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GENERALIZED SPECTRAL RADIUS AND ITS MAX ALGEBRA VERSION

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Abstract. Let $\Sigma \subset \mathbb{C}^{n\times n}$ and $\Psi \subset \mathbb{R}_+^{n\times n}$ be bounded subsets and let $\rho(\Sigma)$ and $\mu(\Psi)$ denote the generalized spectral radius of Σ and the max algebra version of the generalized spectral radius of Ψ, respectively. We apply a single matrix description of $\mu(\Psi)$ to give a new elementary and straightforward proof of the Berger-Wang formula in max algebra and consequently a new short proof of the original Berger-Wang formula in the case of bounded subsets of $n \times n$ non-negative matrices. We also obtain a new description of $\mu(\Psi)$ in terms of the Schur-Hadamard product and prove new trace and max-trace descriptions of $\mu(\Psi)$ and $\rho(\Sigma)$. In particular, we show that

$$\mu(\Psi) = \limsup_{m\to\infty} [\sup_{A\in\Psi_\otimes^m} \mathrm{tr}_\otimes(A)]^{1/m} = \limsup_{m\to\infty} [\sup_{A\in\Psi_\otimes^m} \mathrm{tr}(A)]^{1/m}$$

and

$$\rho(\Sigma) = \limsup_{m\to\infty} [\sup_{B\in\Sigma^m} \mathrm{tr}(|B|)]^{1/m} = \limsup_{m\to\infty} [\sup_{B\in\Sigma^m} \mathrm{tr}_\otimes(|B|)]^{1/m},$$

where $\mathrm{tr}_\otimes(A) = \max_{i=1,\dots,n} a_{ii}$ and $|B| = [|b_{ij}|]$.

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1. Introduction

The algebraic system max algebra and its isomorphic versions provide an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimisation, mathematical physics, DNA analysis, ...(see e.g. [5], [1], [2], [18] and the references cited there). Max algebra's usefulness arises from a fact that these non-linear problems become linear when described in the max algebra language. Moreover, recently max algebra techniques were used to solve certain linear algebra problems (see e.g. [9], [12]).

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The max algebra consists of the set of non-negative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab, where $a, b \ge 0$. Let $A = [a_{ij}]$ be a $n \times n$ non-negative matrix, i.e., $a_{ij} \ge 0$ for all $i, j = 1, \ldots, n$. We may denote the entries a_{ij} also by A_{ij}. Let $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) be the set of all $n \times n$ real (complex) matrices and $\mathbb{R}_+^{n\times n}$ the set of all $n \times n$ non-negative matrices. The operations between matrices and vectors in the max algebra are defined by analogy with the usual linear algebra. The product of $n \times n$ non-negative matrices A and B in the max algebra is denoted by $A \otimes B$, where $(A \otimes B)_{ij} = \max_{k=1,\ldots,n} a_{ik}b_{kj}$ and the sum $A \oplus B$ in the max algebra is defined by $(A \oplus B)_{ij} = \max\{a_{ij}, b_{ij}\}$. The notation $A_\otimes^2$ means $A \otimes A$, and $A_\otimes^k$ denotes the k-th max power of A. If $x = [x_i] \in \mathbb{R}^n$ is a non-negative vector, then the notation $A \otimes x$ means $[A \otimes x]_i = \max_{j=1,\ldots,n} a_{ij}x_j$. The usual associative and distributive laws hold in this algebra. The ordinary product between matrices and vectors, ordinary matrix powers and the spectral radius are denoted by AB, Ax, A^k and $\rho(A)$, respectively.

The role of the spectral radius of $A \in \mathbb{R}_+^{n\times n}$ in max algebra is played by the maximum cycle geometric mean $\mu(A)$, which is defined by

$$\mu(A) = \max\Big\{(a_{i_1 i_2}\cdots a_{i_k i_1})^{1/k} : k \le n \text{ and } i_1, \ldots, i_k \in \{1, \ldots, n\} \text{ mutually distinct}\Big\}.$$

There are many different descriptions of the maximum cycle geometric mean $\mu(A)$ (see e.g. [16] and the references cited there). It is known that $\mu(A)$ is the largest max eigenvalue of A. Moreover, if A is irreducible, then $\mu(A)$ is the unique max eigenvalue and every max eigenvector is positive (see e.g. [2, Theorem 2], [5], [1]). Also, the max version of Gelfand formula holds, i.e.,

$$\mu(A) = \lim_{m\to\infty} \|A_\otimes^m\|^{1/m}$$

for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ (see e.g. [16] and the references cited there). Thus $\mu(A_\otimes^k) = \mu(A)^k$ for all $k \in \mathbb{N}$.

Let Σ be a bounded set of $n \times n$ complex matrices. For $m \ge 1$, let

$$\Sigma^m = \{A_1 A_2 \cdots A_m : A_i \in \Sigma\}.$$

The generalized spectral radius of Σ is defined by

$$\rho(\Sigma) = \limsup_{m\to\infty} \,[\sup_{A\in\Sigma^m} \rho(A)]^{1/m} \tag{1}$$

and is equal to

$$\rho(\Sigma) = \sup_{m\in\mathbb{N}} \,[\sup_{A\in\Sigma^m} \rho(A)]^{1/m}.$$

The joint spectral radius of Σ is defined by

$$\hat\rho(\Sigma) = \lim_{m\to\infty} [\sup_{A\in\Sigma^m} \|A\|]^{1/m}, \tag{2}$$

where $\|\cdot\|$ is any vector norm on $\mathbb{C}^{n\times n}$. It is well known that $\rho(\Sigma) = \hat\rho(\Sigma)$ for a bounded set Σ of complex $n \times n$ matrices (see e.g. [3], [8], [7] and the references cited there).

This equality is called the Berger-Wang formula or also the generalized spectral radius theorem. For infinite dimensional generalizations see e.g. [20], [21].

The theory of the generalized and the joint spectral radius has many important applications for instance to discrete and differential inclusions, wavelets, invariant subspace theory (see e.g. [3], [7], [22], [20], [21] and the references cited there). In particular, $\hat{\rho}(\Sigma)$ plays a central role in determining stability in convergence properties of discrete and differential inclusions. In this theory the quantity $\log \hat{\rho}(\Sigma)$ is known as the maximal Lyapunov exponent (see e.g. [22]).

Let Ψ be a bounded set of $n \times n$ non-negative matrices. For $m \geq 1$, let

$$\Psi_{\otimes}^{m} = \{A_1 \otimes A_2 \otimes \cdots \otimes A_m : A_i \in \Psi\}.$$

The max algebra version of the generalized spectral radius $\mu(\Psi)$ of Ψ, is defined by

$$\mu(\Psi) = \limsup_{m\to\infty} [\sup_{A\in\Psi_{\otimes}^{m}} \mu(A)]^{1/m}$$

and is equal to

$$\mu(\Psi) = \sup_{m\in\mathbb{N}} [\sup_{A\in\Psi_{\otimes}^{m}} \mu(A)]^{1/m}.$$

Also the max algebra version of the Berger-Wang formula holds, i.e., $\mu(\Psi)$ is equal to the max algebra version of the joint spectral radius $\hat{\mu}(\Psi)$ of Ψ, which is defined by

$$\hat{\mu}(\Psi) = \lim_{m\to\infty} [\sup_{A\in\Psi_{\otimes}^{m}} \|A\|]^{1/m},$$

where $\|\cdot\|$ denotes an arbitrary vector norm on $\mathbb{R}^{n\times n}$ (see e.g. [16], [14] or (3) below). The quantity $\log \mu(\Psi)$ measures the worst case cycle time of certain discrete event systems and it is sometimes called the worst case Lyapunov exponent (see e.g. [1], [11], [4], [18], [13] and the references cited there).

The paper is organized in the following way. In section 2 we apply a single matrix description of $\mu(\Psi)$ to give a new elementary and straightforward proof of the Berger-Wang formula in max algebra and consequently a new short proof of the original Berger-Wang formula in the case of bounded subsets $\Psi \subset \mathbb{R}_{+}^{n\times n}$ (Corollaries 2.3 and 2.4). We give new short proofs of the known results on the continuity in the Haussdorf distance of maps $\Psi \mapsto \mu(\Psi)$ and $\Psi \mapsto \rho(\Psi)$ (Proposition 2.5 and Remark 2.6). We also obtain a new description of $\mu(\Psi)$ in terms of the Schur-Hadamard product (Theorem 2.7), i.e., we show that

$$\mu(\Psi) = \sup\left\{\rho(\Psi \circ \Gamma) : \Gamma \subset \mathbb{R}_{+}^{n\times n} \text{ bounded }, \rho(\Gamma) \leq 1\right\},$$

where $\Psi \circ \Gamma = \{A \circ B : A \in \Psi, B \in \Gamma\}$ and $(A \circ B)_{ij} = a_{ij}b_{ij}$ for $i, j \in \{1, \ldots, n\}$. In the last section we prove new trace and max-trace descriptions of $\mu(\Psi)$ and $\rho(\Sigma)$ for bounded subsets $\Psi \subset \mathbb{R}_{+}^{n\times n}$ and $\Sigma \subset \mathbb{C}^{n\times n}$ (Corollary 3.2, Theorem 3.3 and Corollary 3.6). In particular, we show that

$$\mu(\Psi) = \limsup_{m\to\infty} [\sup_{A\in\Psi_{\otimes}^{m}} \mathrm{tr}_{\otimes}(A)]^{1/m} = \limsup_{m\to\infty} [\sup_{A\in\Psi_{\otimes}^{m}} \mathrm{tr}(A)]^{1/m}$$

and

$$\rho(\Sigma) = \limsup_{m\to\infty} [\sup_{B\in\Sigma^m} \mathrm{tr}(|B|)]^{1/m} = \limsup_{m\to\infty} [\sup_{B\in\Sigma^m} \mathrm{tr}_{\otimes}(|B|)]^{1/m},$$

where $\mathrm{tr}_{\otimes}(A) = \max_{i=1,\dots,n} a_{ii}$ and $|B| = [|b_{ij}|]$.

2. A single matrix description of $\mu(\Psi)$ and its applications

In this section we prove a description of the max algebra version of the generalized spectral radius $\mu(\Psi)$ in terms of a single matrix. Moreover, we apply this result to obtain new elementary proofs of some known and new results.

If $\Psi \subset \mathbb{R}_+^{n\times n}$ is a bounded subset, then we define the matrix $S(\Psi)$ by

$$(S(\Psi))_{ij} = \sup\{a_{ij} : A \in \Psi\},$$

i.e., $S(\Psi) = \bigoplus_{A\in\Psi} A$. The following result was previously known in the case of finite sets Ψ ([11], [13]). Even though the proof is similar to the proof from [13], we include it for the sake of completeness.

Proposition 2.1. *If $\Psi \subset \mathbb{R}_+^{n\times n}$ is a bounded set, then*

$$\mu(\Psi) = \mu(S(\Psi)).$$

Proof. First we prove $\mu(\Psi) \le \mu(S(\Psi))$. For arbitrary $A \in \Psi_{\otimes}^m$ we have $A \le S(\Psi)_{\otimes}^m$. Therefore $\mu(A) \le \mu(S(\Psi)_{\otimes}^m) = \mu(S(\Psi))^m$, which implies $\mu(\Psi) \le \mu(S(\Psi))$.

For the proof of $\mu(S(\Psi)) \le \mu(\Psi)$ we can assume $\mu(S(\Psi)) > 0$. Let $\varepsilon > 0$ be arbitrary and let $i_1, i_2, \dots, i_k \in \{1, \dots, n\}$ be such that $\mu(S(\Psi)) = (s_{i_1 i_2} s_{i_2 i_3} \cdots s_{i_k i_1})^{1/k}$, where s_{ij} are the entries of $S(\Psi)$. Then there exist $j_1, \dots, j_k$ and $A_{j_1}, \dots, A_{j_k} \in \Psi$ such that

$$\mu(S(\Psi))^k = s_{i_1 i_2} \cdots s_{i_k i_1} \le (A_{j_1})_{i_1 i_2} \cdots (A_{j_k})_{i_k i_1} + \varepsilon \le (A_{j_1} \otimes \cdots \otimes A_{j_k})_{i_1 i_1} + \varepsilon \le \mu(M) + \varepsilon,$$

where $M = A_{j_1} \otimes \cdots \otimes A_{j_k}$. For all $r \in \mathbb{N}$ we thus have

$$\mu(M_{\otimes}^r) = \mu(M)^r \ge (\mu(S(\Psi))^k - \varepsilon)^r.$$

This implies $\mu(\Psi)^k \ge \mu(S(\Psi))^k - \varepsilon$. Therefore we also have $\mu(\Psi) \ge \mu(S(\Psi))$, which completes the proof. □

For $A \in \mathbb{C}^{n\times n}$ we write $\|A\|_\infty = \max\{|a_{ij}| : 1 \le i, j \le n\}$. If $\Psi \subset \mathbb{R}_+^{n\times n}$ is a bounded subset, then we also write $\|\Psi\|_\infty = \sup\{\|A\|_\infty : A \in \Psi\}$. We have $\mu(\Psi) = \mu(S(\Psi)) \le \|S(\Psi)\|_\infty = \|\Psi\|_\infty$. It follows from definitions and Proposition 2.1 that

$$\mu(\Psi) = \sup\Big\{ \Big((A_1)_{i_1 i_2} \cdots (A_k)_{i_k i_1}\Big)^{1/k} : k \in \mathbb{N}, i_1, \dots, i_k \in \{1, \dots, n\}, A_1, \dots, A_k \in \Psi \Big\}.$$

It is also easy to see that we can require that $i_1, \dots, i_k$ are mutually distinct, so in particular $k \le n$. Thus we have

$$\mu(\Psi) = \sup\Big\{ \Big((A_1)_{i_1 i_2} \cdots (A_k)_{i_k i_1}\Big)^{1/k} : k \le n, A_1, \dots, A_k \in \Psi,$$

$$i_1,\dots,i_k\in\{1,\dots,n\}\text{ mutually distinct}\Big\}.$$

For $k\in\mathbb{N}$ let

$$\begin{aligned}c_k(\Psi) &= \sup\big\{\|A_1\otimes\cdots\otimes A_k\|_\infty : A_1,\dots,A_k\in\Psi\big\}\\ &= \sup\Big\{(A_1)_{i_0i_1}\cdots(A_k)_{i_{k-1}i_k} : i_0,\dots,i_k\in\{1,\dots,n\}, A_1,\dots,A_k\in\Psi\Big\}.\end{aligned}$$

The max version of the joint spectral radius $\hat{\mu}(\Psi)$ equals to

$$\hat{\mu}(\Psi)=\lim_{k\to\infty}c_k(\Psi)^{1/k}=\inf_{k\in\mathbb{N}}c_k(\Psi)^{1/k} \tag{3}$$

(the limit exists and is equal to the infimum, since $c_{k+l}(\Psi)\le c_k(\Psi)c_l(\Psi)$ for all $k,l\in\mathbb{N}$).

In what follows we give a new elementary proof of the max version of the Berger-Wang formula and consequently a new proof of the Berger-Wang formula in the case of bounded sets of non-negative $n\times n$ matrices. The proof of the max version of the Berger-Wang formula is much shorter than the proof in [14] and more straightforward than the one in [16], where the original Berger-Wang formula was used.

Lemma 2.2. *Let $\Psi\subset\mathbb{R}_+^{n\times n}$ be a bounded subset. For $k\ge n$ we have*

$$\mu(\Psi)^k\le c_k(\Psi)\le\|\Psi\|_\infty^n\cdot\mu(\Psi)^{k-n}.$$

Proof. The first inequality is clear.

We show the second inequality by induction on n. For $n=1$ clearly $c_k(\Psi)=\mu(\Psi)^k$ for all $k\in\mathbb{N}$. Let $n\ge 2$, $i_0,\dots,i_k\in\{1,\dots,n\}$ and $A_1,\dots,A_k\in\Psi$. Let $m=\max\{j : i_j=i_0\}$. If $m=0$ then

$$\begin{aligned}(A_1)_{i_0i_1}\cdots(A_k)_{i_{k-1}i_k} &= (A_1)_{i_0i_1}\cdot\Big((A_2)_{i_1i_2}\cdots(A_k)_{i_{k-1}i_k}\Big)\\ &\le \|\Psi\|_\infty\cdot\Big((A_2)_{i_1i_2}\cdots(A_k)_{i_{k-1}i_k}\Big)\\ &\le \|\Psi\|_\infty\cdot\|\Psi\|_\infty^{n-1}\cdot\mu(\Psi)^{k-1-(n-1)}=\|\Psi\|_\infty^n\cdot\mu(\Psi)^{k-n}.\end{aligned}$$

by the induction assumption, since $i_1,\dots,i_k\in\{1,\dots,n\}\setminus\{i_0\}$. Note that the induction assumption has been applied to $(n-1)\times(n-1)$ submatrices (without the i_0th row and column).

If $0<m\le k-n$ then we have similarly

$$\begin{aligned}&(A_1)_{i_0i_1}\cdots(A_k)_{i_{k-1}i_k}\\ =\ &\Big((A_1)_{i_0i_1}\cdots(A_m)_{i_{m-1}i_m}\Big)\cdot(A_{m+1})_{i_mi_{m+1}}\Big((A_{m+2})_{i_{m+1}i_{m+2}}\cdots(A_k)_{i_{k-1}i_k}\Big)\\ \le\ &\mu(\Psi)^m\cdot\|\Psi\|_\infty\cdot\|\Psi\|_\infty^{n-1}\mu(\Psi)^{k-m-1-(n-1)}=\|\Psi\|_\infty^n\cdot\mu(\Psi)^{k-n}.\end{aligned}$$

Finally, if $k-n<m\le k$ then

$$(A_1)_{i_0i_1}\cdots(A_k)_{i_{k-1}i_k}=\Big((A_1)_{i_0i_1}\cdots(A_m)_{i_{m-1}i_m}\Big)\cdot\Big((A_{m+1})_{i_mi_{m+1}}\cdots(A_k)_{i_{k-1}i_k}\Big)$$

$$\le \mu(\Psi)^m \|\Psi\|_\infty^{k-m} \le \|\Psi\|_\infty^n \cdot \mu(\Psi)^{k-n}.$$

This completes the proof. □

Corollary 2.3. (The max version of the Berger-Wang formula). *If $\Psi \subset \mathbb{R}_+^{n\times n}$ is a bounded subset, then $\mu(\Psi) = \hat{\mu}(\Psi)$.*

The previous result implies the Berger-Wang formula in the case of bounded sets of non-negative $n \times n$ matrices.

Corollary 2.4. *If $\Psi \subset \mathbb{R}_+^{n\times n}$ is a bounded subset, then $\rho(\Psi) = \hat{\rho}(\Psi)$.*

Proof. It was proved in [16, Proposition 2.3] and [14, Theorem 3(ii)] that

$$n^{-1}\rho(\Psi) \le \mu(\Psi) \le \rho(\Psi) \;\text{ and }\; n^{-1}\hat{\rho}(\Psi) \le \hat{\mu}(\Psi) \le \hat{\rho}(\Psi). \tag{4}$$

Since $\rho(\Psi^m) = \rho(\Psi)^m$ and $\hat{\rho}(\Psi^m) = \hat{\rho}(\Psi)^m$ it follows from (4) and Corollary 2.3 that

$$\rho(\Psi) = \lim_{m\to\infty} \mu(\Psi^m)^{1/m} = \lim_{m\to\infty} \hat{\mu}(\Psi^m)^{1/m} = \hat{\rho}(\Psi),$$

which completes the proof. □

Next we give a new elementary proof of the fact that the map $\Psi \mapsto \mu(\Psi)$ is continuous in the Haussdorf distance, which again simplifies the known proofs (see [15], [18]) substantially. Recall that the Haussdorf distance $\mathrm{dist}\,\{\Psi, \Sigma\}$ for bounded subsets $\Psi, \Sigma \subset \mathbb{R}_+^{n\times n}$ is defined by

$$\mathrm{dist}\,\{\Psi, \Sigma\} = \max\{\delta(\Psi, \Sigma), \delta(\Sigma, \Psi)\},$$

$$\delta(\Psi, \Sigma) = \sup_{A\in\Psi} \inf_{B\in\Sigma} \mathrm{dist}\,\{A, B\} \qquad \text{and} \qquad \mathrm{dist}\,\{A, B\} = \|A - B\|_\infty.$$

Proposition 2.5. *The function $\Psi \mapsto \mu(\Psi)$ is continuous on the set of all bounded subsets of $\mathbb{R}_+^{n\times n}$.*

Proof. Clearly the mapping $\Psi \mapsto S(\Psi)$ is continuous and $\mu(\Psi) = \mu(S(\Psi))$, so it is sufficient to show the continuity of the function $A \mapsto \mu(A)$ for a matrix $A \in \mathbb{R}_+^{n\times n}$.

Let $A, B_m \in \mathbb{R}_+^{n\times n} \quad (m \in \mathbb{N})$ and $\mathrm{dist}\,\{A, B_m\} \to 0$. Then

$$(B_m)_{i_1 i_2} \cdots (B_m)_{i_k i_1} \to a_{i_1 i_2} \cdots a_{i_k i_1}$$

for all $k \le n, i_1, \dots, i_k \in \{1, \dots, n\}$. So

$$\mu(B_m) = \max\Big\{((B_m)_{i_1 i_2} \cdots (B_m)_{i_k i_1})^{1/k} : k \le n, i_1, \dots, i_k \in \{1, \dots, n\} \text{ mutually distinct}\Big\}$$

$$\to \max\Big\{(a_{i_1 i_2} \cdots a_{i_k i_1})^{1/k} : k \le n, i_1, \dots, i_k \in \{1, \dots, n\} \text{ mutually distinct}\Big\} = \mu(A).$$

So the function $A \mapsto \mu(A)$ is continuous and so is the mapping $\Psi \mapsto \mu(\Psi)$. □

Remark 2.6. Given a bounded subset $\Psi \subset \mathbb{R}_+^{n\times n}$, it follows from (4) and $\rho(\Psi) = \lim_{m\to\infty} \mu(\Psi^m)^{1/m}$ that

$$\rho(\Psi) = \sup_{m\in\mathbb{N}} \mu(\Psi^m)^{1/m} = \inf_{m\in\mathbb{N}} (n\mu(\Psi^m))^{1/m}.$$

Using Proposition 2.5 it follows that the function $\Psi \mapsto \rho(\Psi)$ is continuous on the set of all bounded subsets of $\mathbb{R}_+^{n\times n}$. See e.g. [18] and [22] for references on more general results on the continuity of the mapping $\Psi \mapsto \rho(\Psi)$.

To conclude this section we obtain new descriptions of $\mu(\Psi)$ in terms of the Schur-Hadamard product. Let $\Psi, \Sigma \subset \mathbb{R}_+^{n\times n}$ be bounded subsets and $t > 0$. Let $\Psi \circ \Sigma = \{A \circ B : A \in \Psi, B \in \Sigma\}$ and $\Psi^{(t)} = \{A^{(t)} : A \in \Psi\}$, where $A \circ B$ denotes the Schur-Hadamard product and $A^{(t)}$ the Schur-Hadamard power, i.e., $A\circ B = [a_{ij}b_{ij}]$, $A^{(t)} = [a_{ij}^t]$. We will also use the notation $A \circ \Sigma$ instead of $\{A\} \circ \Sigma$. The matrix $[1]_{i,j=1}^n$ is denoted by J.

It was proved in [17, Corollary 5.3] that

$$\rho(\Psi \circ \Sigma) \le \rho(\Psi)\rho(\Sigma) \tag{5}$$

(see [19] for closely related results). It was also shown in [10] and [17] that for $A \in \mathbb{R}_+^{n\times n}$ we have

$$\mu(A) = \sup\{\rho(A\circ B) : B \in \mathbb{R}_+^{n\times n}, \rho(B) \le 1\} = \sup\left\{\frac{\rho(A\circ B)}{\rho(B)} : B \in \mathbb{R}_+^{n\times n}, \rho(B) > 0\right\}$$

and

$$\begin{aligned} \mu(A) &= \sup\{\rho(A\circ\Sigma) : \Sigma \subset \mathbb{R}_+^{n\times n} \text{ bounded }, \rho(\Sigma) \le 1\} \\ &= \sup\left\{\tfrac{\rho(A\circ\Sigma)}{\rho(\Sigma)} : \Sigma \subset \mathbb{R}_+^{n\times n} \text{ bounded }, \rho(\Sigma) > 0\right\}. \end{aligned}$$

It follows from Proposition 2.1 that

$$\begin{aligned} \mu(\Psi) = \mu(S(\Psi)) &= \sup\{\rho(S(\Psi)\circ B) : B \in \mathbb{R}_+^{n\times n}, \rho(B) \le 1\} \\ &= \sup\{\rho(S(\Psi)\circ\Sigma) : \Sigma \subset \mathbb{R}_+^{n\times n} \text{ bounded }, \rho(\Sigma) \le 1\}. \end{aligned} \tag{6}$$

In [16] Inequality (4) was used to prove

$$\mu(\Psi) = \lim_{t\to\infty} \rho(\Psi^{(t)})^{1/t} = \inf_{t\in(0,\infty)} \rho(\Psi^{(t)})^{1/t}. \tag{7}$$

Next we give a new description of $\mu(\Psi)$, which sharpens (5).

Theorem 2.7. *Let $\Psi, \Sigma \subset \mathbb{R}_+^{n\times n}$ be bounded subsets. Then*

$$\rho(\Psi \circ \Sigma) \le \mu(\Psi)\rho(\Sigma) \tag{8}$$

and

$$\begin{aligned} \mu(\Psi) &= \sup\left\{\rho(\Psi\circ\Sigma) : \Sigma \subset \mathbb{R}_+^{n\times n} \text{ bounded }, \rho(\Sigma) \le 1\right\} \\ &= \sup\{\tfrac{\rho(\Psi\circ\Sigma)}{\rho(\Sigma)} : \Sigma \subset \mathbb{R}_+^{n\times n} \text{ bounded }, \rho(\Sigma) > 0\}. \end{aligned} \tag{9}$$

Proof. The second equality in (9) follows from positive homogenicity of $\rho(\cdot)$ and the fact that $\rho(\Sigma) = 0$ implies $\rho(\Psi \circ \Sigma) = 0$.

Next we prove the inequality (8). Since $A \leq S(\Psi)$ for all $A \in \Psi$, we have

$$(A_1 \circ B_1) \cdots (A_m \circ B_m) \leq (S(\Psi) \circ B_1) \cdots (S(\Psi) \circ B_m)$$

for all $A_1, \ldots, A_m \in \Psi$ and $B_1, \ldots, B_m \in \Sigma$. This implies $\rho(\Psi \circ \Sigma) \leq \rho(S(\Psi) \circ \Sigma)$. Now Inequality (8) follows from (6).

To complete the proof let us denote

$$\mu_2(\Psi) = \sup \left\{ \frac{\rho(\Psi \circ \Sigma)}{\rho(\Sigma)} : \Sigma \subset \mathbb{R}_+^{n\times n} \text{ bounded }, \rho(\Sigma) > 0 \right\}.$$

By choosing $\Sigma = \{J\}$, we obtain $\rho(\Psi) \leq n\mu_2(\Psi)$. We only need to prove that $\mu(\Psi) \leq \mu_2(\Psi)$, since $\mu(\Psi) \geq \mu_2(\Psi)$ follows from (8).

If $\mu_2(\Psi) = 0$, then $0 = n\mu_2(\Psi) \geq \rho(\Psi) \geq \mu(\Psi)$ and therefore $\mu(\Psi) = 0$.

Assume $\mu_2(\Psi) > 0$ and $m \in \mathbb{N}$. Since $\Psi^{(m)} \subset \Psi \circ \Psi^{(m-1)}$, we have $\rho(\Psi^{(m)}) \leq \rho(\Psi \circ \Psi^{(m-1)})$. Thus

$$\rho(\Psi^{(m)}) \leq \mu_2(\Psi)\rho(\Psi^{(m-1)}) \leq \mu_2(\Psi)^2\rho(\Psi^{(m-2)}) \leq \cdots \leq \mu_2(\Psi)^{m-1}\rho(\Psi).$$

Therefore

$$\rho(\Psi^{(m)})^{\frac{1}{m}} \leq \mu_2(\Psi)^{\frac{m-1}{m}} \rho(\Psi)^{\frac{1}{m}}.$$

Letting $m \to \infty$, we obtain $\mu(\Psi) \leq \mu_2(\Psi)$ by (7), since $\rho(\Psi) \geq \mu_2(\Psi) > 0$. This completes the proof. □

Remark 2.8. Alternatively, one can prove Inequality (8) in the following way. It is not hard to see that for all $k \geq n$ we have $d_k(\Psi \circ \Sigma) \leq c_k(\Psi)d_k(\Sigma)$, where $d_k(\Psi) = \sup\{\|A_1 \cdots A_k\|_\infty : A_1, \ldots, A_k \in \Psi\}$. This implies (8) by Corollaries 2.3 and 2.4.

3. The trace and max-trace descriptions

In this final section we give a new trace description of $\mu(\Psi)$ and a max-trace description of $\rho(\Sigma)$. It was proved in [6] and [23] that for a finite set $\Sigma \subset \mathbb{C}^{n\times n}$ we have

$$\rho(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} |\mathrm{tr}(A)|]^{1/m}. \tag{10}$$

This result holds also for bounded sets. For completeness, we include a new short proof of this fact.

Theorem 3.1. *If $\Sigma \subset \mathbb{C}^{n\times n}$ is a bounded subset, then Equality* (10) *holds.*

Proof. For each $A \in \Sigma^m$ we have $|\mathrm{tr}\,(A)| \leq n\rho(A)$ and so

$$\limsup_{m\to\infty} \sup_{A\in\Sigma^m} |\mathrm{tr}\,(A)|^{1/m} \leq \rho(\Sigma).$$

To prove the opposite inequality we may assume that $\rho(\Sigma) = 1$.

Let $\varepsilon \in (0,1)$. Then there exists $m \in \mathbb{N}$ and $A \in \Sigma^m$ such that $\rho(A) > (1-\varepsilon)^m$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A (according to their algebraic multiplicities).

There exists (infinitely many) $k \in \mathbb{N}$ such that $\operatorname{Re} \lambda_j^k \geq \frac{|\lambda_j|^k}{2}$ for all $j = 1, \ldots, n$. For such a k we have

$$|\operatorname{tr} A^k| = |\sum_{j=1}^n \lambda_j^k| \geq \sum_{j=1}^n \operatorname{Re} \lambda_j^k \geq \frac{1}{2} \max_j |\lambda_j^k| = \frac{\rho(A^k)}{2} > \frac{(1-\varepsilon)^{mk}}{2}.$$

Thus

$$\limsup_{m\to\infty} \sup_{A\in\Sigma^m} |\operatorname{tr}(A)|^{1/m} \geq 1-\varepsilon.$$

Since ε was arbitrary, Equality (10) is proved. □

For $A \in \mathbb{C}^{n\times n}$, the inequalities

$$|\operatorname{tr}(A)| \leq \operatorname{tr}(|A|) \leq n\|A\|_\infty$$

together with the previous theorem and Berger-Wang formula imply

$$\rho(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} \operatorname{tr}(|A|)]^{1/m}, \tag{11}$$

where $|A| = [|a_{ij}|]$.

Let us define the max-trace of $A \in \mathbb{R}_+^{n\times n}$ by $\operatorname{tr}_\otimes(A) = \max_{i=1,\ldots n} a_{ii}$. The inequalities (11) and

$$\operatorname{tr}_\otimes(A) \leq \operatorname{tr}(A) \leq n \operatorname{tr}_\otimes(A) \tag{12}$$

imply the following result.

Corollary 3.2. *If $\Sigma \subset \mathbb{C}^{n\times n}$ is a bounded subset, then we have*

$$\rho(\Sigma) = \limsup_{m\to\infty}[\sup_{A\in\Sigma^m} \operatorname{tr}_\otimes(|A|)]^{1/m}.$$

Theorem 3.3. *Let $\Psi \subset \mathbb{R}_+^{n\times n}$ be a bounded subset. Then*

$$\mu(\Psi) = \limsup_{m\to\infty} [\sup_{A\in\Psi_\otimes^m} \operatorname{tr}_\otimes(A)]^{1/m} = \limsup_{m\to\infty} [\sup_{A\in\Psi_\otimes^m} \operatorname{tr}(A)]^{1/m} \tag{13}$$

Proof. The second equality in (13) is valid by (12).

Since $\operatorname{tr}_\otimes(A) \leq \mu(A)$ for all $A \in \Psi_\otimes^m$, we have

$$\limsup_{m\to\infty} [\sup_{A\in\Psi_\otimes^m} \operatorname{tr}_\otimes(A)]^{1/m} \leq \mu(\Psi).$$

To prove the reverse inequality we will show that

$$\rho(\Psi^{(t)})^{1/t} \leq n^{1/t} \limsup_{m\to\infty} [\sup_{A\in\Psi_\otimes^m} \operatorname{tr}_\otimes(A)]^{1/m} \tag{14}$$

for all $t > 0$. Indeed, the inequality

$$A_1 \cdots A_m \leq n^{m-1} A_1 \otimes \cdots \otimes A_m$$

implies that

$$\mathrm{tr}_{\otimes}(A_1^{(t)} \cdots A_m^{(t)}) \le n^{m-1}\mathrm{tr}_{\otimes}(A_1^{(t)} \otimes \cdots \otimes A_m^{(t)}) = n^{m-1}\mathrm{tr}_{\otimes}(A_1 \otimes \cdots \otimes A_m)^t$$

for all $A_1, \ldots, A_m \in \Psi$ and $t > 0$. This implies (14) by Corollary 3.2. Letting $t \to \infty$ in (14) and applying (7) completes the proof. □

Corollary 3.4. *Let $A \in \mathbb{R}_+^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$. Then*

$$\mu(A) = \limsup_{m\to\infty} \mathrm{tr}_{\otimes}(A_{\otimes}^m)^{1/m} = \limsup_{m\to\infty} \mathrm{tr}(A_{\otimes}^m)^{1/m} \quad \text{and} \tag{15}$$

$$\rho(B) = \limsup_{m\to\infty} \mathrm{tr}_{\otimes}(|B^m|)^{1/m}.$$

Remark 3.5. The result (15) is not surprising since the definition of $\mu(A)$ implies that

$$\mu(A) = \max_{m=1,\ldots,n} \mathrm{tr}_{\otimes}(A_{\otimes}^m)^{1/m}.$$

Applying Proposition 2.1 we obtain also the following result.

Corollary 3.6. *If $\Psi \subset \mathbb{R}_+^{n\times n}$ is a bounded subset, then*

$$\mu(\Psi) = \limsup_{m\to\infty} \mathrm{tr}_{\otimes}(S(\Psi)_{\otimes}^m)^{1/m} = \max_{m=1,\ldots,n} \mathrm{tr}_{\otimes}(S(\Psi)_{\otimes}^m)^{1/m} \quad \text{and}$$

$$\mu(\Psi) = \limsup_{m\to\infty} \mathrm{tr}(S(\Psi)_{\otimes}^m)^{1/m}.$$

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