A REMARK ON THE EXISTENCE OF STEADY NAVIER-STOKES FLOWS IN 2D SEMI-INFINITE CHANNEL INVOLVING THE GENERAL OUTFLOW CONDITION

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Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. We consider the steady Navier-Stokes equations in a 2-dimensional unbounded multiply connected domain Ω under the general outflow condition. Let T be a 2-dimensional straight channel $\mathbb{R}\times(-1,1)$. We suppose that $\Omega\cap\{x_1<0\}$ is bounded and that $\Omega\cap\{x_1>-1\}=T\cap\{x_1>-1\}$. Let V be a Poiseuille flow in T and μ the flux of V. We look for a solution which tends to V as $x_1\to\infty$. Assuming that the domain and the boundary data are symmetric with respect to the x_1-axis, and that the axis intersects every component of the boundary, we have shown the existence of solutions if the flux is small (Morimoto-Fujita [8]). Some improvement will be reported in this note. We also show certain regularity and asymptotic properties of the solutions.

Keywords: stationary Navier-Stokes equations, non-vanishing outflow, 2-dimensional semi-infinite channel, symmetry

MSC 2000: 35Q30, 76D05

1. Introduction

Let Ω be an unbounded domain in $\mathbb{R}^2$, the x_1x_2 plane, satisfying the following two conditions:

(I) $\Omega\cap\{(x_1,x_2);\ x_1>-1\}=\{(x_1,x_2);\ x_1>-1,-1<x_2<1\}$ and $\Omega\cap\{(x_1,x_2);\ x_1<0\}$ is bounded.

(II) The boundary of Ω is smooth and is composed of an infinite component γ_0 and finite components $\gamma_1,\gamma_2,\ldots,\gamma_N$, the latter being simply closed curves. Namely,

$$\partial\Omega=\gamma_0\cup\gamma_1\cup\ldots\cup\gamma_N.$$

γ_0 can be regarded as the outer boundary of Ω while γ_i $(1\leqslant i\leqslant N)$ as the inner boundary.

We consider the boundary value problem of the Navier-Stokes equations

$$\text{(NS)} \qquad \begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \end{cases}$$

with the boundary conditions

$$\text{(BC)} \qquad \begin{cases} \boldsymbol{u} = \boldsymbol{\beta} & \text{on} \quad \partial\Omega, \\ \boldsymbol{u} \to \mu\boldsymbol{U} & \text{as} \quad x_1 \to \infty \end{cases}$$

where $\boldsymbol{f}$ is a function defined on Ω, $\boldsymbol{\beta}$ is a given function on $\partial\Omega = \gamma_0 \cup \gamma_1 \cup \ldots \cup \gamma_N$ satisfying $\boldsymbol{\beta} = 0$ on γ_0. $\boldsymbol{U}$ is the standard Poiseuille flow in $T = \mathbb{R}^1 \times (-1, 1)$:

$$\text{(U)} \qquad \boldsymbol{U} = \tfrac{3}{4}(1 - x_2^2, 0),$$

and μ is a nonzero constant. For the boundary value $\boldsymbol{\beta}$, we suppose

$$\int_{\partial\Omega} \boldsymbol{\beta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = -\mu,$$

$\boldsymbol{n}$ being the unit outward normal vector to $\partial\Omega$.

Suppose that the domain Ω is symmetric with respect to the x_1-axis, that every γ_i ($0 \leqslant i \leqslant N$) intersects the x_1-axis and that the boundary value $\boldsymbol{\beta}$ and the external force $\boldsymbol{f}$ are also symmetric with respect to the x_1-axis. Here the vector field $\boldsymbol{\varphi}(\boldsymbol{x}) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2))$ is called symmetric with respect to the x_1-axis if $\varphi_1(x_1, x_2)$ is an even function of x_2 and $\varphi_2(x_1, x_2)$ is an odd function of x_2, that is,

$$\varphi_1(x_1, x_2) = \varphi_1(x_1, -x_2), \ \varphi_2(x_1, x_2) = -\varphi_2(x_1, -x_2)$$

holds. Under these assumptions we can show the existence of a solution to (NS), (BC) if $\nu - |\mu|\sigma^S > 0$ holds. Here σ^S is a constant depending only on the channel T.

In Section 2, we state the notation and the results concerning the existence of a solution (Theorem 1), the extension of the boundary value (Lemma 1), and the regularity and the asymptotic behavior of the flow (Theorem 2). In Section 3, some preliminaries, and in Section 4, the proof of Theorem 1 are given. The proof of Theorem 2 is similar to that in [7] and is omitted.

Let us recall that for a certain "full" infinite channel in $\mathbb{R}^2$, symmetric with respect to the axis and with symmetric data, we have obtained the existence of a stationary flow under the general outflow condition ([7]).

For a simply connected channel-like domain or a multiply connected domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ under a stringent outflow condition, that is, under the condition that the flux is zero on every connected component of the boundary, there are many works. Among others, we refer to Amick [1], [2]. We use his argument in order to show the existence result.

Let $\boldsymbol{C}_0^\infty(\Omega)$ be the set of all smooth vector valued functions with compact support in Ω. Let $L^2(\Omega)$ be the set of all vector valued square integrable functions in Ω with the inner product $(\cdot,\cdot)_\Omega$ and the norm $||\cdot||_\Omega$. If it is clear from the context, we write simply $(\cdot,\cdot)$ and $||\cdot||$. Let $L^p(\Omega)$ be the set of all vector valued functions $\boldsymbol{u}$ such that $|\boldsymbol{u}|^p$ is integrable in Ω. The norm is denoted by $||\boldsymbol{u}||_{L^p(\Omega)}$ or simply by $||\boldsymbol{u}||_p$. $W^{m,p}(\Omega)$ are the Sobolev spaces; $H^m(\Omega) = W^{m,2}(\Omega)$. $H_0^m(\Omega)$ is the closure of $\boldsymbol{C}_0^\infty(\Omega)$ in $H^m(\Omega)$. Let $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ be the set of all smooth solenoidal vector valued functions with compact support in Ω. $H_\sigma(\Omega)$ is the closure of $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)$. $V(\Omega)$ is the completion of $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ in the Dirichlet norm $||\nabla\cdot||$. Let $\boldsymbol{C}_{0,\sigma}^{\infty,S}(\Omega)$ be the set of all functions in $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ symmetric with respect to the x_1-axis. $V^S(\Omega)$ is the completion of $\boldsymbol{C}_{0,\sigma}^{\infty,S}(\Omega)$ in the Dirichlet norm $||\nabla\cdot||$.

By definition, $\boldsymbol{u}$ is called a weak solution to the problem (NS), (BC) if

(i) $\boldsymbol{u} - \mu\boldsymbol{U} \in H^1(\Omega)$,

(ii) $\operatorname{div}\boldsymbol{u} = 0$ in Ω,

(iii) $\nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})$ $(\forall\boldsymbol{v}\in\boldsymbol{C}_{0,\sigma}^\infty(\Omega))$, and $\boldsymbol{u}$ satisfies the boundary condition,

(iv) $\boldsymbol{u} = \boldsymbol{\beta}$ on $\gamma_0\cup\gamma_1\cup\ldots\cup\gamma_N$ in the trace sense.

Our main result is as follows.

Theorem 1. *Suppose that $\Omega\subset\mathbb{R}^2$ is symmetric with respect to the x_1-axis satisfying* (I) *and* (II), *and that every γ_i $(0\leqslant i\leqslant N)$ intersects the x_1-axis. Assume that the boundary value $\boldsymbol{\beta}\in H^{3/2}(\partial\Omega)$ is also symmetric with respect to the x_1-axis, $\boldsymbol{\beta} = 0$ on γ_0 and*

$$\int_{\partial\Omega}\boldsymbol{\beta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = -\mu.$$

Suppose further that $\boldsymbol{f}\in L^2(\Omega)$ is symmetric with respect to the x_1-axis. Let

$$\sigma^S \equiv \sup_{\boldsymbol{\varphi}\in V^S(T)}\frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi},\boldsymbol{U})_T}{||\nabla\boldsymbol{\varphi}||_T^2}. \tag{1}$$

There exists a symmetric weak solution to (NS), (BC) *provided the inequality $\nu - |\mu|\sigma^S > 0$ holds true.*

R e m a r k 1. The constant σ^S does not depend on the domain Ω, but depends on the channel T. This constant is studied in detail by Amick [2]. Nevertherless, we shall give an elementary proof of $\sigma^S > 0$ (Lemma 4). In the previous paper [8], we have shown that there exists a solution if $\nu - |\mu|c_0 > 0$, where

$$c_0 \equiv \sup_{\boldsymbol{w}\in V^S(\Omega)}\frac{|((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{U})_\Omega|}{||\nabla\boldsymbol{w}||_\Omega^2}. \tag{2}$$

Let

$$c^S \equiv \sup_{\boldsymbol{w}\in V^S(\Omega)} \frac{((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{U})_\Omega}{\|\nabla\boldsymbol{w}\|_\Omega^2}. \tag{3}$$

Then, it is obvious that $c_0 \geqslant c^S$. In the next section, we will show that the constant c^S is larger than or equal to σ^S and therefore positive. Furthermore, if the domain Ω is contained in the channel T, then $\sigma^S = c^S$ (Lemma 5).

For the regularity and the asymptotic behavior of the flow, we obtain the following result, with the notation

$$\Omega^R = \{(x_1,x_2)\in\Omega;\ x_1 > R\},$$
$$\Omega_0 = \Omega\cap\{(x_1,x_2);\ x_1 < 0\},$$
$$\Omega_1 = \Omega\cap\{(x_1,x_2);\ x_1 > 0\}.$$

Theorem 2. *Suppose $\boldsymbol{f}=0$. The weak solution $\boldsymbol{u}$ to* (NS), (BC) *obtained in Theorem* 1 *is continuous in Ω_1, and converges to $\mu\boldsymbol{U}$ uniformly at infinity, that is,*

$$\sup_{\Omega^R} |\boldsymbol{u}(\boldsymbol{x}) - \mu\boldsymbol{U}(\boldsymbol{x})| \to 0\ (R\to\infty). \tag{4}$$

R e m a r k 2. Theorem 2 holds for nonzero $\boldsymbol{f}\in L^2(\Omega)$ with compact support.

3. Preliminaries

The next lemma plays an important role for the proof of Theorem 1. The proof of the lemma is similar to Fujita [4] with resort to the virtual drain method and is omitted.

Lemma 1. *Suppose that $\Omega\subset\mathbb{R}^2$ is symmetric with respect to the x_1-axis satisfying conditions* (I) *and* (II), *that every γ_i $(0\leqslant i\leqslant N)$ intersects the x_1-axis and that the boundary value $\boldsymbol{\beta}_0\in H^{3/2}(\partial\Omega)$ is symmetric with respect to the x_1-axis, has compact support in $\{(x_1,x_2)|x_1<0\}\cap\partial\Omega$ and $\int_{\partial\Omega}\boldsymbol{\beta}_0\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0$. Then for every $\varepsilon>0$ there exists a symmetric solenoidal extension $\boldsymbol{b}_\varepsilon$ of $\boldsymbol{\beta}_0$ such that $\boldsymbol{b}_\varepsilon\in H^2(\Omega)$ and*

$$|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{b}_\varepsilon)_\Omega| \leqslant \varepsilon\|\nabla\boldsymbol{v}\|_\Omega^2\quad(\forall\boldsymbol{v}\in V^S(\Omega)). \tag{5}$$

R e m a r k 3. The boundary integral

$$\int_{\gamma_i}\boldsymbol{\beta}_0\cdot\boldsymbol{n}\,\mathrm{d}\sigma\quad(0\leqslant i\leqslant N)$$

is not necessarily zero.

R e m a r k 4. It should be noted that we can take $\boldsymbol{b}_\varepsilon$ with compact support contained in $\overline{\Omega}_0$.

Now, we show some properties of the constant σ^S. First, we study the case without assuming the symmetry. Put

$$\sigma(U) \equiv \sup_{\boldsymbol{\varphi}\in V(T)} \frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{U})_T}{||\nabla\boldsymbol{\varphi}||_T^2}, \quad c(U) \equiv \sup_{\boldsymbol{w}\in V(\Omega)} \frac{((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{U})_\Omega}{||\nabla\boldsymbol{w}||_\Omega^2}. \tag{6}$$

Lemma 2. *$\sigma(U)$ does not depend on the direction of $\boldsymbol{U}$ and is positive, i.e.,*

$$\sigma(U) = \sigma(-U) > 0$$

and in particular,

$$\sigma(\mu U) = |\mu|\sigma(U) \quad (\mu \in \mathbb{R}).$$

P r o o f. Let $\boldsymbol{v} = (v_1, v_2)$ be an arbitrary element in $V(T)$. If we define $\boldsymbol{w} = (w_1, w_2)$ by

$$w_1(x_1, x_2) = -v_1(-x_1, x_2), \quad w_2(x_1, x_2) = v_2(-x_1, x_2),$$

then it is easy to see that $\boldsymbol{w} \in V(T)$. Write $\mathcal{F}\boldsymbol{v} = \boldsymbol{w}$. The operator $\mathcal{F}\colon V(T) \to V(T)$ is one to one, onto and isometric. By a simple calculation, we get

$$((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{U})_T = -((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{U})_T.$$

This proves $\sigma(U) = \sigma(-U) \geqslant 0$.

Now let us prove $\sigma(U) > 0$. Let f be the stream function of $\boldsymbol{v} = (v_1, v_2)$, i.e., $v_1 = D_2 f$, $v_2 = -D_1 f$. Here, D_i denotes the differentiation $\partial/\partial x_i$ with respect to x_i. Then

$$((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{U})_T = 3/2 \int_T x_2 v_1 v_2 \,\mathrm{d}x = -3/2 \int_T x_2 D_1 f D_2 f \,\mathrm{d}x$$

where $\boldsymbol{U} = (U_1, U_2) = 3/4(1 - x_2^2, 0)$.

Suppose that $\int_T x_2 D_1 f D_2 f \,\mathrm{d}x = 0$ holds true for all nonzero f in $C_0^\infty(T)$. Let $f, g \in C_0^\infty(T)$, $f, g \neq 0$. Then

$$0 = \int_T x_2 D_1(f+g) D_2(f+g) \,\mathrm{d}x = \int_T x_2 (D_1 f D_2 g + D_2 f D_1 g) \,\mathrm{d}x.$$

Integrating by parts, we obtain

$$\int_T f(2x_2 D_1 D_2 g + D_1 g) \,\mathrm{d}x = 0.$$

Since f is arbitrary, $2x_2 D_1 D_2 g + D_1 g$ should be zero in T. This contradicts $g \neq 0$. □

Lemma 3. *The constant $\sigma(U)$ is less than or equal to $c(U)$, i.e.,*

$$\sigma(U) \leqslant c(U).$$

Furthermore, if the domain Ω is contained in the channel T, then $\sigma(U)$ is equal to $c(U)$.

Proof. Let $\boldsymbol{\varphi} \in \boldsymbol{C}^{\infty}_{0,\sigma}(T)$, $k \in \mathbb{R}$ and put $\boldsymbol{v}_{\varphi}(x_1, x_2) = \boldsymbol{\varphi}(x_1 - k, x_2)$. Then $\operatorname{div}\boldsymbol{v}_{\varphi} = 0$. We choose k so large that the support of $\boldsymbol{v}_{\varphi}$ is contained in Ω. Therefore $\boldsymbol{v}_{\varphi}$ can be considered belonging to $V(\Omega)$. Since $((\boldsymbol{v}_{\varphi} \cdot \nabla)\boldsymbol{v}_{\varphi}, \boldsymbol{U})_{\Omega} = ((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{U})_T$ holds, we have

$$\frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{U})_T}{||\nabla\boldsymbol{\varphi}||^2_T} = \frac{((\boldsymbol{v}_{\varphi} \cdot \nabla)\boldsymbol{v}_{\varphi}, \boldsymbol{U})_{\Omega}}{||\nabla\boldsymbol{v}_{\varphi}||^2_{\Omega}} \leqslant c(U).$$

Therefore $\sigma(U) \leqslant c(U)$.

Next we consider the case $\Omega \subset T$. Let $\boldsymbol{v}$ be an arbitrary element in $V(\Omega)$ and put

$$\boldsymbol{\varphi}(x) = \begin{cases} \boldsymbol{v}(x) & x \in \Omega \\ 0 & x \in T \setminus \Omega. \end{cases}$$

Then $\boldsymbol{\varphi} \in V(T)$ and

$$\frac{((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{U})_{\Omega}}{||\nabla\boldsymbol{v}||^2_{\Omega}} = \frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{U})_T}{||\nabla\boldsymbol{\varphi}||^2_T} \leqslant \sigma(U).$$

Therefore $c(U) \leqslant \sigma(U)$, and $c(U) = \sigma(U)$ holds true. □

Since the operator $\mathcal{F}$ preserves the symmetry, we can prove the next two lemmas similarly to the above ones.

Lemma 4. *We have*

$$\sigma(U) \geqslant \sigma^S(U), \qquad \sigma^S(U) = \sigma^S(-U) > 0$$

and in particular,

$$\sigma^S(\mu U) = |\mu|\sigma^S(U).$$

Lemma 5. *$\sigma^S(U) \leqslant c^S(U)$ holds true. Furthermore, if the domain Ω is contained in the channel T, then $\sigma^S(U)$ is equal to $c^S(U)$.*

Hereafter we write simply σ, σ^S instead of $\sigma(U)$, $\sigma^S(U)$.

Now we construct "an approximation" $\boldsymbol{s} \in \boldsymbol{C}_{0,\sigma}^{\infty,S}(\Omega)$ of the Poiseuille flow $\boldsymbol{U}$. We follow the argument of Amick [1]. Let $\varphi(t) = 3/4(t - t^3/3)$ be a stream function of $\boldsymbol{U}$,

$$\boldsymbol{U} = \nabla^{\perp}\varphi(x_2) = (D_2\varphi, -D_1\varphi) = (\varphi'(x_2), 0)$$

and $\varrho(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$. Let δ_0, κ_0 be real numbers such that $\delta_0 > 0, 1/4 > \kappa_0 > 0$. We take a function $j(t) \in C_0^{\infty}[0, \infty)$ having the following properties:

$$\begin{aligned}
&0 \leqslant j(t) \leqslant 1/t,\\
&j(t) = 0 \quad (0 \leqslant t \leqslant \kappa_0\delta_0,\ (1-\kappa_0)\delta_0 \leqslant t),\\
&j(t) = 1/t \quad (2\kappa_0\delta_0 \leqslant t \leqslant (1-2\kappa_0)\delta_0).
\end{aligned}$$

Put $h(t) = 1 - \int_0^t j(s)\,\mathrm{d}s / \int_0^{\infty} j(s)\,\mathrm{d}s$. Then

$$\begin{aligned}
h(t) &\equiv 1 \text{ for } 0 \leqslant t \leqslant \kappa_0\delta_0,\\
h(t) &\equiv 0 \text{ for } \delta_0 \leqslant t,\\
t \cdot h'(t) &\to 0 \text{ uniformly as } \kappa_0 \to 0.
\end{aligned}$$

See Fujita [3] for details. Put $\widetilde{\boldsymbol{U}} = \nabla^{\perp}\{h(\varrho(x))\varphi(x_2)\}$. Then $\operatorname{div}\widetilde{\boldsymbol{U}} = 0$ in Ω , $\widetilde{\boldsymbol{U}}|_{\partial\Omega} = \boldsymbol{U}|_{\partial\Omega}$ and

$$\widetilde{\boldsymbol{U}}(x) = h(\varrho(x))\boldsymbol{U} + \varphi(x_2)h'(\varrho(x))\nabla^{\perp}\varrho.$$

For x in Ω_1 we have

$$\widetilde{\boldsymbol{U}}(x) = h(\varrho(x))\boldsymbol{U} + \varphi(x_2)h'(\varrho(x))(-\operatorname{sgn} x_2, 0),$$

since $\varrho(x) = 1 \pm x_2$ in Ω_1. Let $\theta(t)$ be a smooth function such that $0 \leqslant \theta(t) \leqslant 1$ $(t \in \mathbb{R})$, $\theta(t) \equiv 0$ for $t \leqslant 1/2$ and $\theta(t) \equiv 1$ for $t \geqslant 1$. Let $\delta > 0$ and put $\theta_\delta(x) = \theta(\delta x_1)$. We introduce the following function:

$$\boldsymbol{s}(x) = \begin{cases} \boldsymbol{U} - \widetilde{\boldsymbol{U}} & \text{in} \quad \Omega_0,\\ \nabla^{\perp}\{(1 - h(\varrho(x)))(1 - \theta_\delta(x)^2)\varphi(x_2)\} & \text{in} \quad \Omega_1. \end{cases} \tag{7}$$

It is easy to see that the support of $\boldsymbol{s}$ is contained in $\Omega \cap \{x_1 < 1/\delta\}$ and $\boldsymbol{s} \in \boldsymbol{C}_{0,\sigma}^{\infty,S}(\Omega)$. The function $\boldsymbol{s}$ is "an approximation" of $\boldsymbol{U}$ in the following sense.

Lemma 6. *There exists a positive constant C_0 such that for every positive ε and δ there exists $\boldsymbol{s} \in \boldsymbol{C}_{0,\sigma}^{\infty,S}(\Omega)$ satisfying the following estimates:*

$$|((\boldsymbol{w} \cdot \nabla)\boldsymbol{w}, \mu\boldsymbol{U})_{\Omega_0} - ((\boldsymbol{w} \cdot \nabla)\boldsymbol{w}, \mu\boldsymbol{s})_{\Omega_0}| \leqslant \varepsilon||\nabla\boldsymbol{w}||^2, \quad \forall \boldsymbol{w} \in V(\Omega) \tag{8}$$

and

$$(9)\qquad |((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\mu\boldsymbol{U})_{\Omega_1}-((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\mu\boldsymbol{s})_{\Omega_1}-((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\theta_\delta^2\mu\boldsymbol{U})_{\Omega_1}| \leqslant (\varepsilon+C_0\delta)||\nabla\boldsymbol{w}||^2,\quad \forall\boldsymbol{w}\in V(\Omega).$$

P r o o f. The left hand side of (8) is

$$|((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\mu\widetilde{\boldsymbol{U}})_{\Omega_0}| \leqslant \int_{\Omega_0}|(\boldsymbol{w}\cdot\nabla)w_1\mu h(\varrho(x))U_1|\,\mathrm{d}x + \int_{\Omega_0}|(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}\cdot\mu\varphi(x_2)h'(\varrho(x))\nabla^\perp\varrho|\,\mathrm{d}x.$$

If we choose δ_0 sufficiently small, the first term on the right hand side is less than $\varepsilon/2||\nabla\boldsymbol{w}||^2$. The second term is

$$(10)\qquad \begin{aligned}|\mu|\int_{\Omega_0}\Big|\frac{\boldsymbol{w}}{\varrho(x)}\Big|\cdot|\nabla\boldsymbol{w}|\cdot|\varphi(x_2)|\cdot|\varrho(x)h'(\varrho(x))|\cdot|\nabla^\perp\varrho|\,\mathrm{d}x \\ \leqslant C\sup|\varrho(x)h'(\varrho(x))|\int_{\Omega_0}\Big|\frac{\boldsymbol{w}}{\varrho(x)}\Big|\,|\nabla\boldsymbol{w}|\,\mathrm{d}x \\ \leqslant C\sup|\varrho(x)h'(\varrho(x))|\,||\nabla\boldsymbol{w}||^2,\end{aligned}$$

where we have used the Hardy type inequality. As $\kappa_0\to 0$, $C\sup|\varrho(x)h'(\varrho(x))|$ becomes less than $\varepsilon/2$, which proves (8).

Now we show (9). According to the definition of $\boldsymbol{s}$, we have

$$((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\mu\boldsymbol{U})_{\Omega_1}-((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\mu\boldsymbol{s})_{\Omega_1}-((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\theta_\delta^2\mu\boldsymbol{U})_{\Omega_1} = \int_{\Omega_1}(\boldsymbol{w}\cdot\nabla)w_1\,\mu\widetilde{U}_1\,(1-\theta_\delta^2)\,\mathrm{d}x - \int_{\Omega_1}(\boldsymbol{w}\cdot\nabla)w_2\,\mu s_2\,\mathrm{d}x.$$

Therefore, the left hand side of (9) is less than

$$\int_{\Omega_1}|(\boldsymbol{w}\cdot\nabla)w_1\mu h(\varrho)U_1(1-\theta_\delta^2)|\,\mathrm{d}x + \int_{\Omega_1}|(\boldsymbol{w}\cdot\nabla)w_1\mu h'(\varrho)\varphi(x_2)|\,\mathrm{d}x + \int_{\Omega_1}|(\boldsymbol{w}\cdot\nabla)w_2\,\mu s_2|\,\mathrm{d}x.$$

If δ_0 is sufficiently small, then $|\mu h(\varrho(x))U_1|$ is small in Ω_1, and the first term is less than $\varepsilon/2||\nabla\boldsymbol{w}||^2$. The second term is estimated as in (10) and, if we take κ_0 sufficiently small, then this term is less than $\varepsilon/2||\nabla\boldsymbol{w}||^2$. On the other hand, since

$$s_2(x) = -D_1\{(1-h(\varrho(x)))(1-\theta(\delta x_1)^2)\varphi(x_2)\},$$

an estimate $|s_2| \leqslant C\delta$ holds where $C = 2\sup|\theta'(\delta x_1)\varphi(x_2)|$. Therefore the third term is estimated by

$$\int_{\Omega_1} |(\boldsymbol{w}\cdot\nabla)w_2\,\mu s_2|\,\mathrm{d}x \leqslant C\delta\int_{\Omega_1}|(\boldsymbol{w}\cdot\nabla)w_2|\,\mathrm{d}x,$$

and we obtain (9). Thus Lemma 6 is proved. □

We need the following theorem for the proof of Theorem 1, which was proved by Amick [1] for a more general case. The constant σ^S is defined in (1).

Theorem 3. *Let*

$$\Gamma(\delta) = \sup_{\boldsymbol{w}\in V^S(\Omega)} \frac{((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta_\delta^2\mu\boldsymbol{U})_{\Omega_1}}{||\nabla\boldsymbol{w}||_\Omega^2}. \tag{11}$$

Then

$$\lim_{\delta\to+0}\Gamma(\delta) = |\mu|\sigma^S.$$

R e m a r k 5. Since σ^S is positive (Lemma 4), $\Gamma(\delta)$ is also positive for sufficiently small positive δ. Furthermore, there exists a sequence $\boldsymbol{w}_j \in V^S(\Omega)$ $(j = 1, 2, \ldots)$ such that $((\boldsymbol{w}_j\nabla)\boldsymbol{w}_j, \mu\theta_\delta^2\boldsymbol{U})_{\Omega_1} > 0$ and

$$\frac{((\boldsymbol{w}_j\cdot\nabla)\boldsymbol{w}_j, \mu\theta_\delta^2\boldsymbol{U})_{\Omega_1}}{||\nabla\boldsymbol{w}_j||_\Omega^2} \to \Gamma(\delta)\ (j\to\infty).$$

4. Proof of Theorem 1

By the assumption $\nu - |\mu|\sigma^S > 0$ and Theorem 3, we can choose $\varepsilon > 0$ and $\delta > 0$ small enough that

$$\Gamma(\delta) - |\mu|\sigma^S \leqslant 4^{-1}(\nu - |\mu|\sigma^S), \qquad 3\varepsilon + C_0\delta \leqslant 4^{-1}(\nu - |\mu|\sigma^S) \tag{12}$$

hold. We fix these ε and δ.

Let $\Omega^n, n = 1, 2, \ldots$, be a sequence of bounded symmetric domains with smooth boundaries such that $\Omega^n \subset \Omega^{n+1}$ and $\Omega^n \to \Omega$ as $n\to\infty$. We suppose that the domain Ω^1 contains $\Omega\cap\{x_1 \leqslant 1/\delta\}$. We consider the stationary Navier-Stokes equations in Ω^n:

$$\left\{\begin{array}{ll}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f} & \text{in } \Omega^n,\\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega^n,\end{array}\right. \tag{NS$_n$}$$

with the boundary conditions

$$(\mathrm{BC})_n \qquad \begin{cases} \boldsymbol{u} = \boldsymbol{\beta} & \text{on } \partial\Omega \cap \partial\Omega^n, \\ \boldsymbol{u} = \mu \boldsymbol{U} & \text{on } \partial\Omega^n \setminus \partial\Omega. \end{cases}$$

A function $\boldsymbol{u}$ is called a weak solution to $(\mathrm{NS})_n$, $(\mathrm{BC})_n$ if $\boldsymbol{u} \in H^1(\Omega^n)$, $\mathrm{div}\boldsymbol{u} = 0$,

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in V(\Omega^n)),$$

and $\boldsymbol{u}$ satisfies the boundary condition $(\mathrm{BC})_n$ in the trace sense.

Let $\boldsymbol{u}_n$ be a symmetric weak solution to $(\mathrm{NS})_n$, $(\mathrm{BC})_n$, the existence of which was established by Fujita [4]. Put $\boldsymbol{u}_n = \boldsymbol{w}_n + \boldsymbol{b} + \mu \boldsymbol{U}$, where $\boldsymbol{b} \in H^2(\Omega)$ is the solenoidal symmetric extension (in Ω) of $\boldsymbol{\beta} - \mu \boldsymbol{U}$. As $\boldsymbol{\beta} - \mu \boldsymbol{U}$ satisfies the hypothesis of Lemma 1, there exists its solenoidal extension $\boldsymbol{b} \in H^2(\Omega)$ for which the following inequality holds:

$$|((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b})_\Omega| \leqslant \varepsilon ||\nabla \boldsymbol{v}||_\Omega^2 \quad (\forall \boldsymbol{v} \in V^S(\Omega)).$$

It should be noted that the support of this extension is compact. Without loss of generality, we can suppose that the support is contained in $\{x_1 < 1/\delta\}$. Then $\mathrm{div}\boldsymbol{w}_n = 0$ and $\boldsymbol{w}_n|_{\partial\Omega^n} = 0$. Therefore $\boldsymbol{w}_n$ belongs to $V^S(\Omega^n)$ and satisfies the equation

$$\begin{aligned} &\nu(\nabla \boldsymbol{w}_n, \nabla \boldsymbol{v}) + ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{v}) + \mu((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{v}) \\ &\quad + ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{v}) = (\boldsymbol{F}, \boldsymbol{v}) - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in V(\Omega^n)), \end{aligned} \tag{13}$$

where $\boldsymbol{F} = \boldsymbol{f} - \{(\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + \mu(\boldsymbol{U} \cdot \nabla)\boldsymbol{b} + \mu(\boldsymbol{b} \cdot \nabla)\boldsymbol{U}\}$. Since $\boldsymbol{w}_n \in V^S(\Omega^n)$, we can substitute $\boldsymbol{v} = \boldsymbol{w}_n$ into (13), obtaining

$$\nu||\nabla \boldsymbol{w}_n||^2 = \mu((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{U}) + ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{b}) + (\boldsymbol{F}, \boldsymbol{w}_n) - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{w}_n). \tag{14}$$

Because $V^S(\Omega^n)$ is a subset of $V^S(\Omega)$ for all n, the following inequality holds true from the definition of $\Gamma(\delta)$:

$$((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \theta_\delta^2 \mu \boldsymbol{U})_{\Omega_1} \leqslant \Gamma(\delta)||\nabla \boldsymbol{v}||^2 \quad (\forall \boldsymbol{v} \in V^S(\Omega^n)).$$

Let $\boldsymbol{s}$ be fixed as in Lemma 6. By (8) and (9) we have

$$\begin{aligned} ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{U})_\Omega &= ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{U})_{\Omega_0} + ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{U})_{\Omega_1} \\ &\leqslant ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{s})_\Omega + ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \mu \theta_\delta^2 \boldsymbol{U})_{\Omega_1} + (2\varepsilon + C_0\delta)||\nabla \boldsymbol{w}_n||^2. \end{aligned} \tag{15}$$

Since $\boldsymbol{s} \in \boldsymbol{C}_{0,\sigma}^{\infty,S}(\Omega^n)$, we substitute $\boldsymbol{v} = \mu \boldsymbol{s}$ in (13), obtaining

$$\begin{aligned} ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{s})_\Omega &= -\nu(\nabla \boldsymbol{w}_n, \nabla(\mu \boldsymbol{s})) - ((\boldsymbol{w}_n \cdot \nabla)\mu \boldsymbol{U} + (\mu \boldsymbol{U} \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{s}) \\ &\quad - ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{s}) + (\boldsymbol{F}, \mu \boldsymbol{s}) - \nu(\nabla \boldsymbol{b}, \nabla \mu \boldsymbol{s}). \end{aligned} \tag{16}$$

The right hand side is linear in $\boldsymbol{w}_n$. Therefore the following inequality holds true with some positive constants k', k'' which do not depend on $\boldsymbol{w}_n$:

$$((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{s})_\Omega \leqslant k' + k''||\nabla \boldsymbol{w}_n||. \tag{17}$$

Hence we have

$$\begin{aligned}\nu||\nabla \boldsymbol{w}_n||^2 &= ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{U})_\Omega + ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{b})_\Omega + (\boldsymbol{F}, \boldsymbol{w}_n)_\Omega - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{w}_n)_\Omega \\ &\leqslant ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \mu \boldsymbol{s})_\Omega + ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \theta_\delta^2 \mu \boldsymbol{U})_{\Omega_1} + (2\varepsilon + C_0\delta)||\nabla \boldsymbol{w}_n||^2 \\ &\quad + ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{b})_\Omega + (\boldsymbol{F}, \boldsymbol{w}_n)_\Omega - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{w}_n)_\Omega \\ &\leqslant k' + k''||\nabla \boldsymbol{w}_n|| + ((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \theta_\delta^2 \mu \boldsymbol{U})_{\Omega_1} + (2\varepsilon + C_0\delta)||\nabla \boldsymbol{w}_n||^2 \\ &\quad + \varepsilon||\nabla \boldsymbol{w}_n||^2 + k'''||\nabla \boldsymbol{w}_n||\end{aligned}$$

where k''' is a positive constant independent of $\boldsymbol{w}_n$. Using the definition of $\Gamma(\delta)$, we have

$$\nu||\nabla \boldsymbol{w}_n||^2 \leqslant k' + (k'' + k''')||\nabla \boldsymbol{w}_n|| + (3\varepsilon + C_0\delta)||\nabla \boldsymbol{w}_n||^2 + \Gamma(\delta)||\nabla \boldsymbol{w}_n||^2. \tag{18}$$

Since ε and δ are chosen as in (12), we obtain

$$2^{-1}(\nu - |\mu|\sigma^S)||\nabla \boldsymbol{w}_n||^2 \leqslant k' + (k'' + k''')||\nabla \boldsymbol{w}_n||.$$

Therefore, there exists a positive constant M such that the following estimate holds for all n:

$$||\nabla \boldsymbol{w}_n|| \leqslant M.$$

The sequence $\boldsymbol{w}_n$ being bounded in $V^S(\Omega)$, we can choose a subsequence $\boldsymbol{w}_{n'}$ which converges weakly in $V^S(\Omega)$. Let $\boldsymbol{w}$ be the limit.

Let $\boldsymbol{\varphi}$ be an arbitrary element in $\boldsymbol{C}^\infty_{0,\sigma}(\Omega)$. There exists an integer n_0 such that the support of $\boldsymbol{\varphi}$ is contained in Ω^{n_0}. Let $n' \geqslant n_0$. Then

$$\begin{aligned}&\nu(\nabla \boldsymbol{w}_{n'}, \nabla \boldsymbol{\varphi}) + ((\boldsymbol{w}_{n'} \cdot \nabla)\boldsymbol{w}_{n'}, \boldsymbol{\varphi}) + \mu((\boldsymbol{w}_{n'} \cdot \nabla)\boldsymbol{U}, \boldsymbol{\varphi}) + \mu((\boldsymbol{U} \cdot \nabla)\boldsymbol{w}_{n'}, \boldsymbol{\varphi}) \\ &\qquad + ((\boldsymbol{w}_{n'} \cdot \nabla)\boldsymbol{b}, \boldsymbol{\varphi}) + ((\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_{n'}, \boldsymbol{\varphi}) = (\boldsymbol{F}, \boldsymbol{\varphi}) - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{\varphi}).\end{aligned} \tag{19}$$

We can select a subsequence which converges strongly in $L^4(\Omega^{n_0})$. We denote this subsequence by the same symbol $\boldsymbol{w}_{n'}$. Letting $n' \to \infty$, we obtain

$$\begin{aligned}&\nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}) + ((\boldsymbol{w} \cdot \nabla)\boldsymbol{w}, \boldsymbol{\varphi}) + \mu((\boldsymbol{w} \cdot \nabla)\boldsymbol{U}, \boldsymbol{\varphi}) + \mu((\boldsymbol{U} \cdot \nabla)\boldsymbol{w}, \boldsymbol{\varphi}) \\ &\qquad + ((\boldsymbol{w} \cdot \nabla)\boldsymbol{b}, \boldsymbol{\varphi}) + ((\boldsymbol{b} \cdot \nabla)\boldsymbol{w}, \boldsymbol{\varphi}) = (\boldsymbol{F}, \boldsymbol{\varphi}) - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{\varphi}).\end{aligned} \tag{20}$$

Therefore $\boldsymbol{u} := \boldsymbol{w} + \boldsymbol{b} + \mu \boldsymbol{U}$ is a symmetric weak solution to (NS), (BC). Theorem 1 is proved. □

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