-curves (cf. (7) or (10_2)).

Assuming that (restrictions of) functions x, w_0^1, w_0^2 can be taken for *global* coordinates on $\mathbf{L}$, these $\mathcal{C}$-curves lying in $\mathbf{L}$ constitute the so called *Mayer field* (in the space x, w_0^1, w_0^2). All necessary ingredients are available as in the classical theory.

Let a $\mathcal{C}$-curve $P(t) \in \mathbf{E}$ be given and *assume that it can be embedded into a Mayer field* in the familiar sense. (If necessary, this assumption can be verified by the *Jacobi criterion* which we delay to an other place.) Let $Q(t) \in \mathbf{M}$ $(0 \leqslant t \leqslant 1)$ be a *near* admissible curve satisfying the boundary conditions $P(0) = Q(0)$, $P(1) = Q(1)$ (which can be made less restrictive, see 5 (vii)). Concerning the term *near*, it should be understood in the sence of the natural topology of $\mathbf{R}^\infty$, however, a close approximation of the coordinates $x, w_0^1, w_0^2, w_1^1, w_2^1, w_0^2$ is sufficient. Using coordinates in the ambient space $\mathbf{M}(2)$, we may write

$$Q(t) = (x(t), w_0^1(t), w_0^2(t), w_1^1(t), w_1^2(t), \ldots) \in \mathbf{M} \subset \mathbf{M}(2),$$

where $x'(t) > 0$. Let moreover

$$R(t) = (x(t), w_0^1(t), w_0^2(t), r_1^1(t), r_1^2(t), \ldots) \in \mathbf{L} \subset \mathbf{M}(2)$$

be its projection into $\mathbf{L}$. (If $w_s^i \equiv \overline{w}_s^i(x, w_0^1, w_0^2)$ are equations of $\mathbf{L}$, we put $r_s^i(t) \equiv \overline{w}_s^i(x(t), w_0^1(t), w_0^2(t))$.) Then

$$\int_Q \alpha - \int_P \alpha = \left(\int_Q \alpha - \int_R \breve{\alpha}\right) + \left(\int_R \breve{\alpha} - \int_P \breve{\alpha}\right) \tag{25}$$

(use $\int_P \breve{\omega} = 0$ for admissible curves) where the second summand on the right hand side vanishes owing to Green's theorem (since R with P make a loop in $\mathbf{L}$) and the first summand can be expressed by the integral $\int \mathcal{E}\, \mathrm{d}x = \int_0^1 \mathcal{E} x'(t)\, \mathrm{d}t$ where

$$\begin{aligned}\mathcal{E} = f(\ldots, w_1^1(t)) &- f(\ldots, r_1^1(t)) - f_1^1(\ldots, r_1^1(t))(w_1^1(t) - r_1^1(t)) \\ &- \frac{b(\ldots, r_1^1(t), r_2^1(t))}{a(\ldots, r_1^1(t), r_2^1(t))}(g(\ldots, w_1^1(t)) \\ &\qquad - g(\ldots, r_1^1(t)) - g_1^1(\ldots, r_1^1(t))(w_1^1(t) - r_1^1(t))\end{aligned}$$

$(\ldots = x, w_0^1, w_0^2)$ is the *Weierstrass function.* The inequality $\int_Q \alpha \geqslant \int_P \alpha$ is ensured if $\mathcal{E} \geqslant 0$. Geometrical interpretation of this inequality is a classical one: the graph of the function $F(w_1^1) = f(\ldots, w_1^1) + \frac{b}{a} g(\ldots, w_1^1)$ is lying over the tangent at the point $w_1^1 = r_1^1$. Note that the fraction $\frac{b}{a}$ is independent of the variable w_1^1 here. The (generalized) *Legendre condition* $f_{11}^{11} - \frac{b}{a} g_{11}^{11} > 0$ easily follows and needs no comments.

7. A peculiar result. Since our degeneracy concept is not strictly limited, we are able to include much more. For instance, modifying a little the previous example, let us choose $w_1^2 = g(x, w_0^1, w_0^2)$, hence

$$X^r(w_1^2 - g) = w_{r+1}^2 - w_r^1 g_0^1 - w_r^2 g_0^2 + \ldots \equiv 0 \quad (r = 0, 1, \ldots)$$

for the equations of the new subspace $\mathbf{m}\colon \mathbf{M} \subset \mathbf{M}(2)$, and let $\alpha = f(x, w_0^1, w_0^2)\,\mathrm{d}x$ be the new density. Assuming $g_0^1 \neq 0$ to exclude the easier case $\mathcal{R}(\Omega) \neq 0$, the initial form, the $\mathcal{PC}$ form, and the $\mathcal{EL}$ conditions can be easily found:

$$\begin{aligned} \pi &= \omega_0^2, \\ \breve{\alpha} &= f\,\mathrm{d}x + \frac{f_0^1}{g_0^1}\omega_0^2, \\ e &= f_0^2 - \frac{f_0^1}{g_0^1} g_0^2 - X\frac{f_0^1}{g_0^1} = 0. \end{aligned}$$

The assumption $f_0^1 g_{00}^{11} \neq g_0^1 f_{00}^{11}$ ensures the *nondegeneracy*. Then the functions x, w_0^1, w_0^2 provide coordinates on the $\mathcal{EL}$ subspace, and Lagrangian subspaces with coordinates x, w_0^1 equipped with the Mayer fields lead to the Weierstrass function

$$\mathcal{E} = f(\ldots, w_0^1) - f(\ldots, r_0^1) - \frac{f_0^1(\ldots, r_0^1)}{g_0^1(\ldots, r_0^1)}(g(\ldots, w_0^1) - g(\ldots, r_0^1))$$

which looks a little strange (but the geometrical interpretation is as fair as before).

8. Three variable functions. Let us deal with the subdiffiety $\Omega = \mathbf{m}^*\Omega(3)$ on the subspace $\mathbf{m}\colon \mathbf{M} \subset \mathbf{M}(3)$ defined by the equations

$$X^r(w_1^3 - g(x, w_0^1, w_0^2, w_0^3, w_1^1, w_1^2)) \equiv 0 \quad (r = 0, 1, \ldots).$$

The functions x, w_r^1, w_r^2, w_0^3 $(r = 0, 1, \ldots)$ provide coordinates on $\mathbf{M}$, and the forms $\omega_r^1, \omega_r^2, \omega_0^3$ $(r = 0, 1, \ldots)$ provide a basis of Ω. Denoting

$$\pi = \omega_0^3 - g_1^1\omega_0^1 - g_1^2\omega_0^2, \quad a^j \equiv g_0^j - Xg_1^j + g_0^3 g_1^j \quad (j = 1, 2),$$

one can obtain $\mathfrak{L}_X\pi = a^1\omega_0^1 + a^2\omega_0^2 + g_0^3\pi$, hence either $a^1 \neq 0$, or $a^2 \neq 0$ (otherwise $\mathcal{R}(\Omega) \neq 0$ which we exclude). If (e.g.) $a^2 \neq 0$ then $\pi^1 = \pi$, $\pi^2 = \omega_0^1$ can be taken for initial forms. Let $\alpha = f(x, w_0^1, w_0^2, w_0^3, w_1^1, w_1^2)\,\mathrm{d}x$ be a given density. Easy calculation gives the $\mathcal{PC}$ form

$$\begin{aligned} \breve{\alpha} &= f\,\mathrm{d}x + f_1^1\omega_0^1 + f_0^2\omega_0^2 + \frac{b^2}{a^2}\pi, \\ \mathrm{d}\breve{\alpha} &\cong (e^1\pi + e^2\omega_0^1) \wedge \mathrm{d}x \pmod{\Omega \wedge \Omega} \end{aligned}$$

where

$$e^1 = f_0^3 - \frac{b^2}{a^2} g_0^3 - X \frac{b^2}{a^2},$$
$$e^2 = b^1 - \frac{b^2}{a^2} a^1,$$
$$b^j \equiv f_0^j - X f_1^j + f_0^3 g_1^j.$$

The top order terms of the $\mathcal{EL}$ conditions

$$e^1 = X \frac{b^2}{a^2} + \ldots = (X g_1^2 \cdot X^2 f_1^2 - X f_1^2 \cdot X^2 g_1^2)/(a^2)^2 + \ldots = 0,$$
$$e^2 = (a^2 b^1 - b^2 a^1)/a^2 = (X g_1^2 \cdot X f_1^1 - X f_1^2 \cdot X g_1^1)/a^2 + \ldots = 0$$

permit to explicitly specify the *nondegenerate* case (not stated here for brevity but see Part IV to follow) when the functions $x, w_0^1, w_0^2, w_0^3, w_1^1, w_1^2, w_2^1$ can be taken as coordinates on the $\mathcal{EL}$ subspace $\mathbf{E} \subset \mathbf{M}$. If a $\mathcal{C}$-curve can be embedded into a Mayer field (on the Lagrange subspace $\mathbf{L} \subset \mathbf{E}$ with coordinates x, w_0^1, w_0^2, w_0^3) then the Weierstrass function

$$\begin{aligned}\mathcal{E} = f(\ldots, w_1^1, w_1^2) - f(\ldots, r_1^1, r_1^2) - \sum f_1^j(\ldots, r_1^1, r_1^2)(w_1^j - r_1^j) \\ + \frac{b^2(\ldots, r_1^1, r_1^2, r_2^1, r_2^2)}{a^2(\ldots, r_1^1, r_1^2, r_2^1, r_2^2)} (g(\ldots, w_1^1, w_1^2) - g(\ldots, r_1^1, r_1^2) \\ - \sum g_1^j(\ldots, r_1^1, r_1^2)(w_1^j - r_1^j))\end{aligned}$$

resolves the problem. The corresponding Legendre criterion means the definiteness of the quadratic form

$$\sum (f_{11}^{jk} + \frac{b^2}{a^2} g_{11}^{jk}) \xi^j \xi^k \qquad \text{(variables } \xi^1, \xi^2)$$

and does not bring any surprise. In accordance with 5 (vii), the fixed end point boundary conditions can be reduced to the equality of coordinates x, w_0^1, w_0^2, w_0^3.

In the same space $\mathbf{M}(3)$, let us mention another subdiffiety $\Omega = \mathbf{m}^* \Omega(3)$ on the smaller subspace $\mathbf{m}\colon \mathbf{M} \subset \mathbf{M}(2)$ defined by the equations

$$X^r (w_1^2 - g(x, w_0^1, w_0^2, w_0^3, w_1^1)) = X^r (w_1^3 - h(x, w_0^1, w_0^2, w_0^3, w_1^1)) \equiv 0 \quad (r = 0, 1, \ldots).$$

Functions x, w_r^1, w_0^2, w_0^3 $(r = 0, 1, \ldots)$ can be taken for coordinates on $\mathbf{M}$. Assuming

$$\triangle = a(Xb + b g_0^2 - a h_0^2) - b(Xa + a h_0^3 - b g_0^3) \neq 0$$

where we have denoted

$$a = g_0^1 - Xg_1^1 + g_0^2 g_1^1 + g_0^3 h_1^1,$$
$$b = h_0^1 - Xh_1^1 + h_0^2 g_1^1 + h_0^3 h_1^1,$$

then $\mathcal{R}(\Omega) = 0$ and $\pi^1 = \pi = b(\omega_0^2 - g_1^1\omega_0^1) - a(\omega_0^3 - h_1^1\omega_0^1)$ can be taken for the single initial form. Let us introduce the density $\alpha = f(x, w_0^1, w_0^2, w_0^3, w_1^1)\,\mathrm{d}x$. A tedious but simple calculation yields the $\mathcal{PC}$ form

$$\breve{\alpha} = f\,\mathrm{d}x + f_1^1\omega_0^1 + A(\omega_0^2 - g_1^1\omega_0^1) - B(\omega_0^3 - h_1^1\omega_0^1)$$

with coefficients $A = \frac{1}{\triangle}(DP + Cb)$, $B = \frac{1}{\triangle}(PQ + Ca)$ where

$$\begin{aligned} P &= Xb + bg_0^2 - ah_0^2, \\ Q &= Xa + ah_0^3 - bg_0^3, \\ D &= f_0^1 - Xf_1^1 + f_0^2 g_1^1 + f_0^3 h_1^1, \\ C &= \Big(f_0^2 - \Big(g_0^2 + X\frac{D}{\triangle}\Big)P + h_0^2 Q + XP\Big)a \\ &\quad + \Big(f_0^3 - g_0^3 P + \Big(h_0^3 + X\frac{D}{\triangle}\Big)b + XQ\Big)b. \end{aligned}$$

Since the formulae are rather clumsy, we state only the top summand of the $\mathcal{EL}$ condition $e^1 = e = -X(C/\triangle) + \ldots$ which indicates the fourth order in the *non-degenerate case*. Then the functions x, w_0^1, w_0^2, w_0^3, w_1^1, w_2^1, w_3^1 can be taken for coordinates on the $\mathcal{EL}$ subspace $\mathbf{E} \subset \mathbf{M}$ and, using x, w_0^1, w_0^2, w_0^3 for coordinates on the Mayer field, the Weierstrass function in a little symbolical transcription

$$\begin{aligned} \mathcal{E} &= F(w_1^1) - F(r_1^1) - F_1^1(r_1^1)(w_1^1 - r_1^1), \\ F &= f + Ag - Bh \end{aligned}$$

(the variables x, w_0^1, w_0^2, w_0^3 in the function F and the parameters r_1^1, r_2^1, r_3^1 in coefficients A, B are omitted) follows quite easily. The Legendre condition briefly expressed by $F_1^1 \neq 0$ is self-evident, and the fixed end point conditions can be weakened to the equality of coordinates x, w_0^1, w_0^2, w_0^3.

9. A second order problem. Let us mention a more instructive example as the determination of initial forms is concerned. We shall deal with the subdiffiety $\Omega = \mathbf{m}^*\Omega(2)$ on the subspace $\mathbf{m}\colon \mathbf{M} \subset \mathbf{M}(2)$ defined by

$$X^2(w_2^2 - g(w_2^1)) = w_{r+2}^2 - w_{r+2}^1 g_2^1 + \ldots \equiv 0 \quad (r = 0, 1, \ldots)$$

where x, w^1_r, w^2_0, w^2_1 $(r = 0, 1, \ldots)$ serve for coordinates on $\mathbf{M}$, and the forms $\omega^1_r, \omega^2_0, \omega^2_1$ $(r = 0, 1, \ldots)$ provide a basis of Ω. Abbreviating $G = g^1_2$, $G^k \equiv X^k G$, let us look for the initial forms. Clearly

$$\mathfrak{L}_X \omega^1_r \equiv \omega^1_{r+1},$$
$$\mathfrak{L}_X \omega^2_0 = \omega^2_1,$$

but

$$\mathfrak{L}_X \omega^2_1 = \omega^2_2 = G\omega^1_2.$$

So we introduce $\omega = \omega^2_1 - G\omega^1_1$ satisfying $\mathfrak{L}_X \omega = -G^1 \omega^1_1$, and moreover $\tilde{\omega} = \omega^2_0 - G\omega^1_0$, $\varpi = \omega + G^1 \omega^1_0$ such that

$$\mathfrak{L}_X \tilde{\omega} = \varpi - G^1 \omega^1_0, \qquad \mathfrak{L}_X \varpi = G^2 \omega^1_0. \tag{26}$$

Finally, denoting $\pi = G^2 \tilde{\omega} + 2G^1 \varpi$ we have the initial form since

$$\mathfrak{L}_X \pi = a\pi + b\varpi \quad (a = G^3/G^2,\ b = 3G^2 - 2G^1 G^3/G^2). \tag{27}$$

As usual, we tacitly suppose $\mathcal{R}(\Omega) = 0$ which means $G^2 \neq 0$, $b \neq 0$.

Let us consider the density $\alpha = f(x, w^1_0, w^2_0)\,\mathrm{d}x$ and search for the $\mathcal{PC}$ form $\breve{\alpha} = \alpha + \breve{\omega}$. Clearly

$$\mathrm{d}\alpha = (f^1_0 \omega^1_0 + f^2_0 \omega^2_0) \wedge \mathrm{d}x = \left(cG^2 \omega^1_0 + \frac{f^2_0}{G^2}(\pi - 2G^1 \varpi)\right) \wedge \mathrm{d}x$$

where $c = (f^1_0 + f^2_0 G)/G^2$. Using (26_2), one can obtain

$$\mathrm{d}(\alpha + c\varpi) \cong \left(\frac{f^2_0}{G^2}\pi - C\varpi\right) \wedge \mathrm{d}x \ (\mathrm{mod}\ \Omega \wedge \Omega), \quad C = 2\frac{f^2_0}{G^2} G^1 + Xc.$$

To delete the right hand term ϖ, we substitute $\varpi = \frac{1}{b}(a\pi + b\varpi) - \frac{a}{b}\pi$ and (27) gives the final result

$$\breve{\alpha} = f\,\mathrm{d}x + c\varpi - \frac{C}{b}\pi,\ \mathrm{d}\breve{\alpha} \cong e\pi \wedge \mathrm{d}x\ (\mathrm{mod}\ \Omega \wedge \Omega),$$
$$e = \frac{f^2_0}{G^2} + a\frac{C}{b} + X\frac{C}{b} = X\frac{C}{b} + \ldots = w^1_6 \frac{g^1_2 g^{11}_{22} f^2_0}{(bG^2)^2}\left(b - 2G^3 \frac{G^1}{G^2}\right) + \ldots$$

where the $\mathcal{EL}$ coefficient e permits to specify the nondegenerate case $g^{11}_{22} f^2_0 \neq 0$ when the functions $x, w^1_0, \ldots, w^1_5, w^2_0, w^2_1$ can be used for coordinates on the $\mathcal{EL}$ subspace

$\mathbf{E} \subset \mathbf{M}$ (this condition ensures the above requirements $G^2 \neq 0$, $b \neq 0$, as well). We will not investigate the Lagrange subspaces $\mathbf{L} \subset \mathbf{E}$ in more detail but if a critical curve can be embedded into a Mayer field with coordinates $x, w_0^1, w_0^2, w_1^1, w_1^2$, the *Weierstrass functions* and the *Legendre condition* can be obtained. In more detail

$$\breve{\alpha} \cong f\,\mathrm{d}x + c(\omega_1^2 - G\omega_1^1) - \frac{C}{b}(\omega_1^2 - G\omega_1^1) \ (\mathrm{mod}\ \omega_0^1, \omega_0^2),$$

hence

$$\mathcal{E} = (c - C/b)(g(w_2^1) - g(r_2^1) - g_2^1(r_2^1)(w_2^1 - r_2^1))$$

where the (obvious) variables of the coefficient $c - C/b$ are not explicitly written down. The function f is latently present in this coefficient. Clearly $(c - C/b)g_{22}^{11} \neq 0$ provides the Legendre condition.

The inverse problem

10. The direct method. For the sake of brevity, we will discuss only the opening problem of Section 6. Recall the (well-known) setting of the inverse problem: we suppose the $\mathcal{EL}$ subspace $\mathbf{e}\colon \mathbf{E} \subset \mathbf{M}$ hence the $\mathcal{EL}$ diffiety (the restriction of $\Omega(2)$ to $\mathbf{M}$) for known but the density $\alpha = f(x, w_0^1, w_0^2, w_1^1)\,\mathrm{d}x$ should be determined. Note that in general the solution need not exist.

It follows at once that the unknown function $f = f(x, w_0^1, w_0^2, w_1^1)$ might be calculated by using the $\mathcal{EL}$ condition

$$f_0^2 - g_0^2 - X\frac{b}{a} = 0, \tag{28}$$

where

$$a = g_0^1 + g_0^2 g_1^1 - X g_1^1,$$
$$b = f_0^1 + f_0^2 g_1^1 - X f_1^1$$

and where the function $g = g(x, w_0^1, w_0^2, w_1^1)$ is given and

$$X = \frac{\partial}{\partial x} + w_1^1\frac{\partial}{\partial w_0^1} + g\frac{\partial}{\partial w_0^2} + w_2^1\frac{\partial}{\partial w_1^1} + h\frac{\partial}{\partial w_2^1} \in \mathcal{T}(\mathbf{E}) \tag{29}$$

(restriction of (20)) with the familiar coefficient $h = \mathbf{e}^* w_3^1$. The condition depends on the variable w_2^1 but the sought solution f must be independent of it. So the derivative of (28) with respect to the variable w_2^1 gives rise to other requirements which altogether need not have a solution. We will not state more details, nor analyse some particular examples, since this direct approach seems to be not the best possible one. This is caused by the fact that together with a possible solution f also all functions $f + Xg$ $(g = g(x, w_0^1, w_0^2))$ are solutions, see 5 (iv). To delete such parasite solutions, it is better to seek the differential $\mathrm{d}\breve{\alpha}$ of the relevant $\mathcal{PC}$ form. Then $\breve{\alpha}$ and $f\,\mathrm{d}x \cong \breve{\alpha}$ (mod Ω) can be easily found.

13. The restriction method. Employing the favourable fact that $\mathrm{d}\breve{\alpha}$ (and even $\breve{\alpha}$) can be expressed by the variables $x, w_0^1, w_0^2, w_1^1, w_2^1$ which serve for coordinates on $\mathbf{E}$, we may deal with the restrictions to $\mathbf{E}$ and identify $\mathrm{d}\breve{\alpha} = \mathbf{e}^*\,\mathrm{d}\breve{\alpha}$. So *we have a space* $\mathbf{E}$ *equipped with the vector field* (29) *and seek the relevant* 2*-form* $\beta = \mathrm{d}\breve{\alpha}$ *on* $\mathbf{E}$. Owing to (23, 24), this form has the properties

I: $\mathrm{d}\beta = 0$,
II: $\beta \cong 0$ (mod $\mathrm{d}x$, $\mathrm{d}w_0^1$, $\mathrm{d}w_0^2$),
III: $X \rfloor \beta = 0$,
IV: $\ell(\mathrm{Adj}\,\beta) = 4$.

Conversely, every such 2*-form* β *on* $\mathbf{E}$ *can be represented as* $\beta = \mathrm{d}\breve{\alpha}$ (more precisely $\beta = \mathbf{e}^*\,\mathrm{d}\breve{\alpha}$) *where* $\breve{\alpha} \in \Phi(\mathbf{M})$ *is a* $\mathcal{PC}$ *form of the kind* (22_1) *such that its* $\mathcal{EL}$ *subspace is identical with the prescribed* $\mathbf{E}$. (Proof: Points I and II ensure the existence of a form $\gamma = A\,\mathrm{d}x + B\,\mathrm{d}w_0^1 + C\,\mathrm{d}w_0^2 + \mathrm{d}w$ satisfying $\mathrm{d}\gamma = \beta$. Let us denote

$$\breve{\alpha} = A\,\mathrm{d}x + B\,\mathrm{d}w_0^1 + C\,\mathrm{d}w_0^2 = f\,\mathrm{d}x + D\omega_0^1 + C\pi \in \Phi(\mathbf{N})$$

where $D = B + Cg_1^1$, $f = A + Dw_1^1 + C(w_1^2 - g_1^1 w_1^1)$, clearly $\mathrm{d}\breve{\alpha} = \beta$ as before. Then, using developments like

$$\mathrm{d}f = Xf\,\mathrm{d}x + (f_0^1 + f_0^2 g_1^1)\omega_0^1 + f_0^2\pi + f_1^1\omega_1^1 + f_2^1\omega_2^1,$$

requirement III (more precisely rewritten as $X \rfloor \mathrm{d}\mathbf{e}^*\breve{\alpha} = 0$) reads

$$X \rfloor (\{\,\mathrm{d}f - XD\omega_0^1 - XC\pi - D\omega_1^1 - C(g_0^2\pi + a\omega_0^1)\} \wedge \mathrm{d}x) = 0$$

whence $\{\ldots\} \cong 0$ (mod $\mathrm{d}x$), that is,

$$f_0^1 + f_0^2 g_1^1 - XD - Ca = f_0^2 - XC - Cg_0^2 = f_1^1 - D = f_2^1 = 0.$$

So we have the form (22_1) and moreover the $\mathcal{EL}$ subspace to this form contains $\mathbf{E}$ (cf. (22_2)). But IV ensures the equality of dimensions and we are done.)

In reality our form $\beta = \mathrm{d}\breve{\alpha}$ is expressible by only four adjoint variables and the independent variable x is not involved (since $\mathrm{d}x \notin \mathrm{Adj}\,\mathrm{d}\breve{\alpha} = \{\pi, \xi, \omega_1^1, \omega_0^1\}$). It is therefore sufficient to determin the restriction $\overline{\beta}$ of β to a fixed hyperplane $x = \mathrm{const}$.

Clearly I′: $d\overline{\beta} = 0$, IV′: $\ell(\mathrm{Adj}\,\overline{\beta}) = 4$ are requirements identical to the above I, IV. Moreover, III ensuring the adjoint variables becomes trivial. However, the restriction II′: $\overline{\beta} \cong 0 \pmod{\mathrm{d}, wz^1, \mathrm{d}w_0^2}$ is much weaker then the original II and must be a little adapted.

To this aim, denoting $\eta = \mathrm{d}x \wedge \mathrm{d}w_0^1 \wedge \mathrm{d}w_0^2$, then II can be expressed by $\eta \wedge \beta = 0$. Using III and (1_1), clearly $\mathfrak{L}_X(\eta\wedge\beta) = \mathfrak{L}_X\eta\wedge\beta = 0$ and even $\mathfrak{L}_X^k(\eta\wedge\beta) = \mathfrak{L}_X^k\eta\wedge\beta \equiv 0$ for any k. For $K \geqslant 1$ large enough, there does exist a dependence $\mathfrak{L}_X^K\eta = \sum f_k\mathfrak{L}_X^k\eta$ (sum over $k = 0, \ldots, K-1$), hence

$$\mathfrak{L}_X^K(\eta \wedge \beta) = \mathfrak{L}_X^K\eta \wedge \beta = \sum f_k\mathfrak{L}_X^k\eta \wedge \beta = \sum f_k\mathfrak{L}_X^k(\eta \wedge \beta).$$

This may be regarded as a linear K-th order differential equation for (the coefficients of) the form $\eta \wedge \beta$. (Hint: use other coordinates such that $X = \frac{\partial}{\partial t}$.) Consequently, if the Cauchy data at a fixed hyperplane $x =$ const. vanish, then the solution $\eta \wedge \beta$ vanishes in the total space. In more explicit terms, consider the requirements

$$\mathfrak{L}_X^k\eta \wedge \overline{\beta} = \mathrm{d}x \wedge \mathfrak{L}_X^k(\mathrm{d}w_0^1 \wedge \mathrm{d}w_0^2) \wedge \overline{\beta} \equiv 0 \quad (k = 0, \ldots, K-1) \tag{30}$$

at the points of a fixed hyperplane $x =$ const. If they are satisfied, then $\overline{\beta}$ extended over the total space (by means of adjoint variables) yields a form β satisfying II.

Altogether taken, *closed* 2-*forms* $\overline{\beta}$ (cf. I′) *expressible by no less than four variables* $w_0^1, w_0^2, w_1^1, w_2^1$ (cf. IV′) *and satisfying* (30) *resolve the inverse problem.*

*__14. A particular example.__ In practice, the $\mathcal{EL}$ subspace is explicitly defined by certain equations $w_1^2 = g(x, w_0^1, w_0^2, w_1^1)$ and $w_3^1 = h(x, w_0^1, w_0^2, w_1^1, w_2^1)$, the unknown form $\overline{\beta}$ is represented as

$$\overline{\beta} = A\,\mathrm{d}w_0^1 \wedge \mathrm{d}w_0^2 + \sum A_{ij}\,\mathrm{d}w_i^1 \wedge \mathrm{d}w_0^j \quad (i, j = 1, 2;\ \det(A_{ij}) \neq 0)$$

and then the requirements (30) can be easily written down. Let us mention the instructive case of equations $w_1^2 = w_0^1, w_3^1 = h(x, w_0^2)$. Then $K = 5$ and the forms $\mathfrak{L}_X^k\eta$ $(k = 1, \ldots, 4)$ are

$$\mathrm{d}w_1^1 \wedge \mathrm{d}w_0^2,\ \mathrm{d}w_2^1 \wedge \mathrm{d}w_0^2 + \mathrm{d}w_1^1 \wedge \mathrm{d}w_0^1,\ \mathrm{d}w_2^1 \wedge \mathrm{d}w_0^1,\ \mathrm{d}w_2^1 \wedge \mathrm{d}w_1^1$$

(we have omitted the left factor $\mathrm{d}x$ and some useless summands). It follows $A_{21} = A_{11} + A_{22} = A_{12} = A = 0$, hence

$$\overline{\beta} = A_{11}(\mathrm{d}w_1^1 \wedge \mathrm{d}w_0^1 - \mathrm{d}w_2^1 \wedge \mathrm{d}w_0^2)$$

where $A_{11} \neq 0$ is a constant. The problem is solvable and the form $\overline{\beta}$ does not depend on the choice of the above function h. This is a very exceptional example, of course.$_*$

*15. **The use of first integrals.** Recall from 5 (ii) that $\mathcal{R}(\mathbf{e}^*\Omega) = \mathbf{e}^*\Omega$, hence $\mathbf{e}^*\Omega = \{\,\mathrm{d}h^1,\, \mathrm{d}h^2,\, \mathrm{d}h^3,\, \mathrm{d}h^4\}$ for appropriate functions $h^i \equiv h^i(x, w_0^1, w_0^2, w_1^1, w_2^1)$. These are adjoint functions to $\mathrm{d}\breve{\alpha}$ on the $\mathcal{EL}$ subspace (since $\mathbf{e}^*\,\mathrm{Adj}\,\mathrm{d}\breve{\alpha} \subset \mathbf{e}^*\Omega$), hence $\mathrm{d}\breve{\alpha}$ can be expressed in terms of them:

$$\mathrm{d}\breve{\alpha} = \sum H^{ij}\,\mathrm{d}h^i \wedge \mathrm{d}h^j \quad \big(i, j = 1, \ldots, 4;\ H^{ij} \equiv H^{ij}(h^1, \ldots, h^4)\big).$$

Passing to the inverse problem, functions h^i may be regarded as known and we have to determine a form $\beta = \mathrm{d}\breve{\alpha}$ satisfying conditions I–IV of Section 13.

In more detail, I is satisfied if and only if

$$H^{ij} \equiv \frac{\partial H^j}{\partial h^i} - \frac{\partial H^i}{\partial h^j} \quad \big(H^i \equiv H^i(h^1, \ldots, h^4)\big) \tag{31}$$

for appropriate functions $H^1, \ldots, H^4$. Condition II can be equivalently expressed as $\beta \cong 0$ (mod $\mathrm{d}x, \omega_0^1, \omega_0^2$) and the congruences

$$\mathrm{d}h^i \cong \frac{\partial h^i}{\partial w_1^1}\omega_1^1 + \frac{\partial h^i}{\partial w_2^1}\omega_2^1 \quad (\mathrm{mod}\ \omega_0^1, \omega_0^2)$$

(use $Xh^i \equiv 0$) give the requirement

$$\sum c^{ij} H^{ij} = 0 \quad \left(c^{ij} \equiv \frac{\partial h^i}{\partial w_1^1}\frac{\partial h^j}{\partial w_2^1} - \frac{\partial h^i}{\partial w_2^1}\frac{\partial h^j}{\partial w_1^1}\right).$$

However, coefficients c^{ij} depend on x (unlike all H^{ij}), hence all requirements

$$\sum \frac{\partial c^{ij}}{\partial x^k} H^{ij} \equiv 0 \qquad (k = 0, 1, \ldots) \tag{32}$$

should be satisfied. In reality only a finite number of them at a fixed value $x =$ const. is sufficient. Condition III is satisfied since $Xh^i \equiv 0$, condition IV can be expressed by $\det(H^{ij}) \neq 0$.

Altogether taken, *the form* $\beta = \sum H^{ij}\,\mathrm{d}h^i \wedge \mathrm{d}h^j$ *with coefficients* (31) *satisfying* (32) *and det* $(H^{ij}) \neq 0$ *resolves the inverse problem.*$_*$

16. A general principle. We shall recall the familiar method which reduces the isoperimetrical problems to the common theory of constrained $\mathcal{C}$-curves. Let a density $\alpha \in \Phi(\mathbf{M})$ and a diffiety $\Omega \subset \Phi(\mathbf{M})$ be given as in Section 4. Let moreover $\beta^1, \ldots, \beta^c \in \Phi(\mathbf{M})$ be given. We are interested in such $\mathcal{A}$-curves $P(t) \in \mathbf{M}$ $(0 \leqslant t \leqslant 1)$ that the integrals

$$\int_P \beta^i = \int_0^1 P^* \beta^i = c^i \quad (i = 1, \ldots, c) \tag{33}$$

are prescribed constants. Then (roughly speaking) we may adjoin new variables $u^1, \ldots, u^c$ to the coordinates of $\mathbf{M}$ to obtain a certain space $\mathbf{N}$ with natural fibration denoted by $\mathbf{n}\colon \mathbf{N} \to \mathbf{M}$. If the space $\mathbf{N}$ is equipped with the density $\mathbf{n}^*\alpha$ and diffiety

$$\Theta = \{\mathbf{n}^*\omega,\ \mathrm{d}u^i - \mathbf{n}^*\beta^i \colon\ \omega \in \Omega;\ i = 1, \ldots, c\} \subset \Phi(\mathbf{N}), \tag{34}$$

then an investigation of the primary $\mathcal{C}$-curves (to α and Ω) satisfying moreover the isoperimetrical constraints (33) can be found equivalent to the study of new $\mathcal{C}$-curves in the space $\mathbf{N}$ for the density $\mathbf{n}^*\alpha$ and with the constraint Θ.

17. Two model examples. In the space $\mathbf{M}(1)$ equipped with diffiety $\Omega(1)$ and density $\alpha = f(x, w_0^1, w_1^1)\,\mathrm{d}x$, let us consider the isoperimetrical requirement (33) with a single form $\beta^1 = g(x, w_0^1, w_1^1)\,\mathrm{d}x$. Introducing an additional variable u^1, we obtain the extended space $\mathbf{N}$ with coordinates x, w_r^1, u^1 $(r = 0, 1, \ldots)$ and diffiety $\Theta \subset \Phi(\mathbf{N})$ generated by contact forms $\omega_r^1 \equiv \mathrm{d}w_r^1 - w_{r+1}^1\,\mathrm{d}x$ $(r = 0, 1, \ldots)$ together with the form $\vartheta^1 = \mathrm{d}u^1 - g\,\mathrm{d}x$ (therefore $X = \frac{\partial}{\partial x} + g\frac{\partial}{\partial u} + \sum^{\infty} w_{r+1}^i \frac{\partial}{\partial w_r^i} \in \mathcal{H}(\Theta)$). One can calculate the initial form $\pi = \vartheta^1 - g_1^1\omega_0^1$, the $\mathcal{PC}$ form and the $\mathcal{EL}$ equation

$$\breve{\alpha} = f\,\mathrm{d}x + f_1^1\omega_0^1 + \frac{F}{G}\pi, \quad X\frac{F}{G} = 0$$

where

$$F = f_0^1 - Xf_1^1, \quad G = g_0^1 - Xg_1^1$$

provided $G \neq 0$ (which ensures $\mathcal{R}(\Theta) = 0$). It follows that the function F is a constant multiple of G on every $\mathcal{C}$-curve. Assuming the $\mathcal{EL}$ equation exactly of the third order (the nondegenerate problem), the functions $x, w_0^1, w_1^1, w_2^1, u^1$ can be taken for coordinates on $\mathbf{E}$, and x, w_0^1, u^1 for coordinates in the Mayer field. The Weierstrass function

$$\mathcal{E} = \widetilde{f}(w_1^1) - \widetilde{f}(r_1^1) - \widetilde{f}_1^1(r_1^1)(w_1^1 - r_1^1), \quad \widetilde{f} = f + \frac{F}{G}g$$

and the Legendre condition $f^{11}_{11} + \frac{F}{G} g^{11}_{11} \neq 0$ are obvious.

Let us adjoin still one isoperimetrical requirement $\beta^2 = h(x, w^1_0, w^1_1)\,\mathrm{d}x$. We obtain a broader space $\mathbf{N}$ with coordinates x, w^1_r, u^1, u^2 and diffiety Θ generated by contact forms together with $\vartheta^1 = \mathrm{d}u^1 - g\,\mathrm{d}x$, $\vartheta^2 = \mathrm{d}u^2 - h\,\mathrm{d}x$ (therefore $X = \frac{\partial}{\partial x} + g\frac{\partial}{\partial u^1} + h\frac{\partial}{\partial u^2} + \sum^{\infty} w^i_{r+1}\frac{\partial}{\partial w^i_r} \in \mathcal{H}(\Theta)$). Assuming $G \neq 0$, we may introduce the new initial form

$$\pi = \frac{H}{G}(\vartheta^1 - g^1_1\omega^1_0) - \vartheta^2 + h^1_1\omega^1_0, \quad H = h^1_0 - Xh^1_1$$

to obtain the $\mathcal{PC}$ form and the $\mathcal{EL}$ condition

$$\breve{\alpha} = f\,\mathrm{d}x + f^1_1\omega^1_0 + \frac{F}{G}(\vartheta^1 - g^1_1\omega^1_0) + \frac{X(F/G)}{X(H/G)}\pi,$$

$$X\left(\frac{X(F/G)}{X(H/G)}\right) = 0.$$

The latter condition is equivalent to a linear dependence $F = \text{const.}\ G + \text{const.}\ H$ on every $\mathcal{C}$-curve. In the nondegenerate case, the functions $x, w^1_0, \ldots, w^1_3, u^1, u^2$ can be taken for coordinates on $\mathbf{E}$ and the extremality conditions easily follow.

18. Constrained example. Let us mention the density $\alpha = f(x, w^1_0, w^2_0, w^1_1)\,\mathrm{d}x$ and the diffiety $\Omega \subset \Phi(\mathbf{N})$ from Section 6 completed moreover by a single isoperimetric requirement with the form $(\beta^1 =)\ \beta = h(x, w^1_0, w^2_0)\,\mathrm{d}x$. Introducing one additional variable $(u^1 =) u$ we obtain the extended space $\mathbf{N}$ with the coordinates x, w^1_r, w^2_0, u $(r = 0, 1, \ldots)$ equipped with the diffiety $\Theta \subset \Phi(\mathbf{N})$ generated by the forms ω^1_r, ω^2_0, $\mathrm{d}u - \beta$ $(r = 0, 1, \ldots)$ (therefore $X = \frac{\partial}{\partial x} + h\frac{\partial}{\partial u} + g\frac{\partial}{\partial w^2_0} + \sum^{\infty} w^i_{r+1}\frac{\partial}{\partial w^i_r} \in \mathcal{H}(\Theta)$). One can find the new initial form

$$\pi = \frac{H}{a}(\omega^2_0 - g^1_1\omega^1_0) - \mathrm{d}u + \beta, \quad H = h^1_0 + h^2_0 g^1_1$$

which yields the $\mathcal{PC}$ form

$$\breve{\alpha} = f\,\mathrm{d}x + f^1_1\omega^1_0 + \frac{b}{a}(\omega^2_0 - g^1_1\omega^1_0) + \frac{e}{E}\pi,$$

where

$$E = X\frac{H}{a} + g^2_0\frac{H}{a} - h^2_0$$

(e is given in (22_2), $E \neq 0$ ensures $\mathcal{R}(\Theta) = 0$) and the $\mathcal{EL}$ condition $X(e/E) = 0$ which means that e is a constant multiple of E on every $\mathcal{C}$-curve).

We will not deal with the extremality conditions; it seems that even the constrained isoperimetrical problems do not bring much novelty.

C o m m e n t s . The actual state of the classical calculus of variations involving some recent applications in symplectical and contact geometry is thoroughly explained in the voluminous monograph [4]. It is instructive to look over the contents: the Lagrange and isoperimetrical problems are investigated with less details than in the familiar textbook [1]. Analogous selection of topics, however, occurs also in [2] with its "royal road" to the problems, and in the modern textbook [5] based on a systematic use of differential forms. We believe that all reasons for such a crude discrimination of the Lagrange problem are unjustifiable at the present time and appropriately adapted Poincaré-Cartan forms provide the best tool: elimination of uncertain coefficients proposed in [3] permits to develop the theory of nondegenerate Lagrange problems exactly in the same manner as in the classical theory without constraints. Moreover, also the inverse problems can be analyzed by using this tool. The remaining "degenerate" problems prove to be more difficult. We refer to the next Part of this article. (Added in proof. General foundations of our methods are available in the recent author's monograph: *The formal theory of differential equations*, Folia Mathematica Facultatis Sci. Mat. Univ., Mathematica 6, Masaryk University, Brno, 1998.)

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