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A STABLE AND OPTIMAL COMPLEXITY SOLUTION METHOD FOR MIXED FINITE ELEMENT DISCRETIZATIONS

Jan Brandts, Rob Stevenson, Utrecht

Abstract. We outline a solution method for mixed finite element discretizations based on dissecting the problem into three separate steps. The first handles the inhomogeneous constraint, the second solves the flux variable from the homogeneous problem, whereas the third step, adjoint to the first, finally gives the Lagrangian multiplier. We concentrate on aspects involved in the first and third step mainly, and advertise a multi-level method that allows for a stable computation of the intermediate and final quantities in optimal computational complexity.

Keywords: mixed finite elements, multi-level solver

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1. Introduction

There are well-known examples in the finite element literature of problems that are cast into the form of a saddle-point problem as a result of applying mixed variational principles. Already in 1973, Babuška [1] handled non-homogeneous Dirichlet boundary conditions for an elliptic problem by introducing a Lagrange multiplier and solving the resulting saddle-point problem. Around the same time, also Brezzi [5] published his abstract theory of approximation of saddle point problems, which led to the development of mixed finite element methods for elliptic equations, starting with the elements of Raviart and Thomas [10] in 1979. Since then, a large amount of attention has been paid to several aspects of saddle-point problems, ranging from the design of stable finite element spaces to the efficient solution of the indefinite linear systems that arise from the discretization [2], [4], [12]. In particular concerning the latter, much progress has been made with the realization that such systems can

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often be solved in three separate steps [6], [8]. The first step handles the inhomogeneous constraint, the second step involves the homogeneous problem, whereas the third step constitutes a problem that is adjoint to the first. In the literature, the emphasis is on the analysis of the second step, whereas for the first and third step either unstable methods are suggested, or stable methods left unanalyzed. In this paper we perform a rigorous analysis of the first and third step, and present recent insights that follow from employing several aspects of the papers [6], [8], [9].

We start by introducing the mixed finite element discretization of a model problem in Section 2, and proceed to illustrate the three separate solution steps. In Section 3 we present a stable method for handling steps one and three, both of optimal computational complexity. We conclude with some further comments in Section 4.

2. Mixed discretization of a model problem

Consider the Poisson problem with, for simplicity, homogeneous Neumann boundary conditions,

$$-\Delta u = f \quad \text{in } \Omega, \quad \nabla u^T \nu = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $f \in L_0^2(\Omega)$, the space of $L^2(\Omega)$ functions with mean zero. For simplicity, we will assume that Ω is a bounded polygonal domain in $\mathbb{R}^2$, although the arguments remain valid for three-dimensional domains. The mixed weak formulation of (1) introduces a second variable $\mathbf{p} = -\nabla u \in \mathbf{H}_0(\mathrm{div}\,;\Omega)$, the space of vectorfields in $[L^2(\Omega)]^2$ with weak divergence in $L^2(\Omega)$ and with vanishing normal trace on $\partial\Omega$. It seeks a pair $(u, \mathbf{p}) \in L_0^2(\Omega) \times \mathbf{H}_0(\mathrm{div}\,;\Omega)$ such that for all $(w, \mathbf{q}) \in L_0^2(\Omega) \times \mathbf{H}_0(\mathrm{div}\,;\Omega)$,

$$(\mathbf{p}, \mathbf{q}) - (u, \mathrm{div}\,\mathbf{q}) \;=\; 0 \quad \text{and} \quad (\mathrm{div}\,\mathbf{p}, w) = (f, w). \tag{2}$$

For the discretization of (2) we use, again for ease of presentation only, the space W_h of piecewise constant functions with mean value zero, and the space $\mathbf{\Gamma}_{0h} = \mathbf{\Gamma}_h \cap \mathbf{H}_0(\mathrm{div}\,;\Omega)$. Here, $\mathbf{\Gamma}_h$ is the lowest order Raviart-Thomas [10] space of all piecewise linear vector fields with constant and continuous normal fluxes on each edge. With this choice, the mixed finite element approximations $(u_h, \mathbf{p}_h) \in W_h \times \mathbf{\Gamma}_{0h}$ satisfy

$$(\mathbf{p}_h, \mathbf{q}_h) - (u_h, \mathrm{div}\,\mathbf{q}_h) \;=\; 0 \quad \text{and} \quad (\mathrm{div}\,\mathbf{p}_h, w_h) = (f, w_h), \tag{3}$$

for all $(w_h, \mathbf{q}_h) \in W_h \times \mathbf{\Gamma}_{0h}$. To conclude, we note that $\mathrm{div}\,\mathbf{\Gamma}_{0h} = W_h$ and moreover that $\mathbf{\Gamma}_{0h}$ and W_h satisfy the Babuška-Brezzi conditions (see also Section 3.2) which guarantee that there exists a unique solution.

2.1. Optimal complexity solution of the mixed system. The system of algebraic equations that results from (3) after choosing a suitable basis, is symmetric indefinite. Various methods have been proposed to solve it. Here we will discuss a method of optimal complexity. It makes use of the well-known property [7],

$$\mathbf{q}_h \in \mathbf{\Gamma}_{0h} \quad \text{and} \quad \operatorname{div} \mathbf{q}_h = 0 \quad \Leftrightarrow \quad \mathbf{q}_h \in \mathbf{curl}\, V_{0h}, \tag{4}$$

where V_{0h} is the space of continuous piecewise linear functions that are zero on the boundary—the usual standard finite element space. This property, together with the first equation in (3), immediately gives that

$$(\mathbf{p}_h, \mathbf{curl}\, v_h) = 0 \quad \text{for all } v_h \in V_{0h}. \tag{5}$$

The key idea is now to split the solution process for the pair $(u_h, \mathbf{p}_h)$ in three separate steps. We will discuss these steps in detail afterwards.

(A) Find a particular solution $\mathbf{r}_h \in \mathbf{\Gamma}_{0h}$ such that $(\operatorname{div} \mathbf{r}_h, w_h) = (f, w_h)$ for all $w_h \in W_h$, or, equivalently, such that $\operatorname{div} \mathbf{r}_h = P_h f$, where P_h denotes $L^2(\Omega)$-projection onto W_h.

(B) Compute the difference $\mathbf{p}_h - \mathbf{r}_h$, which by (4) equals $\mathbf{curl}\, \omega_h$ for some $\omega_h \in V_{0h}$, by solving the positive definite system $(\mathbf{curl}\, \omega_h, \mathbf{curl}\, v_h) = (\mathbf{p}_h - \mathbf{r}_h, \mathbf{curl}\, v_h) = -(\mathbf{r}_h, \mathbf{curl}\, v_h)$, where the latter (and crucial) equality is due to (5).

(C) Compute $u_h \in W_h$ from the system $(u_h, \operatorname{div} \mathbf{q}_h) = (\mathbf{p}_h, \mathbf{q}_h), \forall \mathbf{q}_h \in \mathbf{\Gamma}_{0h}$. This system, though usually overdetermined, admits a unique solution.

Step (B) is similar to solving a Poisson problem using standard nodal linear elements, since $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot) = (\nabla\cdot, \nabla\cdot)$. For the discretization of the Poisson problem with continuous piecewise linear elements, optimal complexity solvers of multi-grid type are available. To obtain an optimal complexity method for step (B) above in a similar fashion, the size of the right-hand side should be bounded uniformly in h. Thus, the procedure in step (A) should yield a uniformly bounded solenoidal component $\mathbf{curl}\, \omega_h$ of the particular solution $\mathbf{r}_h$. For this, it is sufficient that $\|\mathbf{r}_h\|_{L^2} \leqslant C\|f\|_{L^2}$ with C independent of h. This point, which as far as we know has been neglected in the literature [6], [8], necessitates the use of a multi-level approach in step (A).

Remark 2.1. If the triangulation of the domain does not have internal nodes, then by (4) the only divergence-free function is the zero function. In that case, step (B) becomes redundant.

Remark 2.2. In three space dimensions, the homogeneous problem that results in step (B) is the so-called curl-curl problem, for which there is also an optimal complexity multi-level solver available [8].

In step (C), which constitutes the adjoint of the operation performed in (A), a similar multi-level approach is necessary since in practice $\mathbf{p}_h$ is not computed exactly in step (B). Instead, a perturbation $\widehat{\mathbf{p}_h}$ is obtained, resulting in a perturbation $\hat{u}_h$ of u_h. Typically, one would like to have that $\|\hat{u}_h - u_h\|_{L^2} \leqslant C\|\widehat{\mathbf{p}_h} - \mathbf{p}_h\|_{L^2}$ with C independent of h. As was shown in [11], this is not the case if more naive solution methods are used.

3. Two procedures for steps (A) and (C)

We will now describe two procedures for steps (A) and (C) above. The first one is based on a simple two-term recursion. The second procedure is a multi-level version of the first. For the first procedure it is not guaranteed that the solenoidal component that is introduced in the particular solution, remains bounded independent of the mesh size, whereas for the second, it is. Both procedures are based on the fact that $\operatorname{div} \mathbf{\Gamma}_{0h} = W_h$, whereas generally $\dim(\mathbf{\Gamma}_{0h}) > \dim(W_h)$. Implicitly, subspaces $\mathbf{Z}_h \subset \mathbf{\Gamma}_{0h}$ are defined such that $\operatorname{div} \mathbf{Z}_h = W_h$ and $\dim(\mathbf{Z}_h) = \dim(W_h)$, which means that $\mathbf{r}_h$ is uniquely determined by $\mathbf{Z}_h$.

3.1. A marching process. A *marching process* for step (A) constructs a particular solution $\mathbf{r}_h$ with $\operatorname{div} \mathbf{r}_h = P_h f$ by matching the prescribed divergence $P_h f$ triangle by triangle in the following way.

(M1) Construct a list $(\ell_j)_{j=1}^{M}$ of triangles such that ℓ_{j+1} shares an edge with ℓ_j, and each triangle occurs in the list at least once.

(M2) Set $\mathbf{r}_h = 0$, $f_h = P_h f$ initially.

(M3) For $j = 1$ to $M-1$, let φ_j be the unique function from $\mathbf{\Gamma}_{0h}$ such that $\operatorname{div} \varphi_j = f_h$ on ℓ_j and $\operatorname{supp}(\varphi_j) = \ell_j \cup \ell_{j+1}$ and set $\mathbf{r}_h := \mathbf{r}_h + \varphi_j$ and $f_h := f_h - \operatorname{div} \varphi_j$.

Remark 3.1. Note that φ_j in (M3) is a multiple of the function in $\mathbf{\Gamma}_{0h}$ with normal flux equal to one on the edge between ℓ_j and ℓ_{j+1} and normal flux zero on all other edges. Clearly, its support is $\ell_j \cup \ell_{j+1}$.

Proposition 3.2. *The algorithm above results in an* $\mathbf{r}_h \in \mathbf{\Gamma}_{0h}$ *with* $\operatorname{div} \mathbf{r}_h = P_h f$.

Proof. Let $K^* = \ell_M$ be the last triangle in the list and let K be a triangle different from K^*. Let k be such, that $\ell_k = K$ and $\ell_j \neq K$ for all $j > k$. The k-th execution of step (M3) sets $f_h = 0$ on K. By definition of k, for all $j > k$ we have $K \cap \operatorname{supp}(\varphi_j) = \emptyset$, so f_h remains zero on K until completion of the algorithm. Since $K \neq K^*$ was chosen arbitrarily, and f_h has mean value zero on Ω, we conclude that $f_h = 0$ also on K^* and hence on Ω. Since $\operatorname{div} \mathbf{r}_h + f_h = P_h f$ during the whole execution of the algorithm, we conclude that $\operatorname{div} \mathbf{r}_h = P_h f$. $\square$

The list $(\ell_j)_{j=1}^M$ can always be chosen such that $M \leqslant 2\dim(W_h)$, which shows that the process has optimal complexity. The procedure (M1)–(M3) defines a linear mapping $W_h \to \mathbf{\Gamma}_{0h}\colon\ f_h \mapsto \mathbf{r}_h$, which we will denote by $\mathbf{div}_h^+$. Proposition 3.2 states that $\operatorname{div}\mathbf{div}_h^+$ is the identity on W_h. Defining $\mathbf{Z}_h$ as the image of $\mathbf{div}_h^+$ in $\mathbf{\Gamma}_{0h}$, $\mathbf{r}_h$ is the unique element in $\mathbf{Z}_h$ that satisfies $(\operatorname{div}\mathbf{r}_h, w_h) = (f, w_h)$ for all $w_h \in W_h$.

The space $\mathbf{Z}_h$ can alternatively be used as a *testspace* in step (C) to solve u_h once $\mathbf{p}_h$ has been computed as $\mathbf{r}_h + \mathbf{curl}\,\omega_h$ in steps (A) and (B). Defining the discrete adjoint $\mathbf{div}_h^*\colon\ W_h \to \mathbf{Z}_h$ of the divergence by the relation

$$(6)\qquad \forall w_h \in W_h,\ \forall \mathbf{z}_h \in \mathbf{Z}_h,\ (\mathbf{div}_h^* w_h, \mathbf{z}_h) = (w_h, \operatorname{div}\mathbf{z}_h)$$

and denoting L^2-orthogonal projection of $\mathbf{\Gamma}_{0h}$ onto $\mathbf{Z}_h$ by $\mathbf{\Pi}_h$, it is not difficult to verify that the solution u_h of the equation $\mathbf{div}_h^* u_h = \mathbf{\Pi}_h\mathbf{p}_h$ results from the following consecutive steps:

(N2) Assign an arbitrary value to $u_h(\ell_1)$.

(N3) For $j = 1$ to $M-1$, let $\varphi_j \in \mathbf{\Gamma}_{0h}$ be such that $\operatorname{supp}(\varphi_j)=\ell_j \cup \ell_{j+1}$ and compute $u_h(\ell_{j+1})$ from $u_h(\ell_j)$ by using the relation $(u_h, \operatorname{div}\varphi_j) = (\mathbf{p}_h, \varphi_j)$.

(N4) Shift the solution obtained to mean zero.

Theorem 3.3. *There exists a constant $C_0 = C_0(h)$ such that*

$$(7)\qquad \forall \mathbf{z}_h \in \mathbf{Z}_h,\ \|\mathbf{z}_h\|_{L^2} \leqslant C_0\|\operatorname{div}\mathbf{z}_h\|_{L^2},$$

or, equivalently, $\forall w \in W_h$, $\|w_h\|_{L^2} \leqslant C_0\|\mathbf{div}_h^* w_h\|_{L^2}$. In particular, for $\mathbf{r}_h = \mathbf{div}_h^+ P_h f$ and for the solutions of the perturbed and exact equations $\mathbf{div}_h^*\tilde{u}_h = \mathbf{\Pi}_h\tilde{\mathbf{p}}_h$ and $\mathbf{div}_h^* u_h = \mathbf{\Pi}_h\mathbf{p}_h$ in step (C), *we have*

$$(8)\qquad \|\mathbf{r}_h\|_{L^2} \leqslant C_0\|f\|_{L^2} \text{ and } \|u_h - \tilde{u}_h\|_{L^2} \leqslant C_0\|\mathbf{p}_h - \tilde{\mathbf{p}}_h\|_{L^2}.$$

Proof. Since $\mathbf{Z}_h = \mathbf{div}_h^+ W_h$ and $\operatorname{div}\mathbf{div}_h^+$ is the identity on W_h, it follows that div is a bijection between the finite dimensional spaces $\mathbf{Z}_h$ and W_h. Obviously, the norm of its inverse equals the norm of the inverse of its adjoint. $\square$

As discussed in Section 2.1, steps (A), (B) and (C) can only be expected to give a method of optimal complexity for solving the mixed system when the procedure $\mathbf{div}_h^+$, or equivalently the space $\mathbf{Z}_h$, is chosen such that (7) is valid with a constant C_0 that is bounded uniformly in h. Unfortunately, as can be deduced from an example in [11], using marching as in this section, it may increase rapidly as h tends to zero.

3.2. A multi-level procedure. We will now study the important practical case of nested sequences of discrete spaces $W_0 \subset W_1 \subset \ldots$ and $\mathbf{\Gamma}_0 \subset \mathbf{\Gamma}_1 \subset \ldots$ corresponding to a sequence of triangulations $(\mathcal{T}_\ell)$. We denote the discrete solution on $\mathcal{T}_\ell$

by $(u_\ell, \mathbf{p}_\ell)$. For simplicity, only spaces arising from uniform refinements of an initial triangulation $\mathcal{T}_0$ are considered. By this we mean that each $\mathcal{T}_\ell$ arises from $\mathcal{T}_{\ell-1}$ by subdividing each triangle $K \in \mathcal{T}_{\ell-1}$ into four congruent subtriangles. Denote orthogonal projection on W_ℓ by P_ℓ. Then $(P_\ell - P_{\ell-1})f$ is orthogonal to $W_{\ell-1}$ and to each constant function, and hence to the characteristic function $\chi_K \in W_{\ell-1} \oplus \mathbb{R}$ of each $K \in \mathcal{T}_{\ell-1}$. This implies that $(P_\ell - P_{\ell-1})f$ has zero mean on each $K \in \mathcal{T}_{\ell-1}$. So, by Remark 2.1, for each $K \in \mathcal{T}_{\ell-1}$ there exists a unique $\mathbf{y}_\ell \in \mathbf{\Gamma}_\ell$ with $\operatorname{supp}(\mathbf{y}_\ell) \subset K$ such that $\operatorname{div} \mathbf{y}_\ell = (P_\ell - P_{\ell-1})f$ on K and zero elsewhere. This leads to the following multi-level method for finding a particular solution in step (A), in which each function $\mathbf{r}_\ell$ is such that $\operatorname{div} \mathbf{r}_\ell = P_\ell f$.

(S1) Use steps (M1)–(M3) to find $\mathbf{r}_0$ such that $\operatorname{div} \mathbf{r}_0 = P_0 f$. Set $\ell = 1$.

(S2) For each $K \in \mathcal{T}_{\ell-1}$, find the function $\mathbf{y}_\ell^K \in \mathbf{\Gamma}_\ell$ with $\operatorname{supp}(\mathbf{y}_\ell^K) \subset K$ such that $\operatorname{div} \mathbf{y}_\ell^K = (P_\ell - P_{\ell-1})f$ on K and zero elsewhere. Afterwards, set $\mathbf{r}_\ell = \mathbf{r}_{\ell-1} + \mathbf{y}_\ell$, where $\mathbf{y}_\ell = \sum\limits_{K \in \mathcal{T}_{\ell-1}} \mathbf{y}_\ell^K$.

(S3) Until some final level is reached, set $\ell := \ell + 1$ and return to step (S2).

Just as in the previous section, this procedure implicitly constructs linear mappings $\mathbf{div}_\ell^+ \colon W_\ell \to \mathbf{\Gamma}_\ell$ with $\operatorname{div} \mathbf{div}_\ell^+$ equal to the identity on W_ℓ and spaces $\mathbf{Z}_\ell = \mathbf{div}_\ell^+(W_\ell)$. For all $\ell \geqslant 1$, the space $\mathbf{Z}_\ell$ can then be written as $\mathbf{Z}_\ell = \mathbf{Z}_{\ell-1} \oplus \mathbf{Y}_\ell$, where $\mathbf{Y}_\ell$ is the span of all functions in $\mathbf{\Gamma}_\ell$ with support contained in some $K \in \mathcal{T}_{\ell-1}$.

Lemma 3.4. *There exists a constant C_∞ such that with $C_\ell = 2^{-\ell} C_\infty (\ell \geqslant 1)$,*

$$\forall \mathbf{y}_\ell \in \mathbf{Y}_\ell, \;\; \|\mathbf{y}_\ell\|_{L^2} \leqslant C_\ell \|\operatorname{div} \mathbf{y}_\ell\|_{L^2}. \tag{9}$$

Proof. The statement follows easily from a homogeneity argument. One may consult [9], where this result was used in a different context. □

Theorem 3.5. *There exists a $\beta > 0$ such that for each $\ell \geqslant 0$,*

$$\forall \mathbf{z}_\ell \in \mathbf{Z}_\ell, \;\; \beta \|\mathbf{z}_\ell\|_{L^2} \leqslant \|\operatorname{div} \mathbf{z}_\ell\|_{L^2}. \tag{10}$$

Proof. Write $\mathbf{z}_\ell \in \mathbf{Z}_\ell$ as $\mathbf{z}_\ell = \sum\limits_{j=0}^{\ell} \mathbf{y}_j$, with $\mathbf{y}_0 \in \mathbf{Z}_0$ and $\mathbf{y}_j \in \mathbf{Y}_j$ for $j \geqslant 1$. Then

$$\|\mathbf{z}_\ell\|_{L^2} \leqslant \sum_{j=0}^{\ell} \|\mathbf{y}_j\|_{L^2} \leqslant \sum_{j=0}^{\ell} C_j \|\operatorname{div} \mathbf{y}_j\|_{L^2} \leqslant \|\operatorname{div} \mathbf{z}_\ell\|_{L^2} \sqrt{C_0^2 + \frac{1}{3} C_\infty^2} \tag{11}$$

where we have used the triangle inequality, Theorem 3.3 applied to $\mathbf{y}_0$, Lemma 3.4 applied to the $\mathbf{y}_j$ with $j \geqslant 1$, the Schwarz inequality, the orthogonality of the divergences of the $\mathbf{y}_j$, and the convergence of the geometric sum. □

This proves the stability of step (A) uniformly in ℓ. As noted before, Theorem 3.5 is equivalent to the statement that for each $\ell \geqslant 0$,

$$\forall w_\ell \in W_\ell,\ \beta \|w_\ell\|_{L^2} \leqslant \|\mathbf{div}_\ell^* w_\ell\|_{L^2}, \tag{12}$$

which takes care of the stability of step (C). Finally, we show how all this is related to the Babuška-Brezzi inf-sup condition for the pairs $\mathbf{Z}_\ell, W_\ell$. For this, recall the definition $\|\mathbf{q}\|_{\mathrm{div}}^2 = \|\operatorname{div} \mathbf{q}\|_{L^2}^2 + \|\mathbf{q}\|_{L^2}^2$.

Theorem 3.6. *The spaces $\mathbf{Z}_\ell, W_\ell$ satisfy the Babuška-Brezzi inf-sup condition*

$$\exists \gamma > 0,\ \forall \ell \geqslant 0,\ \forall w_\ell \in W_\ell,\ \gamma \|w_\ell\|_{L^2} \leqslant \sup_{0 \neq \mathbf{z}_\ell \in \mathbf{Z}_\ell} \frac{(\operatorname{div} \mathbf{z}_\ell, w_\ell)}{\|\mathbf{z}_\ell\|_{\mathrm{div}}}. \tag{13}$$

Proof. Theorem 3.5 shows that for all $\mathbf{q} \in \mathbf{Z}_\ell$, $(1 + \beta^{-2})^{-1/2} \|\mathbf{z}_\ell\|_{\mathrm{div}} \leqslant \|\operatorname{div} \mathbf{z}_\ell\|_{L^2}$, and using this, (13) follows by choosing $\mathbf{z}_\ell = \mathbf{div}_\ell^+ w_\ell$ for given nonzero w_ℓ. □

In fact, if (13) holds for some pair of spaces $\mathbf{Z}_\ell, W_\ell$ with $\operatorname{div} \mathbf{Z}_\ell = W_\ell$ then there exists a $\beta > 0$ such that (12) holds. Indeed, using that $\|\mathbf{z}_\ell\|_{L^2} \leqslant \|\mathbf{z}_\ell\|_{\mathrm{div}}$, we obtain

$$\gamma \|w_\ell\|_{L^2} \leqslant \sup_{0 \neq \mathbf{z}_\ell \in \mathbf{Z}_\ell} \frac{(\operatorname{div} \mathbf{z}_\ell, w_\ell)}{\|\mathbf{z}_\ell\|_{L^2}} \leqslant \sup_{0 \neq \mathbf{z}_\ell \in \mathbf{Z}_\ell} \frac{(\mathbf{z}_\ell, \mathbf{div}_\ell^* w_\ell)}{\|\mathbf{z}_\ell\|_{L^2}} = \|\mathbf{div}_\ell^* w_\ell\|_{L^2}. \tag{14}$$

If $\mathbf{Z}_\ell$ and W_ℓ are finite dimensional, (12) is again equivalent with (10). This shows that alternatively, the Babuška-Brezzi inf-sup condition could have been taken as a starting point in proving the stability of the multi-level solvers.

It is interesting to note that since there are no nonzero divergence-free functions in $\mathbf{Z}_\ell$, also the Babuška-Brezzi ellipticity condition is satisfied. So, the spaces $\mathbf{Z}_\ell, W_\ell$ themselves form a stable pair for the mixed discretization of the Poisson equation as in (3). Even though this allows for an optimal complexity and direct solver, the spaces $\mathbf{Z}_\ell$ unfortunately lack approximation properties.

For the Laplace equation, things simplify considerably, and the consequences will be briefly outlined in Section 4.1. In Section 4.2 we note that Babuška's saddle point problem [1] can be treated similarly.

4.1. Solving the mixed discretization of the Laplace equation. Consider the Laplace equation with Dirichlet boundary data, that are assumed to have mean zero without loss of generality,

$$-\Delta u = 0 \quad \text{in } \Omega, \quad \text{and} \quad u = g \quad \text{on } \partial\Omega \text{ with } \langle g, 1\rangle = 0. \tag{15}$$

Its mixed finite element formulation seeks $(u_h, \mathbf{p}_h)$ in $W_h \times \mathbf{\Gamma}_{0h}$ satisfying

$$(\mathbf{p}_h, \mathbf{q}_h) - (u_h, \operatorname{div} \mathbf{q}_h) \;=\; \langle g, \mathbf{q}_h^T \nu\rangle \quad \text{and} \quad (\operatorname{div} \mathbf{p}_h, w_h) = 0, \tag{16}$$

for all $(w_h, \mathbf{q}_h) \in W_h \times \mathbf{\Gamma}_{0h}$, where here $\mathbf{\Gamma}_{0h}$ denotes the subspace of Raviart-Thomas functions with mean zero normal traces. By a variant of (4) we have that $\mathbf{p}_h = \mathbf{curl}\,\omega_h$ for some $\omega_h \in V_h$, where V_h is the space of continuous piecewise linear functions, so step (B) reduces to finding a solution ω_h of

$$\forall v_h \in V_h, \quad (\mathbf{curl}\,\omega_h, \mathbf{curl}\,v_h) \;=\; \langle g, \mathbf{curl}\,v_h^T \nu\rangle. \tag{17}$$

This system also produces (modulo a constant) the standard finite element approximation ω_h of the solution ω of the Laplace equation

$$-\Delta\omega = 0 \quad \text{in } \Omega, \quad \nabla\omega^T\nu = \frac{\partial}{\partial\tau} g \quad \text{on } \partial\Omega, \tag{18}$$

and as observed in [3], ω is related to u in the sense that the pair (ω, u) solves the Cauchy-Riemann equations. Testing the left equation of (16) in the same spaces $\mathbf{Z}_\ell$ as in Section 3.2, the boundary term vanishes because each $\mathbf{z}_\ell \in \mathbf{Z}_\ell$ has normal trace zero on $\partial\Omega$. So, given the standard approximation ω_h of ω, the multi-level method can be used to solve the mixed approximation u_h of u from $\mathbf{div}_\ell^* u_h = \mathbf{\Pi}_h \mathbf{curl}\,\omega_h$ in a stable way and in optimal complexity. See [3] for more details.

4.2. The Poisson equation with inhomogeneous boundary data. Consider the Poisson equation $-\Delta u = f$ with inhomogeneous Dirichlet boundary condition $u = g$ on $\partial\Omega$. Let $\gamma\colon H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ be the trace operator. Then the Poisson problem can be written as a saddle point problem by looking for the pair $(u, \lambda) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ such that for all $(v, \mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$,

$$(\nabla u, \nabla v) - \langle\gamma(v), \lambda\rangle = (f, v) \quad \text{and} \quad \langle\gamma(u), \mu\rangle = \langle g, \mu\rangle. \tag{19}$$

Note that the trace operator takes the place of the divergence in the previous section. Discretizing this in V_h and $W_h = \gamma(V_h)$ gives the mixed discrete problem of finding $(u_h, \lambda_h) \in V_h \times W_h$ such that for all $(v_h, \mu_h) \in V_h \times W_h$,

$$(20) \qquad (\nabla u_h, \nabla v_h) - \langle \gamma(v_h), \lambda_h \rangle = (f, v_h) \quad \text{and} \quad \langle \gamma(u_h), \mu_h \rangle = \langle g, \mu_h \rangle.$$

Similar to before, this problem can be solved in three separate steps: finding a particular solution satisfying the second equation, solving the homogeneous problem in V_{0h}, and finally computing the Lagrangian multiplier. It can be shown that a naive choice for the particular solution may hamper the overall solution process and that a similar multi-level method should be used instead. An abstract treatment of the methods presented in this paper is in preparation.

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Authors' addresses: *Jan H. Brandts*, Utrecht University, Mathematical Institute, P. O. Box 80.010, 3508 TA, Utrecht, Netherlands, e-mail: `brandts@math.nu.nl`; *Rob Stevenson*, Utrecht University, Mathematical Institute, P. O. Box 80.010, 3508 TA, Utrecht, Netherlands, e-mail: `stevenso@math.uu.nl`.