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On joint numerical radius

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ON JOINT NUMERICAL RADIUS

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Abstract. Let $T_1, \dots, T_n$ be bounded linear operators on a complex Hilbert space H. We study the question whether it is possible to find a unit vector $x \in H$ such that $|\langle T_j x, x\rangle|$ is large for all j. Thus we are looking for a generalization of a well-known fact for $n = 1$ that the numerical radius $w(T)$ of a single operator T satisfies $w(T) \geq \|T\|/2$.

1. Introduction

Let H be a complex Hilbert space. Denote by $B(H)$ the set of all bounded linear operators on H. The numerical range of an operator $T \in B(H)$ is defined by

$$W(T) = \bigl\{\langle Tx, x\rangle : x \in H, \|x\| = 1\bigr\}$$

and the numerical radius by

$$w(T) = \sup\bigl\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\bigr\} = \sup\bigl\{|\lambda| : \lambda \in W(T)\bigr\}.$$

It is well known that $W(T)$ is a convex subset of the complex plane $\mathbb{C}$. Moreover,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1}$$

The second inequality in (1) is trivial, the first one is less obvious and more interesting. It means that for each $T \in B(H)$ and each $\varepsilon > 0$ there exists a unit vector $x \in H$ such that

$$|\langle Tx, x\rangle| \geq \frac{1}{2}\|T\| - \varepsilon$$

(if $\dim H < \infty$ then there exists a unit vector $x \in H$ with $|\langle Tx, x\rangle| \geq \frac{\|T\|}{2}$ since the numerical range $W(T)$ is closed in this case). Note also that for real Hilbert spaces the first inequality in (1) is not true (consider the matrix $\bigl(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\bigr)$).

Let $T_1, \dots, T_n \in B(H)$ be an n-tuple of operators. The joint numerical range of $T_1, \dots, T_n$ is the subset of $\mathbb{C}^n$ defined by

$$W(T_1, \dots, T_n) = \bigl\{\bigl(\langle T_1 x, x\rangle, \dots, \langle T_n x, x\rangle\bigr) : x \in H, \|x\| = 1\bigr\}.$$

The aim of this paper is to study the following question:

Problem 1. Does there exists a unit vector $x \in H$ such that $|\langle T_j x, x\rangle|$ is "large" for all $j = 1, \dots, n$?

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Since each operator T_j can be written as $T_j = A_j + iB_j$ with selfadjoint operators $A_j = \frac{1}{2}(T_j + T_j^*)$ and $B_j = \frac{1}{2i}(T_j - T_j^*)$ and

$$|\langle T_j x, x\rangle| = \bigl|\langle A_j x, x\rangle + i\langle B_j x, x\rangle\bigr| \geq \max\bigl\{|\langle A_j x, x\rangle|, |\langle B_j x, x\rangle|\bigr\},$$

Problem 1 can be reduced to the case of n-tuples of selfadjoint operators. Moreover, it is possible to consider only finite-dimensional spaces, since

$$W(T_1,\dots,T_n) = \bigcup_P W(PT_1P,\dots,PT_nP)$$

where P runs over all finite-rank orthogonal projection (in fact, it is sufficient to consider only projections of rank $\leq n+1$).

If the operators T_j are not only selfadjoint but also positive semidefinite, then it is possible to reduce the problem to the corresponding question for the norms (even for infinitely many operators).

Theorem 2. Let $T_1, T_2, \dots \in B(H)$ be positive semidefinite operators, let $c_j \geq 0$ satisfy $\sum_{j=1}^\infty c_j < 1$. Then there exists a unit vector $x \in H$ such that

$$\langle T_j x, x\rangle \geq c_j \|T_j\|$$

for all $j \in \mathbb{N}$.

Proof. By [M], p.334 for the square roots $T_j^{1/2}$ there exists a unit vector $x \in H$ such that

$$\|T_j^{1/2} x\| \geq \sqrt{c_j}\|T_j^{1/2}\|$$

for all j. So

$$\langle T_j x, x\rangle = \|T_j^{1/2}x\|^2 \geq c_j\|T_j^{1/2}\|^2 = c_j\|T_j\|$$

for all $j \in \mathbb{N}$. □

Corollary 3. Let $T_1,\dots,T_n \in B(H)$ be positive semidefinite operators, let $\varepsilon > 0$. Then there exists a unit vector $x \in H$ such that

$$\langle T_j x, x\rangle \geq \frac{1}{n}\|T_j\| - \varepsilon$$

for all $j = 1,\dots,n$.

If the operators T_j are not positive semidefinite but only selfadjoint then the situation is more complicated. We give an exact answer for $n = 2$ and $n = 3$. The main result of Section 2 will be

Theorem 4. Let $T_1, T_2, T_3 \in B(H)$ be selfadjoint operators and $\varepsilon > 0$. Then:

(i) there exists a unit vector $x \in H$ such that

$$|\langle T_j x, x\rangle| \geq \frac{1}{3}\|T_j\| - \varepsilon \qquad (j = 1, 2);$$

(ii) there exists a unit vector $y \in H$ such that

$$|\langle T_j y, y\rangle| \geq \frac{1}{5}\|T_j\| - \varepsilon \qquad (j = 1, 2, 3).$$

The estimates in Theorem 4 are the best possible.

For $n \geq 4$ the situation is more complicated. Among other technical difficulties, the joint numerical range of an n-tuple of selfadjoint operators is in general not convex. For $n \geq 4$ we give only some estimates how large values of $|\langle T_j x, x\rangle|$ in Problem 1 can be obtained.

The results can be also applied to other types of numerical ranges — the essential numerical range and the algebraic numerical range of n-tuples of elements in a unital Banach algebra.

2. Cases $n = 2, 3$

Let $T_1, T_2, T_3 \in B(H)$ be selfadjoint operators. The numerical range $W(T_1, T_2)$ is always a convex set — it reduces to the convexity of the numerical range of a single operator $W(T_1 + iT_2)$. If $\dim H \geq 3$ then the numerical range $W(T_1, T_2, T_3)$ is also convex, see e.g. [AT], [FT], [GJK]. The convexity may be used for solving Problem 1.

For $u \in \mathbb{R}^n$ we write $u = (u_1, \dots, u_n)$.

Lemma 5. Let $K \subset [-1, 1]^2$ be a convex set, let $u, v \in K$ satisfy $u_1 = 1 = v_2$. Then there exists $w \in K$ such that $|w_1| \geq 1/3$ and $|w_2| \geq 1/3$.

Proof. If $u_2 < -1/3$ then set $w = u$.

If $v_1 < -1/3$ then set $w = v$.

If both $u_2 \geq -1/3$ and $v_1 \geq -1/3$ then $w := \frac{u+v}{2}$ satisfies

$$w_1 = \frac{u_1 + v_1}{2} = \frac{1 + v_1}{2} \geq 1/3$$

and similarly,

$$w_2 = \frac{u_2 + v_2}{2} = \frac{u_2 + 1}{2} \geq 1/3.$$

□

Lemma 6. Let $K \subset [-1, 1]^3$ be a convex set, let $u, v, w \in K$ satisfy $u_1 = v_2 = w_3 = 1$. Then there exists $x = (x_1, x_2, x_3) \in K$ such that $|x_j| \geq 1/5 \quad (j = 1, 2, 3)$.

Proof. Let

$$M = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_j| \geq 1/5 \quad (j = 1, 2, 3)\big\}.$$

Suppose on the contrary that $K \cap M = \emptyset$. Consider the matrix

$$\begin{pmatrix} 1 & u_2 & u_3 \\ v_1 & 1 & v_3 \\ w_1 & w_2 & 1 \end{pmatrix}. \tag{2}$$

Since $u, v, w \notin M$, in each row of matrix (2) there exists an entry with modulus $< 1/5$ (we call such entries small).

We distinguish two cases:

A. There exists a column of (2) with two small entries.

Without loss of generality we may assume that $|u_3| < 1/5$ and $|v_3| < 1/5$. Moreover, either w_1 or w_2 is small; without loss of generality we may assume that $|w_2| < 1/5$.

Let $a=\frac{v+w}{2}$. We have $|a_2|=|\frac{1+w_2}{2}|\ge\frac{1-1/5}{2}\ge 1/5$ and $|a_3|=\left|\frac{v_3+1}{2}\right|\ge 1/5$. So $|a_1|=\left|\frac{v_1+w_1}{2}\right|<1/5$ and

$$|v_1+w_1|<\frac{2}{5}. \tag{3}$$

Let $b=\frac{u+v+w}{3}\in K$. Then $|b_3|=\left|\frac{u_3+v_3+1}{3}\right|\ge 1/5$ and, by (3), $b_1=\frac{1+v_1+w_1}{3}\ge\frac{1-2/5}{3}=\frac{1}{5}$. So $|b_2|=|\frac{1+u_2+w_2}{3}|<1/5$ and

$$u_2+w_2<-\frac{2}{5}. \tag{4}$$

Finally, let $x=\frac{2u+w}{3}\in K$. We have $|x_1|=|\frac{2+w_1}{3}|\ge\frac{1}{3}>\frac{1}{5}$, $|x_2|=|\frac{2u_2+w_2}{3}|\ge\frac{1}{3}(|2u_2+2w_2|-|w_2|)\ge\frac{1}{3}(\frac{4}{5}-\frac{1}{5})=\frac{1}{5}$ by (4), and $|x_3|=|\frac{2u_3+1}{3}|\ge\frac{1}{5}$.

So $x\in M$, a contradiction.

Case B. In each column there is one small entry.

Without loss of generality we may assume that $|u_2|<1/5$, $|v_3|<1/5$ and $|w_1|<1/5$.

Consider the vector $a=\frac{2u+v}{3}\in K$. Then $a_1=\frac{2+v_1}{3}\ge\frac{2-1}{3}=\frac{1}{3}>\frac{1}{5}$ and $a_2=\frac{2u_2+1}{3}>\frac{1-2/5}{3}=\frac{1}{5}$. So $|a_3|=|\frac{2u_3+v_3}{3}|<\frac{1}{5}$ and

$$|u_3|\le\frac{1}{2}\big(|2u_3+v_3|+|v_3|\big)<\frac{1}{2}\Big(\frac{3}{5}+\frac{1}{5}\Big)=\frac{2}{5}.$$

Symmetrically, $|v_1|<\frac{2}{5}$ and $|w_2|<\frac{2}{5}$.

Let $b=\frac{u+v}{2}\in K$. Then $b_1=\frac{1+v_1}{2}>\frac{1-2/5}{2}>\frac{1}{5}$ and $b_2=\frac{u_2+1}{2}>\frac{1}{5}$. So $|b_3|=|\frac{u_3+v_3}{2}|<\frac{1}{5}$ and

$$|u_3+v_3|<\frac{2}{5}. \tag{5}$$

Symmetrically, $|v_1+w_1|<\frac{2}{5}$ and $|u_2+w_2|<\frac{2}{5}$.

Let $x=\frac{u+v+w}{3}\in K$. Then $x_1=\frac{1+v_1+w_1}{3}>\frac{1-2/5}{3}=\frac{1}{5}$, and similarly, $x_2>\frac{1}{5}$, $x_3>\frac{1}{5}$. Hence $x\in M$, a contradiction. □

Lemmas 5 and 6 are particular cases of the following conjecture:

Conjecture 7. Let $n\in\mathbb{N}$ and let $K\subset[-1,1]^n$ be a convex set. Let $u_j=(u_{j1},\dots,u_{jn})\in K$ satisfy $u_{jj}=1\quad(j=1,\dots,n)$. Then there exists $v=(v_1,\dots,v_n)\in K$ such that $|v_j|\ge\frac{1}{2n-1}\quad(j=1,\dots,n)$.

Conjecture 7 is a particular case of the famous still open plank problem [B], whether a bounded convex subset of $\mathbb{R}^n$ can be covered by a finite number of planks such that the sum of their relative widths is less than 1. For details see [Ba].

The estimate $\frac{1}{2n-1}$ in Conjecture 7 cannot be improved as the following example shows:

Example 8. Let $n\in\mathbb{N}$ and let $u_j=(u_{j1},\dots,u_{jn})\in\mathbb{R}^n$ be defined by $u_{jj}=1\quad(j=1,\dots,n)$, $u_{jk}=\frac{-1}{2n-1}\quad(j,k=1,\dots,n,j\ne k)$. Let K be the convex hull of the vectors $u_1,\dots,u_n$.

Let $v\in K$, $v=(v_1,\dots,v_n)$ be an arbitrary vector. Then $v=\sum_{j=1}^n\alpha_ju_j$ for some $\alpha_j\ge 0$, $\sum_{j=1}^n\alpha_j=1$. So there exists $k\in\{1,\dots,n\}$ such that $\alpha_k\le\frac{1}{n}$. Then $v_k=\sum_{j=1}^n\alpha_ju_{jk}=\alpha_k+(1-\alpha_k)\frac{-1}{2n-1}=\alpha_k\big(1+\frac{1}{2n-1}\big)-\frac{1}{2n-1}$. So

$$\frac{-1}{2n-1}\le v_k\le\frac{1}{n}\Big(1+\frac{1}{2n-1}\Big)-\frac{1}{2n-1}=\frac{1}{2n-1}.$$

So $\min_{1\le k\le n}|v_k|\le \frac{1}{2n-1}$ for each $v\in K$.

Lemmas 5 and 6 imply the following statement about the joint numerical radius mentioned in Introduction.

Theorem 9. Let $\dim H<\infty$, let $T_1,T_2,T_3\in B(H)$ be selfadjoint operators. Then:

(i) there exists a unit vector $x\in H$ such that

$$|\langle T_jx,x\rangle|\ge \frac{1}{3}\|T_j\| \qquad (j=1,2);$$

(ii) there exists a unit vector $y\in H$ such that

$$|\langle T_jy,y\rangle|\ge \frac{1}{5}\|T_j\| \qquad (j=1,2,3).$$

Proof. (i) If $\|T_j\|\in\sigma(T_j)$ then set $A_j=\frac{T_j}{\|T_j\|}$. If $-\|T_j\|\in\sigma(T_j)$ then set $A_j=\frac{-T_j}{\|T_j\|}$. Then $\|A_j\|=1$ and $1\in\sigma(A_j)\subset W(A_j)$. So there exist unit vectors $x_j\in H$ such that $\langle A_jx_j,x_j\rangle = 1 \quad (j=1,2)$. Consider the convex set $W(A_1,A_2)$ and elements

$$\bigl(\langle A_1x_1,x_1\rangle,\langle A_2x_1,x_1\rangle\bigr),\ \bigl(\langle A_1x_2,x_2\rangle,\langle A_2x_2,x_2\rangle\bigr)\in W(A_1,A_2).$$

By Lemma 5, there exists a unit vector $x\in H$ such that $|\langle A_jx,x\rangle|\ge\frac13 \quad (j=1,2)$ and so $|\langle T_1x,x\rangle|\ge\frac{\|T_j\|}{3} \quad (j=1,2)$.

(ii) If $\dim H\ge 3$ then $W(T_1,T_2,T_3)$ is a convex set and the statement can be proved similarly as above using Lemma 6 instead of Lemma 5. If $\dim H=1$ then the statement is trivial.

Suppose that $\dim H=2$. Let $\tilde H=H\oplus\mathbb{C}$ and $\tilde T_j=T_j\oplus 0\in B(\tilde H) \quad (j=1,2,3)$.

It is easy to see that $W(\tilde T_1,\tilde T_2,\tilde T_3)=\{t\mu: 0\le t\le 1,\mu\in W(T_1,T_2,T_3)\}$. We have proved that there exists $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in W(\tilde T_1,\tilde T_2,\tilde T_3)$ with $|\lambda_j|\ge\frac{\|\tilde T_j\|}{5}=\frac{\|T_j\|}{5} \quad (j=1,2,3)$. It is easy to see that there exists $\mu\in W(T_1,T_2,T_3)$ with $|\mu_j|\ge\frac{\|T_j\|}{5} \quad (j=1,2,3)$. □

These estimates are the best possible.

Example 10. Let $n\in\mathbb{N}$, let $\dim H=n$, let $T_1,\dots,T_n\in B(H)$ be the diagonal matrices

$$\begin{aligned} T_1&=\operatorname{diag}\Bigl(1,\frac{-1}{2n-1},\dots,\frac{-1}{2n-1}\Bigr),\\ T_2&=\operatorname{diag}\Bigl(\frac{-1}{2n-1},1,\frac{-1}{2n-1},\dots,\frac{-1}{2n-1}\Bigr),\\ &\vdots\\ T_n&=\operatorname{diag}\Bigl(\frac{-1}{2n-1},\dots,\frac{-1}{2n-1},1\Bigr).\end{aligned}$$

Then $T_1,\dots,T_n$ are commuting selfadjoint operators, $\|T_j\|=1$ and

$$W(T_1,\dots,T_n)=\operatorname{conv}\Bigl\{\Bigl(1,\frac{-1}{2n-1},\dots,\frac{-1}{2n-1}\Bigr),\dots,\Bigl(\frac{-1}{2n-1},\dots,\frac{-1}{2n-1},1\Bigr)\Bigr\}.$$

By Example 8, for each $v\in W(T_1,\dots,T_n)$ we have $\min_{1\le j\le n}|v_j|\le\frac{1}{2n-1}$.

Corollary 11. Let $\dim H<\infty$, let $T_1,T_2,T_3\in B(H)$. Then:

(i) there exists a unit vector $x \in H$ such that

$$|\langle T_j x, x\rangle| \geq \frac{1}{6}\|T_j\| \qquad (j = 1, 2);$$

(ii) there exists a unit vector $y \in H$ such that

$$|\langle T_j y, y\rangle| \geq \frac{1}{10}\|T_j\| \qquad (j = 1, 2, 3).$$

Proof. (i) Write $T_j = A_j + iB_j$ with selfadjoint operators A_j, B_j. Then $\|A_j\| \geq \frac{\|T_j\|}{2}$ or $\|B_j\| \geq \frac{\|T_j\|}{2}$. For each j choose either A_j or B_j with bigger norm and apply Theorem 9.

(ii) can be proved similarly. □

Remark 12. We do not know what are the best constants in Corollary 11. For $n = 2$ it lies between 1/6 and 1/4 as the following example shows. Let

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to show that for each unit vector x either $|\langle T_1 x, x\rangle| \leq 1/4$ or $|\langle T_2 x, x\rangle| \leq 1/4$.

Similarly, one can show that for $n = 3$ the best constant in Corollary 11 lies between 1/10 and 1/6.

3. Case $n \geq 4$

The following lemma is a weaker version of Conjecture 7.

Lemma 13. Let $n \in \mathbb{N}$ and let $K \subset [0,1]^n$ be a convex set. Let $u_j = (u_{j1}, \dots, u_{jn}) \in K$ satisfy $u_{jj} = 1 \quad (j = 1, \dots, n)$. Then there exists $v = (v_1, \dots, v_n) \in K$ such that $|v_j| \geq \frac{1}{2n^2} \quad (j = 1, \dots, n)$.

Proof. Let $M = [0,1]^n$. Clearly M is a convex set with width $(M) = 1$, where

$$\text{width}\,(M) = \inf\Big\{\sup_{v\in M}\langle v, f\rangle - \inf_{v\in M}\langle v, f\rangle : f \in \mathbb{R}^n, \|f\| = 1\Big\}.$$

Indeed, for $f = (f_1, \dots, f_n) \in \mathbb{R}^n, \|f\| = \Big(\sum_{j=1}^n f_j^2\Big)^{1/2} = 1$ let $J = \big\{j \in \{1, \dots, n\} : f_j \geq 0\big\}$. Then $\sup_{v\in M}\langle v, f\rangle = \sum_{j\in J} f_j$ and $\inf_{v\in M}\langle v, f\rangle = \sum_{j\notin J} f_j$. Hence $\sup_{v\in M}\langle v, f\rangle - \inf_{v\in M}\langle v, f\rangle = \sum_{j=1}^n |f_j| \geq \sum_{j=1}^n |f_j|^2 = 1$ and width $M \geq 1$. Considering the vector $f = (1, 0, \dots, 0)$ we get width $M = 1$.

For $j = 1, \dots, n$ let $L_j = \Big\{(t_1, \dots, t_n) \in \mathbb{R}^n : \Big|\sum_{k=1}^n t_k u_{kj}\Big| < \frac{1}{2n}\Big\}$. Then width $(L_j) = \frac{n^{-1}}{(\sum_{k=1}^n u_{kj}^2)^{1/2}} \leq \frac{1}{n}$. So $\sum_{j=1}^n \text{width}\,(L_j) \leq 1$. By [B], there exists $t = (t_1, \dots, t_n) \in M$ such that $t \notin \bigcup_{j=1}^n L_j$.

Let $s = \frac{t}{\sum_{j=1}^n t_j}$. Then $\sum_{k=1}^n s_k = 1$ and for each $j = 1, \dots, n$ we have

$$\Big|\sum_{k=1}^n s_k u_{kj}\Big| = \frac{\big|\sum_{k=1}^n t_k u_{kj}\big|}{\sum_{k=1}^n t_k} \geq \frac{1}{2n^2}.$$

So $v=\sum_{k=1}^n s_k u_k \in K$ and

$$|v_j| \ge \frac{1}{2n^2} \qquad (j=1,\dots,n).$$

□

Corollary 14. Let $\dim H<\infty$ and $T_1,\dots,T_n\in B(H)$ be commuting selfadjoint operators. Then there exists a unit vector $x\in H$ such that

$$|\langle T_j x,x\rangle| \ge \frac{\|T_j\|}{2n^2} \qquad (j=1,\dots,n).$$

Proof. The numerical range $W(T_1,\dots,T_n)=\operatorname{conv}\sigma(T_1,\dots,T_n)$ is a convex set. For each $j=1,\dots,n$ there exists a unit vector $x_j\in H$ with $|\langle T_jx_j,x_j\rangle|=\|T_j\|$, so there exists $u_j\in W(T_1,\dots,T_n)$ with $|u_{jj}|=\|T_j\|$.

Using Lemma 13 we can show as in the proof of Theorem 9 that there exists $v\in W(T_1,\dots,T_n)$ with $|v_j|\ge \frac{\|T_j\|}{2n^2} \qquad (j=1,\dots,n)$. □

Lemma 13 can be also applied for other types of numerical ranges.

Let H be an infinite-dimensional Hilbert space and let $T_1,\dots,T_n\in B(H)$. The essential numerical range $W_e(T_1,\dots,T_n)$ is the set of all $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ such that there exists an orthonormal sequence $(x_k)\subset H$ with

$$\lambda_j=\lim_{k\to\infty}\langle T_jx_k,x_k\rangle.$$

An important property of the the essential numerical range is that it is always a closed convex set, see [LP].

For a single selfadjoint operator $S\in B(H)$ we have $\sup\{|\mu|:\mu\in W_e(S)\}=\|S\|_e$, the essential norm of S. So an easy application of Lemma 13 (Lemmas 5 and 6, respectively) gives

Theorem 15. Let H be an infinite-dimensional Hilbert space, let $T_1,\dots,T_n\in B(H)$ be selfadjoint operators. Then there exists an orthonormal sequence $(x_k)\subset H$ such that $a_j:=\lim_{k\to\infty}\langle T_jx_k,x_k\rangle$ exists and $|a_j|\ge\frac{\|T_j\|_e}{2n^2}$ for all $j=1,\dots,n$.

For $n=2$ and $n=3$ there exists an orthonormal sequence $(x_k)\subset H$ such that $|a_j|\ge \frac{\|T_j\|_e}{3} \quad (j=1,2)$, and $|a_j|\ge\frac{\|T_j\|_e}{5} \quad (j=1,2,3)$, respectively.

Corollary 16. Let $n\in\mathbb{N}$, $\varepsilon>0$, let $T_1,\dots,T_n\in B(H)$ be arbitrary operators. Then there exists an orthonormal sequence $(x_k)\subset H$ such that $a_j:=\lim_{k\to\infty}\langle T_jx_k,x_k\rangle$ exists and $|a_j|\ge \frac{\|T_j\|_e}{4n^2}$ for all $j=1,\dots,n$.

For $n=2$ and $n=3$ there exists an orthonormal sequence $(x_k)\subset H$ such that $|a_j|\ge \frac{\|T_j\|_e}{6} \quad (j=1,2)$, and $|a_j|\ge\frac{\|T_j\|_e}{10} \quad (j=1,2,3)$, respectively.

Another situation where the results can be applied is the algebraic numerical range.

Let $\mathcal{A}$ be a unital Banach algebra, let $a_1,\dots,a_n\in\mathcal{A}$. The algebraic numerical range is defined by

$$V(a_1,\dots,a_n,\mathcal{A})=\big\{(f(a_1),\dots,f(a_n)):f\in\mathcal{A}^*,\|f\|=1=f(1_{\mathcal{A}})\big\},$$

where $1_{\mathcal{A}}$ denotes the unit in $\mathcal{A}$.

It is well known that $V(a_1,\dots,a_n,\mathcal{A})$ is always a closed convex subset of $\mathbb{C}^n$. For a single element $a_1\in\mathcal{A}$ we have

$$\sup\{|\mu| : \mu\in V(a_1,\mathcal{A})\}\ge \frac{\|a_1\|}{e}$$

(where $e=2.71...$), see [BD], p. 34.

Corollary 17. Let $\mathcal{A}$ be a unital Banach algebra, let $a_1,\dots,a_n\in\mathcal{A}$. Then there exists $f\in\mathcal{A}^*$, $\|f\|=1=f(1_{\mathcal{A}})$ such that

$$|f(a_j)|\ge\frac{\|a_j\|}{2n^2e}\qquad(j=1,\dots,n).$$

For $n=2$ ($n=3$) we have

$$|f(a_j)|\ge\frac{\|a_j\|}{3e}\qquad(j=1,2)$$

and

$$|f(a_j)|\ge\frac{\|a_j\|}{5e}\qquad(j=1,2,3),$$

respectively.

Proof. For $j=1,\dots,n$ there exists $f_j\in\mathcal{A}^*$ with $\|f_j\|=1=f_j(1_{\mathcal{A}})$, $|f(a_j)|\ge\frac{\|a_j\|}{e}$. Let α_j be the complex unit such that $f(\alpha_j a_j)\ge\frac{\|a_j\|}{e}$. The numerical range $V(\alpha_1a_1,\dots\alpha_na_n,\mathcal{A})$ is a convex set, and so is the set $K:=\{(\operatorname{Re}\lambda_1,\dots,\operatorname{Re}\lambda_n):(\lambda_1,\dots,\lambda_n)\in V(\alpha_1a_1,\dots,\alpha_na_n,\mathcal{A})\}$. By Lemma 13, there exists $\mu\in K\subset\mathbb{R}^n$ with $|\mu_j|\ge\frac{\|a_j\|}{2n^2e}$ for all j. So there exists $\lambda\in V(a_1,\dots,a_n,\mathcal{A})$ with $|\lambda_j|\ge\frac{\|a_j\|}{2n^2e}\quad(j=1,\dots,n)$.

4. Non-convex case

In this section we consider the general case of a sequence $T_1,T_2,\dots$ of operators on a Hilbert space H.

Let $c_j\ge0$, $\sum_{j=1}^\infty c_j<1$. By [M], p. 353, there exist unit vectors $x,y\in H$ such that

$$|\langle T_jx,y\rangle|\ge c_j\|T_j\| \tag{6}$$

for all $j\in\mathbb{N}$. By the polarization formula,

$$4\langle T_jx,y\rangle=\langle T_j(x+y),x+y\rangle-\langle T_j(x-y),x-y\rangle+i\langle T_j(x+iy),x+iy\rangle-i\langle T_j(x-iy),x-iy\rangle.$$

Set $u_1=\frac{x+y}{\|x+y\|}$, $u_2=\frac{x-y}{\|x-y\|}$, $u_3=\frac{x+iy}{\|x+iy\|}$, $u_4=\frac{x-iy}{\|x-iy\|}$. So by (6), there are four unit vectors $u_1,\dots,u_4\in H$ such that

$$\max_{1\le k\le4}|\langle T_ju_k,u_k\rangle|\ge\frac{c_j}{4}\|T_j\|$$

for all $j\in\mathbb{N}$. However, it is much more difficult to find a single unit vector $u\in H$ such that $|\langle T_ju,u\rangle|$ is large for all j, as it was required in Problem 1.

We give only a modest estimate in this case.

We need the following lemma.

Lemma 18. Let $b\le0$, $0<\varepsilon<1$. Then

$$m\Big(\big\{t\in[0,2\pi):b\le\cos t\le b+\varepsilon\big\}\Big)\le\pi\sqrt{2\varepsilon},$$

where m denotes the Lebesgue measure.

Proof. It is a matter of routine to show that the maximum of the function

$$b \mapsto m\Big(\big\{t \in [0,2\pi) : b \le \cos t \le b+\varepsilon\big\}\Big)$$

is attained for $b=-1$. For $0 \le t < 2\pi$ we have

$$-1 \le \cos t \le -1+\varepsilon \Longleftrightarrow \pi - t_0 \le t \le \pi + t_0,$$

where $0 < t_0 < \frac{\pi}{2}$ and $\sin t_0 = \sqrt{1-(1-\varepsilon)^2} = \sqrt{2\varepsilon - \varepsilon^2} \le \sqrt{2\varepsilon}$. Thus

$$m\Big(\big\{t \in [0,2\pi) : b \le \cos t \le b+\varepsilon\big\}\Big) \le$$

$$m\Big(\big\{t \in [0,2\pi) : \cos t \le -1+\varepsilon\big\}\Big) = 2t_0 \le \pi \sin t_0 \le \pi\sqrt{2\varepsilon}.$$

□

Theorem 19. Let $A_j \in B(H) \quad (j=1,2,\dots)$ be selfadjoint operators. Let $\sum_{j=1}^\infty c_j^{1/3} < 1$. Then there exists a unit vector $u \in H$ such that

$$|\langle A_j u, u\rangle| \ge \frac{c_j}{4}\|A_j\|$$

for all $j \in \mathbb{N}$.

Proof. Without loss of generality we may assume that $\|A_j\| = 1$ for all j. As mentioned above, there exist $x, y \in H$, $\|x\| = 1 = \|y\|$ such that

$$|\langle A_j x, y\rangle| \ge c_j^{1/3} \qquad (j \in \mathbb{N}).$$

For $0 \le t < 2\pi$ set $v(t) = x + e^{it}y$. Then for each $j \in \mathbb{N}$,

$$\big\langle A_j v(T), v(t)\big\rangle = \langle A_j x, x\rangle + \langle A_j y, y\rangle + 2\operatorname{Re} e^{-it}\langle A_j x, y\rangle.$$

Let $M_j = \big\{t \in [0,2\pi) : |\langle A_j v(t), v(t)\rangle| < c_j\big\}$. We have

$$t \in M_j \Longleftrightarrow |\operatorname{Re}(a_j + r_j e^{i(s-t)}| < c_j,$$

where $a_j = \langle A_j x, x\rangle + \langle A_j y, y\rangle$, $r_j = 2|\langle A_j x, y\rangle|$ and $2\langle A_j x, y\rangle = r_j e^{is}$. So $r_j \ge 2c_j^{1/3}$.

So

$$t \in M_j \Longleftrightarrow \Big|\operatorname{Re}\Big(\frac{a_j}{r_j} + e^{i(s-t)}\Big)\Big| < \frac{c_j}{r_j}$$

$$\Longleftrightarrow -\frac{a_j}{r_j} - \frac{c_j}{r_j} \le \cos(s-t) \le -\frac{a_j}{r_j} + \frac{c_j}{r_j}.$$

By Lemma 18 for $b = -\frac{a_j}{r_j} - \frac{c_j}{r_j}$ and $\varepsilon = \frac{2c_j}{r_j}$ we have

$$m(M_j) \le \pi\Big(\frac{4c_j}{r_j}\Big)^{1/2} \le 2\pi\Big(\frac{c_j}{2c_j^{1/3}}\Big)^{1/2} = \pi\sqrt{2}c_j^{1/3}.$$

So

$$m\Big(\bigcup_{j=1}^\infty M_j\Big) \le \sum_{j=1}^\infty m(M_j) \le \pi\sqrt{2}\sum_{j=1}^\infty c_j^{1/3} < 2\pi.$$

Hence there exists $t \in [0,2\pi) \setminus \bigcup_{j=1}^\infty M_j$. For this t we have $\big|\langle A_j v(t), v(t)\rangle\big| \ge c_j$ for all $j \in \mathbb{N}$. Let $u = \frac{v(t)}{\|v(t)\|}$. Then $\|u\| = 1$ and $|\langle A_j u, u\rangle| \ge \frac{c_j}{4} \quad (j \in \mathbb{N})$. □

Corollary 20. Let $T_1, \dots, T_n \in B(H)$, $\varepsilon > 0$. Then there exists a unit vector $x \in H$ such that

$$|\langle T_j x, x\rangle| \geq \frac{\|T_j\|}{8n^3} - \varepsilon.$$

Proof. If $T_1, \dots, T_n$ are selfadjoint operators, by Theorem 19 we get the existence of a unit vector $x \in H$ with $|\langle T_j x, x\rangle| \geq \frac{\|T_j\|}{4n^3} - \varepsilon$ for $j = 1, \dots, n$.

If $T_1, \dots, T_n$ are general non-selfadjoint operators then we can consider either the real or imaginary part of each T_j with greater norm and obtain Corollary 20 in the usual way. □

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