



ACADEMY of SCIENCES of the CZECH REPUBLIC

INSTITUTE of MATHEMATICS

**The commuting graph of bounded linear  
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*Calin Grigore Ambrozie*

*Janko Bračič*

*Bojan Kuzma*

*Vladimír Müller*

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# THE COMMUTING GRAPH OF BOUNDED LINEAR OPERATORS ON A HILBERT SPACE

C. AMBROZIE<sup>†</sup>, J. BRAČIČ<sup>‡</sup>, B. KUZMA<sup>‡</sup>, AND V. MÜLLER<sup>†</sup>

ABSTRACT. An operator  $T$  on the separable infinite-dimensional Hilbert space is constructed such that the commutant of every operator which is not a scalar multiple of the identity operator and commutes with  $T$  coincides with the commutant of  $T$ . On the other hand, it is shown that for several classes of operators it is possible to construct a finite sequence of operators, starting at a given operator from the class and ending in a rank-one projection such that each operator in the sequence commutes with its predecessor. The classes which we study are: finite-rank operators, normal operators, partial isometries, and  $\mathcal{C}_0$  contractions. It is also shown that for any given set of *yes/no* conditions between points in some finite set, there always exist operators on a finite-dimensional Hilbert space such that their commutativity **relations** exactly satisfy those conditions.

## 1. INTRODUCTION

Commutativity is certainly one of the most important relations in  $\mathcal{B}(\mathcal{H})$ , the algebra of all bounded linear operators on a complex Hilbert spaces  $\mathcal{H}$ . Recently a new approach in the study of it has emerged in terms of the *commuting graph*. The roots of this approach can be traced back at least as far as to the work of Brauer and Fowler [7] on the distance between two involutions in a finite group, where the distance means the smallest positive integer  $d$  such that a chain of  $d$  successively commuting elements exist starting with the first involution and ending with the second one.

Formally, the commuting graph,  $\Gamma(\mathcal{B}(\mathcal{H}))$ , of  $\mathcal{B}(\mathcal{H})$  is a simple (i.e., undirected and loopless) graph **whose** vertex set consists of all non-scalar operators, that is, operators in  $\mathcal{B}(\mathcal{H})$  which are not scalar multiples of the identity operator, and where two distinct vertices  $X, Y$  form an edge  $X \leftrightarrow Y$  if and only if  $X \in \{Y\}'$ , where  $\{Y\}' = \{Z \in \mathcal{B}(\mathcal{H}); ZY = YZ\}$  is the commutant of  $Y$ .

The basic problems with commuting graph is to find paths and distances between vertices. Recall that a path of length  $k$  which connects  $A, B \in \Gamma(\mathcal{B}(\mathcal{H}))$  is any sequence of  $k+1$  pairwise distinct non-scalar operators  $A = X_0, X_1, \dots, X_k = B$  in  $\mathcal{B}(\mathcal{H})$  such that  $X_i$  commutes with  $X_{i+1}$ , for every  $i = 0, \dots, k-1$ . We denote such a sequence by  $A \leftrightarrow X_1 \leftrightarrow \dots \leftrightarrow X_{k-1} \leftrightarrow B$ . The distance between  $A$  and  $B$ , denoted by  $d(A, B)$ , is the minimal length of paths which connect  $A$  and  $B$ . If there is no path connecting  $A$  and  $B$ , then we set  $d(A, B) = \infty$ . Observe

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that the distance in a commuting graph is a metric. A commuting graph  $\Gamma(\mathcal{B}(\mathcal{H}))$  is connected if every two non-scalar operators can be connected by a finite path. The supremum of all distances between any two non-scalar operators is called the diameter of  $\Gamma(\mathcal{B}(\mathcal{H}))$ . Recently it has been proved by Akbari, Mohammadian, Radjavi, and Raja [1] that  $\Gamma(\mathcal{B}(\mathcal{H}))$  is connected with diameter 4 if  $2 < \dim \mathcal{H} < \infty$ . On the other hand, if  $\dim \mathcal{H} = 2$ , then  $\Gamma(\mathcal{B}(\mathcal{H}))$  is not connected.

The aim of our paper is to study the connectedness of  $\Gamma(\mathcal{B}(\mathcal{H}))$  in the case when  $\mathcal{H}$  is an infinite-dimensional complex Hilbert space. We will show that  $\Gamma(\mathcal{B}(\mathcal{H}))$  is not connected for the separable infinite dimensional Hilbert space  $\mathcal{H}$  and it is connected with the diameter two for a non-separable  $\mathcal{H}$ . We will also study distances between various types of operators on the separable Hilbert space:  $\mathcal{C}_0$  contractions, normal operators, partial isometries, and unilateral shifts. Note that the standard forward shift is known to have the smallest possible commutant - the strongly closed unital algebra which is generated by the operator (see Halmos' book [10]). Nonetheless, there exists a path of finite length in the commuting graph which connects the forward shift with a rank-one operator.

In the end of the paper we show that any finite simple graph can be realized as a subgraph of  $\mathcal{B}(\mathcal{H})$ , with  $\dim \mathcal{H} < \infty$  sufficiently large, in a sense that there is an edge between two vertices if and only if the corresponding two operators, which represent the given vertices, commute.

## 2. DIAMETERS AND DISTANCES

The answer to the question about the diameter of  $\Gamma(\mathcal{B}(\mathcal{H}))$  is almost trivial if  $\mathcal{H}$  is not separable.

**Theorem 2.1.** *Let  $\mathcal{H}$  be a non-separable Hilbert space. If  $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$  is a finite or countable set of operators, then there exists an orthogonal projection  $P \neq 0, I$  in  $\mathcal{A}'$ , the commutant of  $\mathcal{A}$ .*

*Proof.* Let  $\mathcal{A} = \{A_n; n = 1, \dots, N\}$  where  $N \in \mathbb{N} \cup \{\infty\}$ . The multiplicative semigroup  $\mathcal{W}(\mathcal{A})$  generated by operators  $I, A_n, A_n^*$  ( $n = 1, \dots, N$ ) is finite or countable. Thus, for a non-zero vector  $x \in \mathcal{H}$ , the closed subspace  $\mathcal{M} = \bigvee \{Wx; W \in \mathcal{W}(\mathcal{A})\}$  is non-trivial and separable. Clearly, it reduces every operator in  $\mathcal{A}$ . Thus, the orthogonal projection  $P$  which maps onto  $\mathcal{M}$  is non-scalar and commutes with every operator in  $\mathcal{A}$ .  $\square$

**Corollary 2.2.** *If  $\mathcal{H}$  is not separable, then  $\Gamma(\mathcal{B}(\mathcal{H}))$  is connected with the diameter two.*

*Proof.* If  $A, B$  are arbitrary non-scalar operators on  $\mathcal{H}$ , then Theorem 2.1 applied on the set  $\mathcal{A} = \{A, B\}$  gives  $d(A, B) \leq 2$ . For linearly independent vectors  $x, y \in \mathcal{H}$ , let  $A_0 = x \otimes x$  and  $B_0 = (x + y) \otimes x$ , where  $u \otimes v$  denotes the operator  $w \mapsto \langle w, v \rangle u$  ( $w \in \mathcal{H}$ ). It is easily seen that  $A_0$  and  $B_0$  do not commute, i.e.,  $d(A_0, B_0) = 2$ .  $\square$

From now on we assume without further notice that  $\mathcal{H}$  is the separable infinite dimensional **complex** Hilbert space. In this case, the commuting graph is not connected. This easily follows from our main theorem, which we state now.

**Theorem 2.3.** *There exists a bounded linear operator  $T \in \mathcal{B}(\mathcal{H})$  such that  $\{A\}' = \{T\}'$  for any non-scalar operator  $A \in \{T\}'$ .*

The proof of Theorem 2.3 will be given in Section 3. In this section we prove that several **classes of** operators in  $\mathcal{B}(\mathcal{H})$  belong to the same connected component in  $\Gamma(\mathcal{B}(\mathcal{H}))$ . We refer to standard textbooks of operator theory for the definitions and basic properties of these classes of operators. In particular, see Sz.-Nagy, Foiaş [12] for the definition of  $\mathcal{C}_0$  contractions and the properties of Nagy-Foiaş functional calculus.

**Theorem 2.4.** *Finite rank operators, operators whose spectrum is disconnected, non-scalar operators which are similar (i) to normal operators or (ii) to  $\mathcal{C}_0$  contractions or (iii) to weighed shifts or (iv) to partial isometries, are in the same connected component of  $\Gamma(\mathcal{B}(\mathcal{H}))$ .*

For the proof we need the following lemmas.

**Lemma 2.5.** *If  $A, B \in \mathcal{B}(\mathcal{H})$  are non-scalar operators of finite rank, then  $d(A, B) \leq 2$ .*

*Proof.* If  $A, B \in \mathcal{B}(\mathcal{H})$  are of finite rank, then  $A^*$  and  $B^*$  are of finite rank, as well. Thus,  $\mathcal{M} = \text{Im } A + \text{Im } A^* + \text{Im } B + \text{Im } B^*$  is a finite dimensional subspace of  $\mathcal{H}$  which reduces  $A$  and  $B$ . If  $P$  is the orthogonal projection onto  $\mathcal{M}$ , then, clearly,  $P \in \{A\}' \cap \{B\}'$ .  $\square$

A weighted shift on  $\ell^2$  is the product of a diagonal operator and a shift. If a weighted shift is bilateral, then it is normal and therefore commutes with its spectral orthogonal projections. It has been proved by Shields and Wallen [16] that a unilateral weighted shift  $W$  with nonzero weights has the smallest possible commutant, that is,  $\{W\}'$  is the strongly closed unital algebra generated by  $W$ . It follows that  $\{W\}'$  is minimal also with respect to set inclusion of commutants: if  $\{X\}' \subseteq \{W\}'$ , for some operator  $X$ , then  $\{X\}' = \{W\}'$ .

**Lemma 2.6.** *Let  $W$  be a weighted forward shift. Then there exists a non-scalar orthogonal projection which commutes with  $W^2$ .*

*Proof.* Assume that  $\{e_n\}_{n=1}^\infty$  is an orthonormal basis of  $\mathcal{H}$  such that  $We_n = w_n e_{n+1}$ , for any  $n$ . Then  $W^2 e_n = w_n w_{n+1} e_{n+2}$  for all  $n$  and therefore  $W^2 = U^*(W_1 \oplus W_2)U$ , where  $U : \ell^2 \rightarrow \ell^2 \oplus \ell^2$  is the unitary operator defined by  $U : e_{2n} \mapsto e_n \oplus 0$  and  $U : e_{2n+1} \mapsto 0 \oplus e_n$ , and  $W_1, W_2$  are weighted shifts given by  $W_1 e_n = w_{2n} w_{2n+1} e_{n+1}$  and  $W_2 e_n = w_{2n+1} w_{2n+2} e_{n+1}$ , respectively. It is obvious that the orthogonal projection  $P = U^*(I \oplus 0)U$  commutes with  $W^2$ .  $\square$

We remark that the Volterra operator  $V$  of integration on  $L^2[0, 1]$  also has the smallest possible commutant, see Read [13] and Sarason [15]. Let  $M \in \mathcal{B}(L^2[0, 1])$  be a **convolution operator**  $(Mf)(x) := (\chi_{[1/2, 1]} * f)(x) = \int_0^1 \chi_{[1/2, 1]}(x) f(x-t) dt$ , where  $\chi_{[1/2, 1]}$  is the characteristic function of  $[1/2, 1]$ . Since  $Vf = 1 * f$ , it easily follows that  $M$  commutes with  $V$ . Also,  $M^2 = 0$ , so there are nonzero vectors  $x \in \ker M$  and  $y \in \ker M^*$  **where  $\ker X$  is the kernel of operator  $X$** . This gives a path connecting  $V$  to a rank-one operator:  $V \leftrightarrow M \leftrightarrow x \otimes y$ .

*Proof of Theorem 2.4.* By Lemma 2.5, two finite rank operators are always connected. Assume next that  $T = SAS^{-1}$  is a completely nonunitary  $\mathcal{C}_0$  contraction and let  $m_T \in H^\infty$  be its

minimal function. By [12, p. 107], every inner function, except scalar multiples of Blaschke factors  $\lambda \mapsto \mu \frac{\alpha - \lambda}{1 - \bar{\alpha}\lambda} \frac{|\alpha|}{\alpha}$  ( $|\alpha| < 1$  and  $|\mu| = 1$ ), decomposes into a product of two nonconstant inner functions. Since a Blaschke factor annihilates only a scalar multiple of  $I$ , function  $m_T$  cannot be a scalar multiple of a Blaschke factor. Thus, there exists a nontrivial decomposition  $m_T(\lambda) = f(\lambda)g(\lambda)$ . By the minimality of  $m_T$ , operators  $f(T)$  and  $g(T)$  are nonzero. However,  $f(T)g(T) = g(T)f(T) = m_T(T) = 0$ . Therefore, there exist nonzero vectors  $x \in \ker g(T)$  and  $y \in \ker g(T)^*$  such that  $A = STS^{-1} \leftrightarrow Sg(T)S^{-1} \leftrightarrow S(x \otimes y)S^{-1}$ , i.e.,  $A$  is connected with a rank-one operator.

Note that algebraic operators are scalar multiples of  $\mathcal{C}_0$  contractions. In particular, if  $A \notin \mathbb{C}I$  is a scalar multiple of a projection or if it is a nilpotent, then  $\frac{1}{2\|A\|}A \in \mathcal{C}_0$  and hence  $A$  is connected to a rank-one operator in  $\Gamma(\mathcal{B}(\mathcal{H}))$ .

If the spectrum of  $A$  is disconnected, then the Riesz functional calculus produces a non-scalar projection  $P$  which commutes with  $A$ . If  $A$  is similar to a normal operator, say  $A = SNS^{-1}$ , then the functional calculus for  $N$  gives a non-scalar orthogonal projection  $P$ . Hence, the projection  $SPS^{-1}$  commutes with  $A$ . Let  $W$  be a weighted shift. If  $A = SW^2S^{-1}$ , then, by Lemma 2.6,  $W^2$  commutes with non-scalar orthogonal projection  $P$ , and we have a path  $A \leftrightarrow SW^2S^{-1} \leftrightarrow SPS^{-1}$ .

Assume that  $A = SUS^{-1}$  for a non-scalar partial isometry  $U$  (i.e.,  $U^*U$  is an orthogonal projection). If  $U$  is invertible, then it is unitary and hence commutes with any of its spectral projections. If both,  $U$  and  $U^*$ , have nontrivial kernels, then we choose unit vectors  $x \in \ker U$  and  $y \in \ker U^*$  to form a rank-one operator  $x \otimes y \in \{U\}'$ . The orthogonal projection  $P$  which maps onto the space generated by  $x$  and  $y$  gives a path  $U \leftrightarrow (x \otimes y) \leftrightarrow P$  of length two. If  $U$  is injective but  $U^*$  is not, then  $U$  is a noninvertible isometry. By [10, Problem 149],  $U$  is either a direct sum of one or more unilateral shifts or a direct sum of a unitary operator and one or more unilateral shifts. If  $U$  decomposes into a direct sum of two or more summands, then we take  $P$  to be the orthogonal projection onto one of the summands and, clearly,  $P$  commutes with  $U$ . If  $U$  is a unilateral shift, then  $U^2$  commutes with a non-scalar orthogonal projection  $P$  (see Lemma 2.6), and we have a path  $U \leftrightarrow U^2 \leftrightarrow P$ . Lastly, if  $U^*$  is injective but  $U$  is not, then we repeat the above arguments with  $U^*$  instead of  $U$ .  $\square$

By Theorem 2.4, projections are in the same connected component of  $\Gamma(\mathcal{B}(\mathcal{H}))$ . Now we will show that the distance between two projections is at most 2. We acknowledge that the ideas of the proof of the following proposition come from a paper by Allan and Zemánek [2] and, in the case of orthogonal projections, from Halmos' paper [11].

**Proposition 2.7.** *If  $P, Q \in \mathcal{B}(\mathcal{H})$  are non-scalar projections, then  $d(P, Q) \leq 2$ . If, additionally,  $P$  and  $Q$  are orthogonal projections, then there exists a non-scalar orthogonal projection  $R$  which commutes with  $P$  and  $Q$ .*

*Proof.* We may assume that  $P$  and  $Q$  do not commute, for otherwise  $d(P, Q) \leq 1$ . It is easy to see that  $(P - Q)^2$  commutes with both  $P$  and  $Q$ . If it is non-scalar, then there is a path  $P \leftrightarrow (P - Q)^2 \leftrightarrow Q$  of length two.

Assume that  $(P - Q)^2 = \alpha I$ . With respect to the (not necessary orthogonal) decomposition  $\mathcal{H} = \text{Im } Q \dot{+} \ker Q$  one has  $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  and  $Q = \begin{pmatrix} I_{\text{Im } Q} & 0 \\ 0 & 0 \end{pmatrix}$ . It is easy to see that  $(P - Q)^2 = \alpha I$  if and only if

$$(2.1) \quad P = \begin{pmatrix} (1-\alpha)I_{\text{Im } Q} & B \\ C & \alpha I_{\ker Q} \end{pmatrix}, \quad BC = \alpha(1-\alpha)I_{\text{Im } Q} \quad \text{and} \quad CB = \alpha(1-\alpha)I_{\ker Q}.$$

Hence, if  $\alpha(1-\alpha) \neq 0$ , then, for any choice of  $Y \in \mathcal{B}(\ker Q)$ , the block diagonal operator  $\begin{pmatrix} \frac{1}{\alpha(1-\alpha)}BYC & 0 \\ 0 & Y \end{pmatrix}$  commutes with  $P$  and  $Q$ . On the other hand, if  $\alpha(1-\alpha) = 0$ , then  $BC = 0 = CB$ . However, as  $P$  and  $Q$  do not commute, at least one among  $B$  and  $C$  has to be nonzero. If  $C \neq 0$ , then let  $x \in \text{Im } Q$  and  $y \in \ker Q$  be such that  $Cx \neq 0$  and  $C^*y \neq 0$ . It is easily seen that  $\begin{pmatrix} x \otimes C^*y & 0 \\ 0 & Cx \otimes y \end{pmatrix}$  is a non-scalar operator which commutes with  $P$  and  $Q$ . Similarly, if  $C = 0$ , and therefore  $B \neq 0$ , there exist vectors  $u \in \ker Q$  and  $v \in \text{Im } Q$  such that  $\begin{pmatrix} Bu \otimes v & 0 \\ 0 & u \otimes B^*v \end{pmatrix}$  is non-scalar and commutes with  $P$  and  $Q$ .

Finally, assume that  $P$  and  $Q$  are orthogonal. Then  $(P - Q)^2$  is self-adjoint. If it is non-scalar, then the Borel functional calculus gives a non-scalar orthogonal projection  $R$  in the second commutant of  $(P - Q)^2$ , which means that  $R$  commutes with  $P$  and  $Q$ . If  $(P - Q)^2$  is scalar, then we have  $C = B^*$  and  $\alpha(1-\alpha) \neq 0$  in (2.1). Now a non-scalar self-adjoint  $Y \in \mathcal{B}(\ker Q)$  produces a non-scalar self adjoint operator  $H = \frac{1}{\alpha(1-\alpha)}BYB^* \oplus Y \in \{P, Q\}'$  and the Borel functional calculus for  $H$  gives the desired non-scalar orthogonal projection.  $\square$

We have already mentioned that any operator whose spectrum is disconnected commutes with a non-scalar projection. Also, every normal operator commutes with a non-scalar orthogonal projection. Thus, Proposition 2.7 has the following simple consequence.

**Corollary 2.8.** *If  $A$  and  $B$  are operators with disconnected spectra or are normal operators, then  $d(A, B) \leq 4$ .*

Actually, the bound in the above corollary is the best possible. Namely, as we show next, one can find two unitary operators (respectively, two compact operators) at distance four in  $\Gamma(\mathcal{B}(\mathcal{H}))$ .

**Example 2.9.** Let  $\{e_n\}_{n=1}^\infty$  be an orthonormal basis for  $\mathcal{H}$ . It is well known that any bounded sequence  $\{\delta_n\}_{n=1}^\infty \subseteq \mathbb{C}$  defines a bounded diagonal operator  $D$  on  $\mathcal{H}$  if we set  $De_n = \delta_n e_n$  and extend this linearly and continuously to the whole space. For our purposes, we assume that  $\delta_n \neq \delta_m$  if  $n \neq m$ . It follows that in this case any operator in the commutant  $\{D\}'$  is diagonal, **and in particular, if  $A \in \mathcal{B}(\mathcal{H})$  is such that** there is a path  $D \leftrightarrow T \leftrightarrow A$  in  $\Gamma(\mathcal{B}(\mathcal{H}))$  then  $T$  is diagonal. Denote  $\tau_n = \langle Te_n, e_n \rangle$  ( $n \in \mathbb{N}$ ) and let  $P = I - \chi_{\{\tau_1\}}(T)$ , i.e.,  $P$  is the orthogonal projection onto  $(\ker(T - \tau_1 I))^\perp$ . Since  $T$  is non-scalar,  $P \neq 0, I$ . By the Fuglede-Putnam Theorem,  $P$  commutes with  $D$  and  $A$ , as well, which means that there is a path  $D \leftrightarrow P \leftrightarrow A$  in  $\Gamma(\mathcal{B}(\mathcal{H}))$ . Note that  $P$  is a spectral projection for  $D$ , as well. Indeed, if  $N = \{n \in \mathbb{N}; \tau_n \neq \tau_1\}$ , then  $P = \chi_{\{\delta_n; n \in N\}}(D)$ .

Let now  $u \in \mathcal{H}$  be an arbitrary vector of norm 1 such that  $\langle u, e_1 \rangle > \frac{\sqrt{2}}{2}$  and  $\langle u, e_n \rangle > 0$  for  $n \geq 2$ , say  $u = \sqrt{3} \sum_{n=1}^\infty \frac{e_n}{2^n}$ . It is obvious that  $U = I - 2u \otimes u$  is an involution, i.e., a selfadjoint

unitary operator. A straightforward computation gives

$$(2.2) \quad \langle Ue_1, e_1 \rangle < 0 \quad \text{and} \quad \langle Ue_n, e_m \rangle < 0 \quad (n \neq m).$$

Operators  $D$  and  $U^*DU$  do not commute as  $U^*DU$  is not diagonal with respect to the orthonormal basis  $\{e_n\}_{n=1}^\infty$ . Let us show that  $D$  and  $U^*DU$  are not at distance 2 in  $\Gamma(\mathcal{B}(\mathcal{H}))$ . If there was a path  $D \leftrightarrow T \leftrightarrow U^*DU$  in  $\Gamma(\mathcal{B}(\mathcal{H}))$ , then, as we have seen above, there would be also a path  $D \leftrightarrow P \leftrightarrow U^*DU$ , where  $P = \chi_{\{\delta_n; n \in \mathbb{N}\}}(D) = \chi_{\{\tau_1\}}(T)$  (see the notation above). It follows that  $UPU^*$  commutes with  $D$  and therefore, by the Fuglede-Putnam Theorem,  $P$  and  $UPU^*$  commute, as well. However, it follows from  $PUPU^* = UPU^*P$  that  $(2\langle u, Pu \rangle - 1)Pu \otimes u = (2\langle u, Pu \rangle - 1)u \otimes Pu$  and consequently  $2\langle u, Pu \rangle = 1$  as  $Pu \otimes u$  and  $u \otimes Pu$  are linearly independent operators. But,  $\langle u, Pu \rangle = \langle \sum_{m=1}^\infty \langle u, e_m \rangle e_m, \sum_{n \in \mathbb{N}} \langle u, e_n \rangle e_n \rangle = \sum_{n \in \mathbb{N}} \langle u, e_n \rangle^2 < \frac{1}{2}$  as  $1 \notin N$  and we have assumed that  $\langle u, e_1 \rangle > \frac{\sqrt{2}}{2}$  and  $u$  has norm 1.

Now we will show that  $D$  and  $U^*DU$  are neither at distance 3. Towards a contradiction assume that there exists a path  $D \leftrightarrow T \leftrightarrow S \leftrightarrow U^*DU$  in  $\Gamma(\mathcal{B}(\mathcal{H}))$ . As before we can see that  $T$  can be replaced by a non-scalar orthogonal projection  $P$ , which is actually a spectral projection for  $T$  and  $D$ , such that  $Pe_1 = 0$ . Thus, we have a path  $D \leftrightarrow P \leftrightarrow S \leftrightarrow UDU^*$  which gives  $UPU^* \leftrightarrow USU^* \leftrightarrow D$ . Again, in this last path we can replace  $USU^*$  by a non-scalar orthogonal projection  $Q$ . Using the Fuglede-Putnam theorem, we see that  $D \leftrightarrow P \leftrightarrow U^*QU \leftrightarrow U^*DU$ . Let  $j$  be such that  $Pe_j = e_j$ . Since  $PU^*QUe_1 = U^*QUPe_1 = 0$  one has  $0 = \langle PU^*QUe_1, e_j \rangle = \langle QUe_1, Ue_j \rangle$ . Let  $Ue_1 = \sum_{n=1}^\infty \alpha_n e_n$  and  $Ue_j = \sum_{\substack{n=1 \\ n \neq j}}^\infty \beta_n e_n + \gamma e_j$ . Then, by (2.2),  $\alpha_n < 0$  and  $\beta_n < 0$

( $n \in \mathbb{N}$ ). Because of  $0 = \langle e_1, e_j \rangle = \langle Ue_1, Ue_j \rangle = \sum_{\substack{n=1 \\ n \neq j}}^\infty \alpha_n \beta_n + \gamma \alpha_j$  we see that  $\gamma > 0$ . Since

$U^*QU$  commutes with  $U^*DU$  we can write  $Q = \sum_{n=1}^\infty \omega_n e_n \otimes e_n$ , where  $\omega_n \in \{0, 1\}$  ( $n \in \mathbb{N}$ ).

Then  $0 = \langle QUe_1, Ue_j \rangle = \sum_{\substack{n=1 \\ n \neq j}}^\infty \omega_n \alpha_n \beta_n + \omega_j \gamma \alpha_j$ . Because of  $\sum_{\substack{n=1 \\ n \neq j}}^\infty \alpha_n \beta_n + \gamma \alpha_j = 0$ , one can have

$\sum_{\substack{n=1 \\ n \neq j}}^\infty \omega_n \alpha_n \beta_n + \omega_j \gamma \alpha_j = 0$  if and only if either  $\omega_n = 0$ , for any  $n$ , or  $\omega_n = 1$ , for any  $n$ , which

contradicts the fact that  $Q \notin \{0, I\}$ .

We have seen that  $D$  and  $U^*DU$  are at distance 4 at least. Since they are normal both of them commute with a non-scalar projection. By Proposition 2.7, projections are at distance at most 2. Hence, we may conclude that  $D$  and  $U^*DU$  are at distance 4. Note that  $D$  and  $U^*DU$  are unitary operators if  $|\delta_n| = 1$  for any  $n$ . Thus, there are two unitary operators which are at distance 4 in  $\Gamma(\mathcal{B}(\mathcal{H}))$ . If  $\delta_n \rightarrow 0$ , then  $D$  and  $U^*DU$  are unitarily equivalent compact operators. Hence, there exists unitarily equivalent normal compact operators at distance 4 in  $\Gamma(\mathcal{B}(\mathcal{H}))$ .



## 3. PROOF OF THEOREM 2.3

We start the section with the following technical lemma.

**Lemma 3.1.** *There exist an increasing sequence  $\{r_k\}_{k=1}^\infty \subseteq \mathbb{N}$  and a function  $h : \mathbb{N} \rightarrow \mathbb{N}_0$  such that*

- (i)  $r_1 = 4$ ,  $4r_k < r_{k+1} < 6r_k$ , for any  $k \in \mathbb{N}$ ;
- (ii)  $h(k) \leq k - 1$ , for any  $k \in \mathbb{N}$ ;
- (iii) for all  $j, n \in \mathbb{N}$  and each  $s \in \{0, 1, \dots, n-1\}$ , there are infinitely many  $k \in \mathbb{N}$  satisfying simultaneously  $h(k) = j$  and  $r_k \equiv s \pmod{n}$ .

*Proof.* Let  $N = \{(n, s); n \in \mathbb{N}, 0 \leq s \leq n-1\} \subseteq \mathbb{N} \times \mathbb{N}_0$ . Since  $N$  is a countable set, there exists a function  $g : \mathbb{N} \rightarrow N$  such that, for each pair  $(n, s) \in N$ , there are infinitely many  $k \in \mathbb{N}$  with  $g(k) = (n, s)$ . Moreover, we may assume that  $n_k \leq k$  if  $g(k) = (n_k, s_k)$ .

We construct a sequence  $\{r_k\}_{k=1}^\infty \subseteq \mathbb{N}$  which satisfies conditions

$$(3.1) \quad r_1 = 4, \quad 4r_k < r_{k+1} < 6r_k, \quad \text{and} \quad r_k \equiv s_k \pmod{n_k}$$

inductively. Let  $m \geq 1$  and suppose that integers  $r_1, \dots, r_m$  satisfying (3.1) have already been constructed. It is obvious that the cardinality of  $\{4r_m+1, \dots, 6r_m-1\}$  is  $2r_m-1 > m+1 \geq n_{m+1}$ . Hence, there exists a positive integer  $r_{m+1} \in \{4r_m+1, \dots, 6r_m-1\}$  such that  $r_{m+1} \equiv s_{m+1} \pmod{n_{m+1}}$  (here  $n_{m+1}$  and  $s_{m+1}$  are the integers such that  $g(m+1) = (n_{m+1}, s_{m+1})$ ).

Now we construct  $h$ . For  $(n, s) \in N$ , let  $M_{n,s} = \{k \in \mathbb{N}; g(k) = (n, s)\}$ . Note that  $r_k \equiv s \pmod{n}$  if  $k \in M_{n,s}$ . Since  $M_{n,s}$  is an infinite set, we can find a function  $h_{n,s} : M_{n,s} \rightarrow \mathbb{N}_0$  such that each  $j \in \mathbb{N}_0$  is attained infinitely many times. Moreover, we may assume that  $h_{n,s}(k) < k$  for all  $k \in M_{n,s}$ . As  $\mathbb{N}$  is a disjoint union of sets  $M_{n,s}$  ( $(n, s) \in N$ ) we may define  $h : \mathbb{N} \rightarrow \mathbb{N}_0$  by  $h(k) = h_{n,s}(k)$  if  $k \in M_{n,s}$ .  $\square$

Let us choose and fix an increasing sequence  $\{r_n\}_{n=1}^\infty \subseteq \mathbb{N}$  and a function  $h : \mathbb{N} \rightarrow \mathbb{N}_0$  which are satisfying the conditions of Lemma 3.1. Set  $r_0 = 0$ . We also choose and fix a sequence of positive numbers  $\{\varepsilon_n\}_{n=1}^\infty$  which is decreasing to 0 sufficiently fast. For our purposes it suffices that  $\varepsilon_1 < 1/2$  and  $\varepsilon_k \leq \left(\frac{\varepsilon_{k-1}}{r_{r_{k+3}}}\right)^{3r_{k+3}}$  for  $k \geq 2$ .

Throughout this section, let  $\{e_n\}_{n=0}^\infty$  be a fixed orthonormal basis for  $\mathcal{H}$ . Denote

$$\mathcal{H}_j = \bigvee \{e_0, \dots, e_j\} \quad (j \geq 0) \quad \text{and} \quad \mathcal{H}_\infty = \bigcup_{j=0}^\infty \mathcal{H}_j.$$

It is obvious that  $x \in \mathcal{H}_\infty$  if and only if  $\langle x, e_n \rangle = 0$  for all but at most finitely many indices  $n$  and hence  $\mathcal{H}_\infty$  is a dense linear manifold in  $\mathcal{H}$ . We are going to define a linear mapping  $T : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$  with a continuous extension to  $\mathcal{H}$  which will fulfill the condition of Theorem 2.3. We recursively define vectors  $Te_n$  ( $n \in \mathbb{N}_0$ ) and an auxiliary sequence  $\{u_n\}_{n=-\infty}^\infty \subseteq \mathcal{H}$  as follows. First we set  $u_0 = e_0$  and, because of technical reasons,  $u_n = 0$  for  $n < 0$ . Then define recursively

$$(3.2a) \quad Te_0 = 0, \quad Te_j = e_{j-1} \quad \text{if } r_k < j < r_{k+1},$$

$$(3.2b) \quad Te_{r_k} = \varepsilon_k e_{r_k-1} + \frac{\sqrt{\varepsilon_k}}{\|u_{h(k)}\|} u_{h(k)},$$

$$(3.2c) \quad u_{r_k} = \frac{1}{\varepsilon_1 \cdots \varepsilon_k} e_{r_k}, \quad \text{and } u_j = T^{r_k-j} u_{r_k} \quad \text{if } r_{k-1} < j < r_k.$$

Now  $T$  is linearly extended to  $\mathcal{H}_\infty$ . Note that  $Te_{r_k}$  in (3.2b) is correctly defined as  $h(k) \leq k-1 \leq r_{k-1}$  and it is easy to see that  $u_j \in \mathcal{H}_j$  ( $j \in \mathbb{N}_0$ ).

Let  $S$  be the shift part of  $T$ , i.e.,  $Se_0 = 0$  and  $Se_j = \omega_j e_{j-1}$  ( $j \in \mathbb{N}$ ), where the sequence of weights is given by

$$(3.3) \quad \omega_{r_k} = \varepsilon_k \quad \text{and} \quad \omega_j = 1 \quad \text{if } j \neq r_k \quad (k \in \mathbb{N}).$$

It is obvious that  $S$  is a linear mapping on  $\mathcal{H}_\infty$  with  $\|S\| = 1$ . Note that  $(T-S)e_j = 0$  if  $j \neq r_k$  ( $k \in \mathbb{N}$ ) and  $(T-S)e_{r_k} = \frac{\sqrt{\varepsilon_k}}{\|u_{h(k)}\|} u_{h(k)}$  ( $k \in \mathbb{N}$ ).

**Lemma 3.2.** *For any  $j \in \mathbb{N}$ , one has*

(i)  $T\mathcal{H}_j \subseteq \mathcal{H}_{j-1}$  and, for  $1 \leq i \leq j$ ,  $T^i e_j = \omega_j \cdots \omega_{j-i+1} e_{j-i} + w$ , where  $w \in \mathcal{H}_{k-1}$  and  $k$  is such that  $r_k \leq j < r_{k+1}$ ;

(ii)  $Tu_j = u_{j-1}$ ;

(iii)  $\|u_{j-1}\| \leq \|u_j\|$ ;

(iv)  $\mathcal{H}_j = \vee\{u_0, \dots, u_j\}$ , which means that  $\{u_j\}_{j=0}^\infty$  is a basis for  $\mathcal{H}_\infty$ .

*Proof.* (i) The inclusion  $T\mathcal{H}_j \subseteq \mathcal{H}_{j-1}$  is straightforward as, by (3.2),  $Te_j \in \mathcal{H}_{j-1}$  for any  $j \in \mathbb{N}$ . Let  $1 \leq i \leq j$ . Then  $T^i e_j = T^{i-1}(S + (T-S))e_j = T^{i-1}(\omega_j e_{j-1} + w')$ , where  $w'$  is either 0 or  $\frac{\sqrt{\varepsilon_k}}{\|u_{h(k)}\|} u_{h(k)}$ . In any case it is in  $\mathcal{H}_{k-1}$ . Now, because of  $T\mathcal{H}_j \subseteq \mathcal{H}_{j-1}$  one easily sees that  $T^i e_j = \omega_j \cdots \omega_{j-i+1} e_{j-i} + w$ , for some  $w \in \mathcal{H}_{k-1}$ .

(ii) If  $r_{k-1} < j-1 < j < r_k$ , then  $Tu_j = u_{j-1}$ , by (3.2c). Also, for  $j = r_k$ , one has  $Tu_{r_k} = T^{r_k-(r_k-1)} u_{r_k} = u_{r_k-1}$ . Let  $j = r_k + 1$ . Since, by (3.2c),

$$u_{r_k+1} = T^{r_k+1-r_k-2} T u_{r_k+1} = \frac{1}{\varepsilon_1 \cdots \varepsilon_{k+1}} T^{r_k+1-r_k-2} (\varepsilon_{k+1} e_{r_k+1-1} + \frac{\sqrt{\varepsilon_{k+1}}}{\|u_{h(k+1)}\|} u_{h(k+1)})$$

and  $T^{r_k+1-r_k-2} u_{h(k+1)} = 0$ , by clause (i) of this lemma (note that  $r_{k+1} - r_k - 2 > r_k$ , by (3.1)), we have

$$Tu_{r_k+1} = TT^{r_k+1-r_k-2} (\frac{1}{\varepsilon_1 \cdots \varepsilon_k} e_{r_k+1-1}) = \frac{1}{\varepsilon_1 \cdots \varepsilon_k} e_{r_k+1-1-(r_k+1-r_k-2+1)} = \frac{1}{\varepsilon_1 \cdots \varepsilon_k} e_{r_k} = u_{r_k}.$$

(iii) Since  $u_0 = e_0$  and  $u_1 = T^{r_1-1} u_{r_1} = T^3(\frac{1}{\varepsilon_1} e_4) = T^2(e_3 + \frac{1}{\sqrt{\varepsilon_1}} e_0) = e_1$  the claim holds for  $j = 1$ . Suppose that the inequality has already been verified for indices up to  $j-1$ . Assume first that  $j = r_k$  ( $k \in \mathbb{N}$ ). By (3.2c),  $u_{r_k-1} = Tu_{r_k} = \frac{1}{\varepsilon_1 \cdots \varepsilon_k} Te_{r_k} = \frac{1}{\varepsilon_1 \cdots \varepsilon_k} (\varepsilon_k e_{r_k-1} + \frac{\sqrt{\varepsilon_k}}{\|u_{h(k)}\|} u_{h(k)})$  and therefore

$$\|u_{r_k-1}\|^2 = \frac{1}{(\varepsilon_1 \cdots \varepsilon_k)^2} (\|\varepsilon_k e_{r_k-1}\|^2 + \|\frac{\sqrt{\varepsilon_k}}{\|u_{h(k)}\|} u_{h(k)}\|^2) = \frac{\varepsilon_k^2 + \varepsilon_k}{(\varepsilon_1 \cdots \varepsilon_k)^2} < \frac{1}{(\varepsilon_1 \cdots \varepsilon_k)^2} = \|u_{r_k}\|^2.$$

If  $r_{k-1} < j < r_k$ , then  $u_j = T^{r_k-j} u_{r_k} = \frac{1}{\varepsilon_1 \cdots \varepsilon_{k-1}} e_j + \frac{1}{\varepsilon_1 \cdots \varepsilon_{k-1} \sqrt{\varepsilon_k} \|u_{h(k)}\|} u_{h(k)-(r_k-j-1)}$  and  $u_{j-1} = Tu_j = \frac{1}{\varepsilon_1 \cdots \varepsilon_{k-1}} e_{j-1} + \frac{1}{\varepsilon_1 \cdots \varepsilon_{k-1} \sqrt{\varepsilon_k} \|u_{h(k)}\|} u_{h(k)-(r_k-s)}$ . Vectors  $e_j$  and  $u_{h(k)-(r_k-j-1)}$  are orthogonal

as  $h(k) - (r_k - j - 1) < j$  because of  $h(k) \leq r_{k-1}$ . Hence, by Pythagoras' theorem and the induction hypothesis, one has  $\|u_{j-1}\| \leq \|u_j\|$ .

(iv) It is obvious that  $\vee\{e_0\} = \vee\{u_0\}$ . Let  $j \in \mathbb{N}$  be arbitrary and assume that  $\vee\{e_0, \dots, e_i\} = \vee\{u_0, \dots, u_i\}$ , for any  $0 \leq i \leq j-1$ . If  $j = r_k$ , for some  $k \in \mathbb{N}_0$ , then, by (3.2c),  $u_j$  is a scalar multiple of  $e_j$  and therefore  $\vee\{e_0, \dots, e_j\} = \vee\{u_0, \dots, u_j\}$ . In the case when  $r_{k-1} < j < r_j$ , one has, by (3.2c) and clause (i) of this lemma,  $u_j = T^{r_k-j}u_{r_k} = \lambda e_j + w$ , where  $\lambda$  is a nonzero number and  $w$  is a vector in  $\mathcal{H}_{j-1}$ . We may conclude again that  $\vee\{e_0, \dots, e_j\} = \vee\{u_0, \dots, u_j\}$ .  $\square$

The mapping  $T$  can be extended to a bounded linear operator on  $\mathcal{H}$ . Indeed, if  $x = \sum_{j=0}^n \alpha_j e_j \in \mathcal{H}_\infty$  is a vector of norm 1, then

$$\|(T - S)x\| \leq \sum_{j=0}^n |\alpha_j| \|(T - S)e_j\| \leq \sum_{k=1}^{\infty} \|(T - S)e_{r_k}\| = \sum_{k=1}^{\infty} \sqrt{\varepsilon_k} \leq 1$$

which means that  $\|T - S\| \leq 1$  and consequently  $\|T\| \leq 2$ . It is common to use the same notation for the extended operator. Thus, from now on  $T$  denotes an operator on  $\mathcal{H}$  satisfying (3.2) with norm not greater than 2.

Let  $\{c_i\}_{i=0}^{\infty}$  be a sequence of complex numbers. For any  $x \in \mathcal{H}_\infty$ , the sum  $\sum_{i=0}^{\infty} c_i T^i x$  **is finite and** represents a vector in  $\mathcal{H}_\infty$ . Thus,  $\sum_{i=0}^{\infty} c_i T^i$  is a well-defined linear mapping on  $\mathcal{H}_\infty$  (which is not necessarily bounded).

**Lemma 3.3.** *Assume that  $\sum_{i=0}^{\infty} c_i T^i$  is a bounded linear mapping on  $\mathcal{H}_\infty$ . Let  $A$  be its unique continuous extension to  $\mathcal{H}$ . Then*

- (i)  $Ae_j = \sum_{i=0}^j c_i \omega_j \cdots \omega_{j-i+1} e_{j-i} + w$ , where  $w \in \mathcal{H}_{k-1}$  and  $k$  is such that  $r_k \leq j < r_{k+1}$ ;
- (ii)  $\sum_{i=0}^{\infty} |c_i|^2 \leq \|A\|^2$ ; and
- (iii) if  $c_i = 0$  ( $0 \leq i < n$ ) and  $c_n \neq 0$  for some  $n \in \mathbb{N}$ , then  $\ker A = \mathcal{H}_{n-1}$ .

*Proof.* (i) One has  $Ae_j = \sum_{i=0}^j c_i T^i e_j + \sum_{i=j+1}^{\infty} c_i T^i e_j$ . By Lemma 3.2 (i),  $\sum_{i=j+1}^{\infty} c_i T^i e_j = 0$  and

$$\sum_{i=0}^j c_i T^i e_j = \sum_{i=0}^j c_i \omega_j \cdots \omega_{j-i+1} e_{j-i} + w \text{ with } w \in \mathcal{H}_{k-1}.$$

(ii) By clause (i) of this lemma, one has  $Ae_{r_k-1} = c_0 e_{r_k-1} + \cdots + c_{r_k-r_{k-1}-1} e_{r_k-1} + v$  ( $v \in \mathcal{H}_{r_{k-1}-1}$ ), for any  $k \in \mathbb{N}$ . Thus,  $\|A\|^2 \geq \|Ae_{r_k-1}\|^2 \geq \|c_0 e_{r_k-1} + \cdots + c_{r_k-r_{k-1}-1} e_{r_k-1}\|^2 = \sum_{j=0}^{r_k-r_{k-1}-1} |c_j|^2$ . Let  $k \rightarrow \infty$  and the assertion follows.

(iii) Without loss of generality we may assume that  $c_n = 1$ . The inclusion  $\mathcal{H}_{n-1} \subseteq \ker A$  is obvious. We prove the opposite inclusion by a contradiction. Assume that there is a vector  $x = \sum_{j=n}^{\infty} \alpha_j e_j \in \ker A$  of norm 1 which is not in  $\mathcal{H}_{n-1}$ . Then  $x \notin \mathcal{H}_\infty$ . Indeed, if  $x$  were in  $\mathcal{H}_\infty$ ,

then there would be an integer  $\ell > n$  such that  $x = \sum_{j=n}^{\ell} \alpha_j e_j$  and  $\alpha_\ell \neq 0$ . By Lemma 3.2 (i), one would have  $0 = T^{\ell-n} Ax = \alpha_\ell T^\ell e_\ell = \alpha_\ell \omega_\ell \cdots \omega_1 e_0 \neq 0$ , **a contradiction. Hence,  $x \notin \mathcal{H}_\infty$ .**

Let  $\Delta = \max\{5, \|A\|, r_{n+1}\}$ . Choose an integer  $m \geq \Delta$  such that  $|\alpha_m| = \max\{|\alpha_j|; j \geq m\}$  and let  $k$  be such that  $r_k \leq m < r_{k+1}$ . Since  $r_{n+1} \leq \Delta \leq m$  one has  $k \geq n+1$ . Let  $s$  be the largest integer in  $\{m, m+1, \dots, r_{k+1}-1\}$  satisfying  $|\alpha_s| \geq \frac{|\alpha_m|}{\Delta^{2(s-m)}}$ . Hence, for  $s \leq j \leq r_{k+1}-1$ , one has

$$(3.4) \quad |\alpha_j| \leq \frac{|\alpha_m|}{\Delta^{2(j-m)}} = \frac{|\alpha_m|}{\Delta^{2(s-m)}} \cdot \frac{1}{\Delta^{2(j-s)}} \leq \frac{|\alpha_s|}{\Delta^{2(j-s)}}.$$

If  $j \geq r_{k+1}$ , then

$$(3.5) \quad |\alpha_j| \leq |\alpha_m| \leq |\alpha_s| \cdot \Delta^{2(s-m)} \leq |\alpha_s| \Delta^{r_{k+1}}.$$

Since  $Ax = 0$  one has  $0 = \langle Ax, e_{s-n} \rangle = \sum_{j=n}^{\infty} \alpha_j \langle Ae_j, e_{s-n} \rangle$ . If index  $j$  is less than  $s$ , then  $j-n < s-n$  and therefore  $\sum_{j=n}^{s-1} \alpha_j \langle Ae_j, e_{s-n} \rangle = 0$ , by Lemma 3.2 (i). Denote  $\Gamma_1 = \sum_{j=s+1}^{r_{k+1}-1} \alpha_j \langle Ae_j, e_{s-n} \rangle$  and  $\Gamma_2 = \sum_{j=r_{k+1}}^{\infty} \alpha_j \langle Ae_j, e_{s-n} \rangle$ . Then

$$(3.6) \quad 0 = \langle Ax, e_{s-n} \rangle = \alpha_s \langle Ae_s, e_{s-n} \rangle + \Gamma_1 + \Gamma_2.$$

Using (3.4), we estimate number  $|\Gamma_1|$  as follows:

$$\begin{aligned} |\Gamma_1| &\leq |\alpha_s| \sum_{j=s+1}^{r_{k+1}-1} \frac{1}{\Delta^{2(j-s)}} |\langle Ae_j, e_{s-n} \rangle| \leq |\alpha_s| \sum_{j=s+1}^{r_{k+1}-1} \frac{1}{\Delta^{2(j-s)}} \sum_{i=n}^j |c_i| |\langle T^i e_j, e_{s-n} \rangle| \\ &= |\alpha_s| \sum_{j=s+1}^{r_{k+1}-1} \frac{|c_{j-s+n}|}{\Delta^{2(j-s)}} \omega_j \cdots \omega_{s-n+1} \leq |\alpha_s| \omega_s \cdots \omega_{s-n+1} \left( \left( \sum_{j=1}^{\infty} \frac{1}{\Delta^{4j}} \right) \left( \sum_{j=n+1}^{\infty} |c_j|^2 \right) \right)^{1/2} \\ &\leq |\alpha_s| \omega_s \cdots \omega_{s-n+1} \frac{\|A\|}{\sqrt{\Delta^4-1}} \leq \frac{|\alpha_s| \omega_s \cdots \omega_{s-n+1}}{4}. \end{aligned}$$

To estimate  $|\Gamma_2|$ , note that Lemma 3.3 (ii) implies  $|c_j| \leq \|A\| \leq \Delta$  for  $j \in \mathbb{N}_0$ . Combining this with (3.5) we get

$$\begin{aligned} |\Gamma_2| &\leq \sum_{j=r_{k+1}}^{\infty} |\alpha_j| |\langle (c_n T^n + \cdots + c_j T^j) e_j, e_{s-n} \rangle| \leq |\alpha_s| \Delta^{r_{k+1}+1} \sum_{j=r_{k+1}}^{\infty} j \max_{n \leq i \leq j} |\langle T^i e_j, e_{s-n} \rangle| \\ &\leq |\alpha_s| \Delta^{r_{k+1}+1} \sum_{p=k+1}^{\infty} (r_p + (r_p + 1) + \cdots + (r_{p+1} - 1)) \max_{\substack{n \leq i \leq j \\ r_p \leq j < r_{p+1}}} |\langle T^i e_j, e_{s-n} \rangle| \\ &\leq |\alpha_s| \Delta^{r_{k+1}+1} \sum_{p=k+1}^{\infty} r_p^2 \max_{\substack{n \leq i \leq j \\ r_p \leq j < r_{p+1}}} |\langle T^i e_j, e_{s-n} \rangle|. \end{aligned}$$

Since  $s - n < r_{k+1} \leq r_p$  one has  $\langle T^i e_j, e_{s-n} \rangle = \langle e_{j-i}, e_{s-n} \rangle = 0$  for  $i = n, \dots, j - r_p$ . If  $i \geq j - r_p + 1$ , then

$$\begin{aligned} |\langle T^j e_i, e_{s-n} \rangle| &= |\langle T^{j-(i-r_p)-1}(\varepsilon_p e_{r_p-1} + \frac{\sqrt{\varepsilon_p}}{\|u_{h(p)}\|} u_{h(p)}), e_{s-n} \rangle| \\ &\leq 2 \|T^{j-(i-r_p)-1}\| \sqrt{\varepsilon_p} \leq 2^{j-(i-r_p)} \sqrt{\varepsilon_p} \leq 2^{r_p} \sqrt{\varepsilon_p}, \end{aligned}$$

by Lemma 3.3 and since  $\|T\| \leq 2$ . Thus,

$$|\Gamma_2| \leq |\alpha_s| \Delta^{r_{k+1}+1} \sum_{p=k+1}^{\infty} 2^{r_p} r_{p+1}^2 \sqrt{\varepsilon_p} \leq 2^{r_{k+1}+1} |\alpha_s| \Delta^{r_{k+1}+1} r_{k+2}^2 \sqrt{\varepsilon_{k+1}} \leq \frac{|\alpha_s| \omega_s \cdots \omega_{s-n+1}}{4},$$

provided that the sequence  $\{\varepsilon_k\}_{k=1}^{\infty}$  decreases to 0 sufficiently fast.

At the end note that  $\alpha_s \langle A e_s, e_{s-n} \rangle = \alpha_s \omega_s \cdots \omega_{s-n+1}$ , by clause (i) of this lemma. Thus, if we use this and the estimates for  $|\Gamma_1|$  and  $|\Gamma_2|$  in (3.6) we get

$$0 = |\langle Ax, e_{s-n} \rangle| \geq |\alpha_s| \omega_s \cdots \omega_{s-n+1} - |\Gamma_1| - |\Gamma_2| > 0,$$

a contradiction.  $\square$

**Lemma 3.4.** *If  $A \in \{T\}'$ , then there exists a sequence  $\{c_i\}_{i=0}^{\infty} \subseteq \mathbb{C}$  such that  $\sum_{i=0}^{\infty} |c_i|^2 < \infty$  and*

$$Ax = \sum_{i=0}^{\infty} c_i T^i x \text{ for any } x \in \mathcal{H}_{\infty}.$$

*Proof.* Since, by Lemma 3.3 (iii),  $\ker T^{j+1} = \mathcal{H}_j$  and  $A$  commutes with  $T$  one has  $A\mathcal{H}_j \subseteq \mathcal{H}_j$  for every  $j \in \mathbb{N}_0$ . By Lemma 3.2 (iv), vectors  $u_0, \dots, u_j$  form a basis for  $\mathcal{H}_j$ , hence, for any  $j \in \mathbb{N}_0$ , there exist numbers  $\alpha_{ij}$  ( $i = 0, \dots, j$ ) such that  $Au_j = \sum_{i=0}^j \alpha_{ij} u_i$ . It follows that

$$\sum_{i=0}^{j-1} \alpha_{i(j-1)} u_i = Au_{j-1} = ATu_j = T Au_j = T \sum_{i=0}^j \alpha_{ij} u_i = \sum_{i=1}^j \alpha_{ij} u_{i-1},$$

which gives  $\alpha_{(i-1)(j-1)} = \alpha_{ij}$  for  $j \in \mathbb{N}$  and  $1 \leq i \leq j$ . Let  $c_j = \alpha_{0j}$ . Then

$$Au_j = \sum_{i=0}^j \alpha_{ij} u_i = \sum_{i=0}^j \alpha_{0(j-i)} u_i = \sum_{i=0}^j c_i u_{j-i} = \sum_{i=0}^{\infty} c_i T^i u_j.$$

By Lemmas 3.2 (iv) and 3.3 (ii), the assertion follows.  $\square$

Note that Lemma 3.4 implies commutativity of  $\{T\}'$ . As we consider the commutativity properties of  $A \in \{T\}'$  there is no loss of generality if we assume that

$$(3.7) \quad Ax = T^n x + \sum_{i=n+1}^{\infty} c_i T^i x \quad (x \in \mathcal{H}_{\infty}),$$

where  $n \in \mathbb{N}$ . From now on, till the end of this section,  $A$  always means an operator in  $\{T\}'$  such that (3.7) holds. **It will be beneficial to define  $c_n = 1$  so that  $Ax = \sum_{i=n}^{\infty} c_i T^i x$ , ( $x \in \mathcal{H}_{\infty}$ ).**

**Lemma 3.5.** *If  $B \in \{A\}'$ , then there exist numbers  $b_{ij}$  ( $i, j \in \mathbb{N}_0$ ) such that*

$$(3.8) \quad Bu_j = \sum_{i=0}^{j+n-1} b_{ij} u_i \quad (j \in \mathbb{N}_0).$$

*In particular,  $B\mathcal{H}_\infty \subseteq \mathcal{H}_\infty$ .*

*Proof.* By Lemma 3.3 (iii),  $\ker A^m = \mathcal{H}_{mn-1}$  for any  $m \in \mathbb{N}$ . Since  $A$  and  $B$  commute we have  $B\mathcal{H}_{mn-1} \subseteq \mathcal{H}_{mn-1}$ . Let  $j \in \mathbb{N}_0$  be an arbitrary index and let  $m \in \mathbb{N}$  be such that  $(m-1)n-1 < j \leq mn-1$ . Then  $u_j \in \mathcal{H}_{mn-1}$  and consequently  $Bu_j \in \mathcal{H}_{mn-1} \subseteq \mathcal{H}_{j+n-1}$ . Since, by Lemma 3.2 (iv), vectors  $u_0, \dots, u_{j+n-1}$  form a basis for  $\mathcal{H}_{j+n-1}$ , there exist numbers  $b_{ij}$  ( $0 \leq i \leq j+n-1$ ) such that (3.8) holds. For  $i \geq j+n$  we set  $b_{ij} = 0$ .  $\square$

If, on  $\mathcal{H}_\infty$ , the operator  $B$  is represented by a matrix  $[b_{ij}]_{i,j=0}^\infty$  with respect to the basis  $\{u_j\}_{j=0}^\infty$ , then, by (3.8), the matrix is zero below the  $(n-1)$ -th subdiagonal. Let  $d \leq n-1$  be the lowest nonzero diagonal in  $[b_{ij}]_{i,j=0}^\infty$ . Thus,  $b_{ij} = 0$  for  $j \in \mathbb{N}_0$ ,  $i > j+d$ , and there exists  $j_0$  such that  $b_{(j_0+d)j_0} \neq 0$ .

**Lemma 3.6.** (i) *The lowest nonzero diagonal of  $[b_{ij}]_{i,j=0}^\infty$  is periodic with period  $n$ , that is,  $b_{(j+d)j} = b_{(j+d+n)(j+n)}$  for all  $j \in \mathbb{N}_0$ .*

(ii)  *$[b_{ij}]_{i,j=0}^\infty$  is upper triangular, that is,  $d \leq 0$ .*

*Proof.* (i) Let  $x_i$  denote the  $i$ -th coordinate of  $x \in \mathcal{H}_\infty$  with respect to the basis  $\{u_j\}_{j=0}^\infty$ . In particular,  $b_{ij} = (Bu_j)_i$ . Since  $b_{ij} = 0$  whenever  $i > j+d$  we have

$$\begin{aligned} b_{(j+d)j} &= (B(u_j + c_{n+1}u_{j-1} + \dots + c_{j+n}u_0))_{j+d} = (BAu_{j+n})_{j+d} = (ABu_{j+n})_{j+d} \\ &= (A(b_{(j+d+n)(j+n)}u_{j+d+n} + b_{(j+d+n-1)(j+n)}u_{j+d+n-1} + \dots + b_{0(j+n)}u_0))_{j+d} \\ &= b_{(j+d+n)(j+n)}. \end{aligned}$$

(ii) To derive a contradiction, suppose that  $d > 0$ . Then there exists  $j \in \mathbb{N}_0$  such that  $b_{(j+d)j} \neq 0$ . Denote this nonzero number by  $b$ . By Lemma 3.1 (iii), there are infinitely many  $k \in \mathbb{N}_0$  such that  $r_k \equiv j+d \pmod{n}$ . Note that, by clause (i) of this lemma,  $b = b_{r_k(r_k-d)}$  for any such  $k > n$ . Moreover, we have  $\|u_{r_k}\| = (\varepsilon_1 \dots \varepsilon_k)^{-1}$  and, by Lemma 3.2 (iii),

$$\|u_{r_k-d}\| \leq \|u_{r_k-1}\| = \left\| \frac{1}{\varepsilon_1 \dots \varepsilon_{k-1}} e_{r_k-1} + \frac{1}{\varepsilon_1 \dots \varepsilon_{k-1} \sqrt{\varepsilon_k} \|u_{h(k)}\|} u_{h(k)} \right\| \leq 2(\varepsilon_1 \dots \varepsilon_{k-1})^{-1} \varepsilon_k^{-1/2}.$$

Also,  $Bu_{r_k-d} = \sum_{i=0}^{(r_k-d)+d} b_{i(r_k-d)} u_i = b_{r_k(r_k-d)} u_{r_k} + w$ , where  $w \in \mathcal{H}_{r_k-1}$ . Thus,

$$\|Bu_{r_k-d}\| \geq |\langle Bu_{r_k-d}, e_{r_k} \rangle| = |\langle b_{r_k(r_k-d)} u_{r_k}, e_{r_k} \rangle| = \frac{|b_{r_k(r_k-d)}|}{\varepsilon_1 \dots \varepsilon_k} = \frac{|b|}{\varepsilon_1 \dots \varepsilon_k}.$$

It follows that  $\|B\| \geq \frac{\|Bu_{r_k-d}\|}{\|u_{r_k-d}\|} \geq \frac{|b|}{2\sqrt{\varepsilon_k}} \rightarrow \infty$ , as  $k \rightarrow \infty$ , which is a contradiction.  $\square$

Now we will show that the entries on the lowest nonzero diagonal of  $[b_{ij}]_{i,j=0}^\infty$  are constant. To prove this we need an estimate on the moduli of numbers  $b_{ij}$ . For  $k, j \in \mathbb{N}_0$ , let

$$\Gamma_{j,k} = \max_{\substack{i+n \leq s \leq r_k \\ 0 \leq i \leq j}} |b_{(s-i)s} - b_{(s-i-n)(s-n)}|$$

(as usual, the maximum over empty set equals zero). Hence,  $\Gamma_{j,k}$  is the maximal difference between two entries of  $[b_{ij}]_{i,j=0}^{\infty}$  both of which lie on the same among the lowest  $j$  diagonals and their positions differ by  $n$  and are bounded above by  $r_k$ . Moreover, write

$$\Delta_{j,k} = \max_{\substack{i \leq s \leq r_k \\ 0 \leq i \leq j}} |b_{(s-i)s}|.$$

By Lemma 3.6,  $\Gamma_{0,k} = 0$  for any  $k \in \mathbb{N}_0$ .

**Lemma 3.7.** (i) For  $j \geq 1$  and  $k \geq 0$ , we have  $\Gamma_{j,k} \leq 2r_k \|A\| \Delta_{j-1,k}$ .

(ii) Let  $k \geq 1$ . If  $r_k \leq i \leq s < r_{k+1} - h(k+1) - 1$ , then  $|b_{is}| \leq \|B\|$ .

(iii) Let  $j, k \geq 0$  be such that  $j, n < r_k$ . Then,  $\Delta_{j,k} \leq r_k \Gamma_{j,k+1} + \|B\|$ .

*Proof.* (i) If  $0 \leq i \leq j$  and  $i+n \leq s \leq r_k$ , then

$$\begin{aligned} (BAu_s)_{s-i-n} &= (B(c_n u_{s-n} + c_{n+1} u_{s-n-1} + \cdots + c_s u_0))_{s-i-n} \\ &= c_n b_{(s-i-n)(s-n)} + c_{n+1} b_{(s-i-n)(s-n-1)} + \cdots + c_{n+i} b_{(s-i-n)(s-n-i)} \end{aligned}$$

and

$$\begin{aligned} (ABu_s)_{s-i-n} &= (b_{ss} Au_s + b_{(s-1)s} Au_{s-1} + \cdots + b_{(s-i)s} Au_{s-i} + \cdots + b_{0s} Au_0)_{s-i-n} \\ &= b_{ss} c_{n+i} + b_{(s-1)s} c_{n+i-1} + \cdots + b_{(s-i)s} c_n \\ &= c_n b_{(s-i)s} + c_{n+1} b_{(s-i+1)s} + \cdots + c_{n+i} b_{ss}. \end{aligned}$$

Since  $AB = BA$  and  $c_n = 1$  we have

$$\begin{aligned} |b_{(s-i)s} - b_{(s-i-n)(s-n)}| &\leq \sum_{t=1}^i |c_{n+t}| (|b_{(s-i-n)(s-n-t)}| + |b_{(s-i+t)s}|) \\ &\leq 2 \sum_{t=1}^i |c_{n+t}| \max_{\substack{v \leq u+j-1 \\ 0 \leq u \leq v \leq r_k}} |b_{uv}| = 2\Delta_{j-1,k} \sum_{t=1}^i |c_{n+t}| \leq 2\Delta_{j-1,k} \|A\| r_k. \end{aligned}$$

(ii) By (3.2c),

$$u_i = T^{r_{k+1}-i} u_{r_{k+1}} = \frac{1}{\varepsilon_1 \cdots \varepsilon_k} T^{r_{k+1}-i-1} \left( e_{r_{k+1}-1} + \frac{1}{\sqrt{\varepsilon_{k+1}} \|u_{h(k+1)}\|} u_{h(k+1)} \right) = \frac{1}{\varepsilon_1 \cdots \varepsilon_k} e_i$$

and, similarly,  $u_s = \frac{1}{\varepsilon_1 \cdots \varepsilon_k} e_s$ . Hence

$$\begin{aligned} \|B\| &\geq \|Be_s\| = \varepsilon_1 \cdots \varepsilon_k \|Bu_s\| = \varepsilon_1 \cdots \varepsilon_k \left\| \sum_{j=0}^s b_{js} u_j \right\| \\ &\geq \varepsilon_1 \cdots \varepsilon_k \left| \left\langle \sum_{j=0}^s b_{js} u_j, e_i \right\rangle \right| = \varepsilon_1 \cdots \varepsilon_k |\langle b_{is} u_i, e_i \rangle| = |b_{is}|. \end{aligned}$$

(iii) Let  $0 \leq i \leq j$  and  $i \leq s \leq r_k$ . Since  $1 \leq n < r_k$  there exists an integer  $m \geq 2$  such that  $r_k \leq mn < 2r_k$ . Then, by the definition of the sequence  $r_k$  and as  $h(k+1) \leq k \leq r_k$ , we have

$$r_k \leq mn + s - i \leq mn + s < 2r_k + s \leq 3r_k \leq 4r_k + 1 - (h(k+1) + 1) \leq r_{k+1} - (h(k+1) + 1).$$

It follows that

$$\begin{aligned} |b_{(s-i)i}| &\leq |b_{(s-i)s} - b_{(s-i+n)(s+n)}| + |b_{(s-i+n)(s+n)} - b_{(s-i+2n)(s+2n)}| + \cdots \\ &\quad \cdots + |b_{(s-i+(m-1)n)(s+(m-1)n)} - b_{(s-i+mn)(s+mn)}| + |b_{(s-i+mn)(s+mn)}| \\ &\leq m\Gamma_{j,k+1} + |b_{(s-i+mn)(s+mn)}| \leq m\Gamma_{j,k+1} + \|B\|, \end{aligned}$$

where the last inequality holds by clause (ii) of this lemma.  $\square$

**Lemma 3.8.** *For all integers  $k \geq \max\{\|A\|, \|B\|, n, 2\}$  and  $0 \leq j \leq s \leq r_k$ , one has*

$$\Delta_{j,k} \leq (k+j)r_k^3 r_{k+1}^3 \cdots r_{k+j}^3,$$

which gives  $|b_{js}| \leq \frac{1}{\varepsilon_{k-1}}$ .

*Proof.* It has been observed above that  $\Gamma_{0,k+1} = 0$ . Since  $n \leq k < r_k$ , by Lemma 3.7 (iii),  $\Delta_{0,k} \leq \|B\|$  for each  $k$  which is large enough. Now we proceed by the induction on  $j$ . By Lemma 3.7,

$$\begin{aligned} \Delta_{j,k} &\leq r_k \Gamma_{j,k+1} + \|B\| \leq 2r_k r_{k+1} \|A\| \Delta_{j-1,k+1} + \|B\| \\ &\leq 2r_k r_{k+1} r_k \Delta_{j-1,k+1} + r_k \leq r_{k+1}^3 \Delta_{j-1,k+1} + r_{k+1}^3. \end{aligned}$$

Hence

$$\begin{aligned} \Delta_{j,k} &\leq r_{k+1}^3 \Delta_{j-1,k+1} + r_{k+1}^3 \leq \cdots \\ &\leq (r_{k+1}^3 r_{k+2}^3 \cdots r_{k+j}^3 \Delta_{0,k+j}) + (r_{k+1}^3 \cdots r_{k+j}^3 + r_{k+1}^3 \cdots r_{k+j-1}^3) + \cdots + r_{k+1}^3 \\ &\leq \|B\| r_{k+1}^3 r_{k+2}^3 \cdots r_{k+j}^3 + j r_{k+1}^3 r_{k+2}^3 \cdots r_{k+j}^3 \leq (k+j) r_{k+1}^3 r_{k+2}^3 \cdots r_{k+j}^3 \leq \frac{1}{\varepsilon_{k-1}}, \end{aligned}$$

provided that the sequence  $\{\varepsilon_k\}_{k=1}^\infty$  decreases to 0 sufficiently fast.  $\square$

Recall that the lowest nonzero diagonal of  $[b_{ij}]_{i,j=0}^\infty$  is the  $|d|$ -th superdiagonal (by Lemma 3.6 (ii),  $d \leq 0$ ). If we replace  $B$  by  $\tilde{B} = B - b_{0d} T^d$ , then we obtain a bounded linear operator in  $\{A\}'$ , which commutes with  $T$  if and only if  $B$  commutes with  $T$ . Let  $[\tilde{b}_{ij}]_{i,j=0}^\infty$  be the matrix of  $\tilde{B}$  restricted to  $\mathcal{H}_\infty$  with respect to the basis  $\{u_j\}_{j=0}^\infty$ . It is clear that  $[\tilde{b}_{ij}]_{i,j=0}^\infty$  is zero below the  $|d|$ -th superdiagonal. Also, it follows by the definition of  $\tilde{B}$  that  $\tilde{b}_{0d} = 0$ .

**Lemma 3.9.** *The  $|d|$ -th superdiagonal of  $[\tilde{b}_{ij}]_{i,j=0}^\infty$  is zero, that is,  $\tilde{b}_{s(s+d)} = 0$  for any  $s \geq 0$ .*

*Proof.* By Lemma 3.6 (i), the lowest nonzero diagonal of  $B$  is periodic with period  $n$ , hence the  $|d|$ -th superdiagonal of  $[\tilde{b}_{ij}]_{i,j=0}^\infty$  is periodic, too, and the period is the same. So it suffices to show that  $\tilde{b}_{s(s+d)} = 0$  for every  $s = 1, \dots, n-1$ .

Suppose on the contrary that, for some  $s \in \{1, \dots, n-1\}$ , we have  $\tilde{b}_{s(s+d)} \neq 0$ . Denote this number by  $b$ . Since  $\tilde{b}_{0d} = 0$  and since  $[\tilde{b}_{ij}]_{i,j=0}^\infty$  is zero below the  $|d|$ -th superdiagonal we have  $\tilde{B}u_d = 0$ . By Lemma 3.1 (iii), there exist infinitely many integers  $k$  such that  $r_k \equiv s+d+1 \pmod{n}$  and simultaneously  $h(k) = d$ . For each such  $k$  which is large enough one has  $u_{r_k} = \frac{1}{\varepsilon_1 \cdots \varepsilon_k} e_{r_k}$ . Note that  $u_{r_{k-1}} = \frac{1}{\varepsilon_1 \cdots \varepsilon_{k-1}} (e_{r_{k-1}} + \frac{1}{\sqrt{\varepsilon_k} \|u_{h(k)}\|} u_{h(k)})$ . Since, by the assumptions,  $u_{h(k)} = u_d$  is annihilated by  $\tilde{B}$ , we have

$$(3.9) \quad \|\tilde{B}u_{r_{k-1}}\| = \frac{1}{\varepsilon_1 \cdots \varepsilon_{k-1}} \|\tilde{B}e_{r_{k-1}}\| \leq \frac{\|\tilde{B}\|}{\varepsilon_1 \cdots \varepsilon_{k-1}}.$$



On the other hand,

$$\begin{aligned} \|\tilde{B}u_{r_k-1}\| &\geq |\langle \tilde{B}u_{r_k-1}, e_0 \rangle| = \left| \langle \tilde{b}_{(r_k-1-d)(r_k-1)} u_{r_k-1-d} + \sum_{t=0}^{r_k-2-d} \tilde{b}_{t(r_k-1)} u_t, e_0 \rangle \right| \\ &= \left| \langle b u_{r_k-1-d} + \sum_{t=0}^{r_k-2-d} \tilde{b}_{t(r_k-1)} u_t, e_0 \rangle \right|, \end{aligned}$$

where the last equality holds because of  $r_k - 1 - d \equiv s \pmod{n}$ . Moreover, for  $r_{k-1} \leq t \leq r_k - 2 - d$ , one has  $u_t = T^{r_k-t} u_{r_k} = T^{r_k-t-1} u_{r_{k-1}} \in T^{r_k-t-1} (\mathbb{C}e_{r_{k-1}} + \mathcal{H}_{d_0}) \subseteq \mathbb{C}e_t$ , which gives  $\langle u_t, e_0 \rangle = 0$ . Furthermore,  $\langle T^d u_{r_k-1}, e_0 \rangle = \frac{1}{\varepsilon_1 \cdots \varepsilon_{k-1} \sqrt{\varepsilon_k} \|u_{d_0}\|}$ . Hence

$$\begin{aligned} \|\tilde{B}u_{r_k-1}\| &\geq \left| \langle b T^d u_{r_k-1} + \sum_{t=0}^{r_{k-1}-1} \tilde{b}_{t(r_k-1)} u_t, e_0 \rangle \right| \\ &\geq \frac{|b|}{\varepsilon_1 \cdots \varepsilon_{k-1} \sqrt{\varepsilon_k} \|u_{d_0}\|} - \sum_{t=0}^{r_{k-1}-1} |\tilde{b}_{t(r_k-1)}| \|u_t\| \\ &\geq \frac{|b|}{\varepsilon_1 \cdots \varepsilon_{k-1} \sqrt{\varepsilon_k} \|u_{r_{k-1}}\|} - r_{k-1} \|u_{r_{k-1}}\| \max_{0 \leq j \leq s < r_k} |\tilde{b}_{j_s}| \\ &\geq \frac{|b|}{\sqrt{\varepsilon_k}} - r_{k-1} \frac{1}{\varepsilon_{k-1}} \frac{1}{\varepsilon_1 \cdots \varepsilon_{k-1}}. \end{aligned}$$

Combined with (3.9) we get

$$\|\tilde{B}\| \geq \varepsilon_1 \cdots \varepsilon_{k-1} \|\tilde{B}u_{r_k-1}\| \geq \frac{|b| \varepsilon_1 \cdots \varepsilon_{k-1}}{\sqrt{\varepsilon_k}} - \frac{r_{k-1}}{\varepsilon_{k-1}} \geq \frac{|b| \varepsilon_1 \cdots \varepsilon_{k-1}}{\sqrt{\varepsilon_k}} - \frac{1}{\varepsilon_{k-1}^2}.$$

We may assume that  $\varepsilon_k \leq \left( \frac{\varepsilon_1 \cdots \varepsilon_{k-1}}{k(2^k + \varepsilon_{k-1}^2)} \right)^2$  and, for  $k \geq \frac{1}{|b|}$ , we obtain that

$$\|\tilde{B}\| \geq \frac{|b| \varepsilon_1 \cdots \varepsilon_{k-1}}{\sqrt{\varepsilon_k}} - \frac{1}{\varepsilon_{k-1}^2} \xrightarrow{k \rightarrow \infty} \infty,$$

provided that  $\{\varepsilon_n\}_{n=1}^\infty$  decreases to 0 sufficiently fast. This gives a contradiction.  $\square$

*Proof of Theorem 2.3.* Let  $A \in \{T\}'$  be a non-scalar operator and let  $B \in \{A\}'$ . By Lemma 3.4,  $\mathcal{H}_\infty$  is invariant for  $A$  and there exists a sequence  $\{c_j\}_{j=0}^\infty \subseteq \mathbb{C}$  such that  $A|_{\mathcal{H}_\infty} = \sum_{j=0}^\infty c_j T^j|_{\mathcal{H}_\infty}$ .

There is no loss of generality if we assume that  $c_0 = 0$  and the first nonzero number in the sequence, say  $c_n$ , is 1. Thus  $Ax = T^n x + \sum_{j=n+1}^\infty c_j T^j x$  for every  $x \in \mathcal{H}_\infty$ . By Lemma 3.5,  $\mathcal{H}_\infty$  is

invariant for  $B$  and, by Lemma 3.6 (ii), the matrix  $[b_{ij}]_{i,j=0}^\infty$  which corresponds to it with respect to the basis  $\{u_j\}_{j=0}^\infty$  is upper-triangular. Furthermore, each diagonal of  $[b_{ij}]_{i,j=0}^\infty$  has constant entries. Indeed, if there would exist the minimal integer  $m \geq 0$  such that the diagonal  $\{b_{j(j+m)}\}_{j=0}^\infty$

is nonconstant, then  $B - \sum_{i=0}^{m-1} b_{0i} T^i$  would be an operator in  $\{A\}'$  with its lowest nonzero diagonal

nonconstant, which is contradicting to Lemma 3.9. Hence,  $B|_{\mathcal{H}_\infty} = \sum_{i=0}^\infty b_{0i} T^i|_{\mathcal{H}_\infty}$ . Consequently,

$BT = TB$  and  $\{A\}' \subseteq \{T\}'$ . Since the commutant of  $T$  is commutative we have  $\{A\}' = \{T\}'$ .  $\square$

As a counterpart to Theorem 2.3, we state the following proposition.

Let  $T$  be a completely nonunitary contraction on  $\mathcal{H}$  such that its spectrum is connected and contains 1. By Foiaş, Percy, Sz.-Nagy [9, Corollary 3], there exists a universal function  $f_0 \in H^\infty$ ,  $\|f_0\|_\infty = 1$ , such that for every such  $T$ ,  $\sigma(f_0(T)) = \overline{\mathbb{D}}$ . Clearly,  $f_0$  is not rational. Then  $A = f_0(T)$  has an  $H^\infty$  functional calculus ([12]) which is isometric as  $\sigma(A)$  is dominant (see Chevreau, Li [8]) and maps surjectively onto the weakly closed algebra  $\mathcal{A}$  which is generated by  $A$ . This algebra turns out to be reflexive and dual (see Bercovici, Foiaş, Percy [3] and Brown, Chevreau [5]) and isomorphic to  $H^\infty$  (see Brown, Chevreau, Percy [6]). Since  $\mathcal{A}$  is a dual algebra it has, by Blecher, Solel [4], a completely isometric, normal, weak\*-weak\* continuous representation  $\iota : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\iota)$  on a Hilbert space  $\mathcal{H}_\iota$  such that  $\iota(\mathcal{A})'' = \iota(\mathcal{A})$ . Whenever this holds for the identical representation  $\iota : \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$ , namely when  $\mathcal{A}'' = \mathcal{A}$ , we obtain  $\{A\}' \not\subseteq \{T\}'$  as it is shown below. Recall that  $r(A) = \max\{|z|; z \in \sigma(A)\}$  is the spectral radius of  $A \in \mathcal{B}(\mathcal{H})$ .

**Proposition 3.10.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be a non-scalar completely nonunitary contraction with  $r(T) = \|T\| = 1$ . Let  $f_0 \in H^\infty$  be any non-rational function such that  $\|f_0\|_\infty \leq 1$  and  $\sigma(f_0(T)) = \overline{\mathbb{D}}$ . Set  $A = f_0(T) \in \{T\}' \setminus \mathbb{C}I$ . Suppose that the unital, weakly closed algebra  $\mathcal{A}$  which is generated by  $A$  satisfies  $\mathcal{A}'' = \mathcal{A}$ . Then there exists an operator  $B \in \mathcal{B}(\mathcal{H})$  such that  $AB = BA$  and  $BT \neq TB$ .*

*Proof.* By [12, Theorem III.2.1],  $T$  has an  $H^\infty$  functional calculus  $\phi_T : H^\infty \rightarrow \mathcal{B}(\mathcal{H})$  such that for every  $f \in H^\infty$  with  $|f(z)| < 1$  ( $z \in \mathbb{D}$ ) the operator  $f(T) = \phi_T(f)$  is also completely nonunitary and  $g(f(T)) = (g \circ f)(T)$  for every  $g \in H^\infty$ . We may assume that  $\phi_T$  is injective — in the noninjective case see the discussion on operators of class  $\mathcal{C}_0$  in Theorem 2.4. As it was mentioned above, by [3, 5, 6, 8],  $\mathcal{A}$  is a dual algebra isomorphic to  $H^\infty$  via the map  $\phi_A : H^\infty \rightarrow \mathcal{A}$  that is isometric (and hence, injective) and onto. If the inclusion  $\{T\}' \subseteq \{A\}'$  is strict, then any  $B \in \{A\}' \setminus \{T\}'$  satisfies the conclusion. Assume therefore that  $\{T\}' = \{A\}'$ , which gives  $\{T\}'' = \{A\}''$ , of course. We will show that this is impossible. Since  $\{A\}'' = \mathcal{A}''$  and  $\mathcal{A}'' = \mathcal{A}$ , we derive  $\{T\}'' = \mathcal{A}$ . For any  $g \in H^\infty$ ,  $\phi_A(g) = g(A) = g(f_0(T)) = (g \circ f_0)(T) = \phi_T(g \circ f_0)$  gives the inclusion of the images  $\text{Im } \phi_A \subseteq \text{Im } \phi_T$ . Since  $\mathcal{A} = \text{Im } \phi_A$  we have  $\mathcal{A} \subseteq \text{Im } \phi_T$ . It is easily seen that  $\text{Im } \phi_T \subseteq \{T\}''$ . Then, because of  $\{T\}'' = \mathcal{A}$ , we get the inclusion  $\text{Im } \phi_T \subseteq \mathcal{A}$ , too. Thus  $\mathcal{A} = \text{Im } \phi_T$ . The mapping  $\alpha = (\phi_T)^{-1} \circ \phi_A : H^\infty \rightarrow H^\infty$  is well defined since  $\text{Im } \phi_A = \text{Im } \phi_T$  and  $\phi_T$  is injective. Moreover  $\alpha$  is surjective, and also injective because  $\phi_A$  is injective. Since  $\phi_A$  is isometric and  $\alpha^{-1} = \phi_A^{-1} \circ \phi_T$ , the map  $\alpha^{-1}$  is continuous. Thus both  $\alpha$  and  $\alpha^{-1}$  are linear, continuous, unital, and, moreover, multiplicative since  $\phi_T$  and  $\phi_A$  are multiplicative. Therefore,  $\alpha$  is an automorphism of  $H^\infty$ . By Rudin [14, Theorem 6.6.5, Corollary 2], there is an automorphism  $\mu$  of  $\mathbb{D}$ , namely a scalar multiple of a Möbius transform,  $\mu(z) = e^{it} \frac{z-a}{1-\bar{a}z}$  ( $z \in \mathbb{D}$ ), such that  $\alpha g = g \circ \mu$  for every  $g \in H^\infty$ . Function  $h = \alpha g$  can be computed also from the equation  $\phi_T(h) = \phi_A(g)$ , that is,  $\phi_T(h) = \phi_T(g \circ f_0)$ : since  $\phi_T$  is injective  $h = g \circ f_0$ ; therefore  $\alpha g = g \circ f_0$ . Then  $g \circ \mu = g \circ f_0$  for every  $g$ , whence  $f_0 = \mu$ , a contradiction.  $\square$

**Remark 3.11.** Comparing the properties of the operator  $T$  which has been constructed in Theorem 2.3 with the conditions of Proposition 3.10 we see that for  $T$  equality  $r(T) = \|T\|$  cannot hold. We can prove this directly.

First we prove that  $\|T\| > 1$ . To this aim, let  $j_0 = 5$  (the reasoning holds as well for any  $j_0 \geq 5$  with  $j_0 \in \mathbb{N} \setminus \{r_k\}_{k=1}^\infty$ ). By (3.1),  $r_1 = 4$ ,  $r_2 > 16$  and so  $r_1 < j_0 < r_2$ . Hence, by (3.2a), we have  $Te_5 = e_4$ . By Lemma 3.1 (iii), there exists an integer  $k \in \mathbb{N}$  such that  $h(k) = r_1$ . Note that  $r_k \neq j_0$ . By (3.2b) and (3.2c), we derive that  $Te_{r_k} = \varepsilon_k e_{r_k-1} + \sqrt{\varepsilon_k} e_{r_1}$ . Now let  $\delta = \delta_k > 0$  such that  $\delta < 2\sqrt{\varepsilon_k}$  and compare the norms of the vectors  $e_5 + \delta e_{r_k}$  and  $T(e_5 + \delta e_{r_k})$ . Since  $r_k \neq 5$  one has  $e_{r_k} \perp e_5$ ,  $e_{r_k-1} \perp e_4$  and hence  $\|e_5 + \delta e_{r_k}\|^2 = 1 + \delta^2$ , while  $\|T(e_5 + \delta e_{r_k})\|^2 = \|e_4 + \delta(\varepsilon_k e_{r_k-1} + \sqrt{\varepsilon_k} e_{r_1})\|^2 = 1 + 2\sqrt{\varepsilon_k}\delta + \varepsilon_k\delta^2 + \varepsilon_k\delta^2 > 1 + 2\sqrt{\varepsilon_k}\delta > 1 + \delta^2$ . Therefore  $\|T\| > 1$ .

Now we prove that  $r(T) \leq 1$ . Indeed, if  $x = \sum_{j=0}^n \alpha_j e_j \in \mathcal{H}_\infty$  has norm 1, then using  $h(k) < k$  and  $r_\ell > 4^\ell$  we get estimates for  $\|(T - S)^k x\|$ , where  $S$  is the shift part of  $T$  (see the paragraph before Lemma 3.2). For  $k = 2$ , one has

$$\begin{aligned} \|(T - S)^2 x\| &\leq \sum_{j=0}^n |\alpha_j| \|(T - S)^2 e_j\| \leq \sum_{k=1}^\infty \|(T - S)^2 e_{r_k}\| = \sum_{k=1}^\infty \sqrt{\varepsilon_k} \left\| (T - S) \frac{u_{h(k)}}{\|u_{h(k)}\|} \right\| \\ &= \sum_{\substack{k, \ell \geq 1 \\ h(k) = r_\ell}} \sqrt{\varepsilon_k} \sqrt{\varepsilon_\ell} \left\| \frac{u_{h(k)}}{\|u_{h(k)}\|} \right\| \leq \sum_{\substack{k, \ell \geq 1 \\ k > 4^\ell}} \sqrt{\varepsilon_\ell} \sqrt{\varepsilon_k} \leq \sum_{\ell \geq 1} \sqrt{\varepsilon_\ell} \sqrt{\varepsilon_{4^\ell}}, \end{aligned}$$

and, for  $k = 3$ , we estimate

$$\|(T - S)^3 x\| \leq \sum_{\ell \geq 1} \sqrt{\varepsilon_\ell} \sqrt{\varepsilon_{4^\ell}} \sqrt{\varepsilon_{4^{4^\ell}}}.$$

For larger  $k$  we get similar estimates. Since the sequence  $\{\varepsilon_n\}_{n=1}^\infty$  is decreasing very fast we obtain that for any  $\delta > 0$  there exists an integer  $k_\delta \geq 1$  such that  $\|(T - S)^k\| \leq \delta^k$ , for all  $k \geq k_\delta$ . We derive, using also the fact  $\|S\| \leq 1$ , that

$$\|T^k\| \leq \sum_{i=0}^k \binom{k}{i} \|S^i (T - S)^{k-i}\| \leq \sum_{i=0}^k \binom{k}{i} \|(T - S)^{k-i}\| \leq (1 + \delta)^k, \quad (k \geq k_\delta).$$

Hence  $r(T) = \lim_{k \rightarrow \infty} \|T^k\|^{1/k} \leq 1 + \delta$ . Since  $\delta$  was arbitrary we have  $r(T) \leq 1$ .

#### 4. REALIZABILITY OF GRAPHS

Does there exist a set of operators that satisfy any in advance given relations induced by commutativity? More precisely, for a given simple finite graph, is it possible to represent its vertices by operators such that two operators commute if and only if the corresponding vertices form an edge? In the graphological terminology this amounts to find an induced subgraph of a commuting graph of operators which is isomorphic to a given graph. Recall that a subgraph  $\hat{\Gamma}$  of a graph  $\Gamma$  is called an induced subgraph if for any pair of vertices  $x, y \in \hat{\Gamma}$ , there is an edge between them in  $\hat{\Gamma}$  precisely when there is an edge between them already in  $\Gamma$ . We say that

a graph  $\Gamma$  is realizable in  $\mathcal{B}(\mathcal{H})$  if  $\Gamma$  is (isomorphic to) an induced subgraph of  $\Gamma(\mathcal{B}(\mathcal{H}))$ , the commuting graph of  $\mathcal{B}(\mathcal{H})$ .

Let us first show that any finite simple graph can be realized by sufficiently large matrices. Actually the proof establishes more: every graph with countably many vertices can be realized in  $\mathcal{B}(\ell^2)$ .

**Proposition 4.1.** *Every finite simple graph  $\Gamma$  is realizable in  $\mathcal{B}(\mathcal{H})$ , where  $\mathcal{H}$  is a Hilbert space of sufficiently large dimension. Realization can be done with orthogonal projections.*

*Proof.* Clearly, complete graphs are isomorphic to subgraphs of  $\mathbb{C}A$  for any given non-scalar operator  $A$ , so they are realizable. In the sequel, we assume that  $\Gamma$  is not complete.

Let  $v_1, \dots, v_m$  be all the vertices of  $\Gamma$ , and let  $E(\Gamma)$  be its edge set. For each  $(i, j) \notin E(\Gamma)$ ,  $i \neq j$ , let  $\mathcal{H}^{(i,j)} = \mathbb{C}^2$  and  $e_1, e_2$  be its standard basis. Denote  $E_{k\ell} = e_k \otimes e_\ell$  ( $k, \ell \in \{1, 2\}$ ). Consider a tuple of  $m$  operators  $T_1^{(i,j)}, \dots, T_m^{(i,j)} : \mathcal{H}^{(i,j)} \rightarrow \mathcal{H}^{(i,j)}$  which is defined by

$$T_\ell^{(i,j)} = \begin{cases} E_{11}; & \ell = i \\ \frac{1}{2}(E_{11} + E_{12} + E_{21} + E_{22}); & \ell = j \\ I; & \ell \notin \{i, j\} \end{cases}.$$

Observe that  $T_i^{(i,j)} = E_{11}$  does not commute with  $T_j^{(i,j)} = \frac{1}{2}(E_{11} + E_{12} + E_{21} + E_{22})$ . Let  $\mathcal{H} = \mathbb{C} \oplus \bigoplus_{\substack{(i,j) \notin E(\Gamma) \\ i \neq j}} \mathcal{H}^{(i,j)}$  and, for  $k = 1, \dots, m$ , define  $T_k = 0 \oplus \bigoplus_{\substack{(i,j) \notin E(\Gamma) \\ i \neq j}} T_k^{(i,j)} \in \mathcal{B}(\mathcal{H})$ . Since  $\Gamma$  is

not complete, there is at least one pair  $(i, j)$  which is not in  $E(\Gamma)$ , which means that each  $T_k$  is a non-scalar operator on  $\mathcal{H}$ . Now, if  $(k, \ell) \notin E(\Gamma)$ ,  $k \neq \ell$ , then the  $(k, \ell)$ -th components of  $T_k$  and of  $T_\ell$  do not commute because they equal to  $T_k^{(k,\ell)} = E_{11}$  and  $T_\ell^{(k,\ell)} = \frac{1}{2}(E_{11} + E_{12} + E_{21} + E_{22})$ , respectively. In particular,  $T_k$  does not commute with  $T_\ell$ . On the other hand, if  $(k, \ell) \in E(\Gamma)$ , then it is easy to see that  $T_k T_\ell = T_\ell T_k$ .  $\square$

If the dimension of  $\mathcal{H}$  is kept fixed, then not every finite simple graph is realizable. We provide a simple example where the main obstruction lies in lack of sufficiently many linearly independent non-scalar matrices.

**Example 4.2** (Non-realizability by small matrices). Consider a graph  $\Gamma$  which is a complement of a line of length  $n^2 + 1$ , that is,  $\Gamma$  is a complete graph on  $n^2 + 2$  vertices  $v_1, \dots, v_{n^2+2}$ , with all edges  $v_i, v_{i+1}$  ( $i = 1, \dots, n^2 + 1$ ) removed. We claim that  $\Gamma$  is not realizable in  $M_n(\mathbb{C})$ . Indeed if it was let  $X_1, \dots, X_{n^2+2} \in M_n(\mathbb{C})$  be the matrices corresponding to vertices  $v_1, \dots, v_{n^2+2}$ . Since there is no edge between  $v_1, v_2$  we see that  $X_1, X_2$  do not commute and therefore are linearly independent. Inductively, since there is no edge between  $v_{i+1}$  and  $v_i$ , but  $v_{i+1}$  connects to every vertex  $v_1, \dots, v_{i-1}$  we see that  $X_{i+1}$  commutes with every  $X_1, \dots, X_{i-1}$  but not with  $X_i$ . Hence  $X_i$  is not a linear combination of  $X_1, \dots, X_{i-1}$  ( $i = 1, \dots, n^2 + 1$ ). Thus,  $X_1, \dots, X_{n^2+1}$  are linearly independent matrices inside  $M_n(\mathbb{C})$ , which is impossible.

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MATHEMATICAL INSTITUTE, CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, 115 67 PRAGUE 1, CZECH REPUBLIC  
or MATHEMATICAL INSTITUTE, BUCHAREST, P.O. BOX 1-764, RO-014700 ROMANIA

*E-mail address:* ambrozie@math.cas.cz

UNIVERSITY OF LJUBLJANA, IMFM, JADRANSKA UL. 19, 1000 LJUBLJANA, SLOVENIA

*E-mail address:* janko.bracic@fmf.uni-lj.si

UNIVERSITY OF PRIMORSKA, GLAGOLJAŠKA 8, 6000 KOPER, SLOVENIA or IMFM, JADRANSKA UL. 19, 1000 LJUBLJANA, SLOVENIA

*E-mail address:* bojan.kuzma@famnit.upr.si

MATHEMATICAL INSTITUTE, CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, 115 67 PRAGUE 1, CZECH REPUBLIC

*E-mail address:* muller@math.cas.cz