An introduction to the mathematical theory of compressible viscous fluids

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Motto:

Die Energie der Welt ist constant;
Die Entropie der Welt strebt einem Maximum zu

Rudolph Clausius, 1822-1888

Eulerian reference system

- $x \in \Omega$ - reference spatial position
  $\Omega \subset R^3$ - a domain occupied by the fluid
- $t \in I \subset R$ - time

- In some cases $\Omega = \Omega(t)$ may change in time
- The time $t_0 = 0$ is typically the ‘initial time”
- Ω may be bounded or unbounded, with a boundary $\partial\Omega$

State variables

State variables are physical quantities that *characterize* the system at a given instant t. Typically, they are numerical functions of the spatial variable x and the time t

Mass density

$$\varrho = \varrho(t, x)$$

$\int_V \varrho(t, x)\ \mathrm{d}x$ - total mass of the fluid at instant t contained in the volume element $V \subset \Omega$

Specific internal energy

$$e = e(t, x)$$

$\int_V (\varrho e)(t,x) \, \mathrm{d}x$ - internal energy of the fluid at instant t contained in the volume element $V \subset \Omega$

Entropy

BASIC PROPERTIES OF SPECIFIC ENTROPY [CALLEN, 1985]

- specific entropy s is a function of ϱ and e, and the *absolute temperature* ϑ is defined as

$$\frac{\partial s}{\partial e} = \frac{1}{\vartheta} > 0$$

- the entropy remains constant as long as material responds elastically
- the entropy tends to zero as the absolute temperature tends to zero

Pressure

$$\frac{\partial e(\varrho, s)}{\partial \varrho} = p\frac{1}{\varrho^2}$$

$$\frac{\partial e(\varrho, \vartheta)}{\partial \varrho} = \frac{p}{\varrho^2} + \frac{\partial e(\varrho, s)}{\partial s}\frac{\partial s(\varrho, \vartheta)}{\partial \varrho}$$

$$= \frac{p}{\varrho^2} + \vartheta\frac{\partial s(\varrho, \vartheta)}{\partial \varrho}$$

Thermodynamic functions

- $p = p(\varrho, \vartheta)$ - pressure
- $e = e(\varrho, \vartheta)$ - (specific) internal energy
- $s = s(\varrho, \vartheta)$ - (specific) entropy

FUNDAMENTAL RELATION - GIBBS' EQUATION

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

The bulk relative motion of the fluid can cause only a small change in the statistical properties of the molecular motion when the characteristic time of the bulk motion is long compared with the characteristic time of the molecular motion

G.K. Batchelor, 1965

Velocity field

$$\mathbf{u} = \mathbf{u}(t, x)$$

$$\int_I \int_S \varrho(t, x)\mathbf{u}(t, x) \cdot \mathbf{n}(x) \ \mathrm{dS}_x \ \mathrm{d}t$$

the total amount of fluid passing through a surface S during a time interval $\mathbf{I}$

Streamlines

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t) = \mathbf{u}(t, \mathbf{X}(t)), \ \mathbf{X}(0) = \mathbf{X}_0$$

Balance Law

- $d = d(t,x)$ - volumetric density
- $\mathbf{F} = \mathbf{F}(t,x)$ - flux vector
- $s = s(t,x)$ - source

$$\int_B d(t_2,x)\,\mathrm{d}x - \int_B d(t_1,x)\,\mathrm{d}x$$

$$= -\int_{t_1}^{t_2}\int_{\partial B} \mathbf{F}(t,x)\cdot\mathbf{n}(x)\,\mathrm{d}S_x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_B s(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

Integral formulation

$$V \equiv (t_1, t_2) \times B$$

$$\lim_{\varepsilon \to 0} \int_V [d(t,x); \mathbf{F}(t,x)] \cdot \nabla_{t,x} \varphi_\varepsilon \, \mathrm{d}x \, \mathrm{d}t = - \lim_{\varepsilon \to 0} \int_V s(t,x) \varphi_\varepsilon \, \mathrm{d}x \, \mathrm{d}t$$

$$\varphi_\varepsilon \in C_0^\infty(V),\ 0 \leq \varphi_\varepsilon \leq 1,\ \varphi_\varepsilon(x) = 1 \text{ for } \mathrm{dist}[x; \partial V] > \varepsilon$$

- s - (signed) measure on $[0, T] \times \overline{\Omega}$

Alternative formulation

Balance law in integral form

$$\int_0^T \int_\Omega [d(t,x); \mathbf{F}(t,x)] \cdot \nabla_{t,x}\varphi \ \mathrm{d}x \ \mathrm{d}t = - \int_0^T \int_\Omega s(t,x)\varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$\varphi \in C_0^\infty((0,T) \times \Omega)$$

Balance law in differential form

$$\partial_t d + \mathrm{div}_x \mathbf{F} = s$$

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Balance of momentum - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho \mathbf{f}$$

Stokes' law

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}$$

- $\mathbb{T}$ - Cauchy stress
- $\mathbb{S}$ - viscous stress
- $\mathbf{f}$ - external force

Kinetic energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + p \right) \mathbf{u} - \mathbb{S} \cdot \mathbf{u} \right)$$

$$= p \mathrm{div}_x \mathbf{u} - \mathbb{S} : \nabla_x \mathbf{u} + \varrho \mathbf{f} \cdot \mathbf{u}$$

Internal energy balance

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}$$

Total energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + e + p \right) \mathbf{u} + \mathbf{q} - \mathbb{S} \cdot \mathbf{u} \right) = \varrho \mathbf{f} \cdot \mathbf{u}$$

Energetically insulated boundary conditions, I

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No-slip

$$[\mathbf{u}]_{\mathrm{tangent}}|_{\partial\Omega} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Energetically insulated boundary conditions, II

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Complete-slip

$$[\mathbb{S} \cdot \mathbf{n}]_{\rm tangent}|_{\partial\Omega} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Energetically insulated boundary conditions, III

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Complete-slip with friction

$$\beta \mathbf{u}_{\text{tangent}} + [\mathbb{S} \cdot \mathbf{n}]_{\text{tangent}}|_{\partial\Omega} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n} + \beta |\mathbf{u}|^2|_{\partial\Omega} = 0$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) (t, \cdot) \ \mathrm{d}x = \int_{\Omega} \varrho \mathbf{f} \cdot \mathbf{u} \ \mathrm{d}x$$

Conservative driving force

$$\mathbf{f} = \nabla_x F, \ \ F = F(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) (t, \cdot) \ \mathrm{d}x = 0$$

Internal energy and entropy

Internal energy balance

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}$$

Gibbs' equation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

Entropy production rate

$$\sigma = (\geq)\frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) \geq 0$$

Navier-Stokes-Fourier system - weak formulation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) (t, \cdot) \ \mathrm{d}x = 0$$

WEAK + REGULARITY = STRONG

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

as soon as

- $$0 < \underline{\varrho} \leq \varrho(t,x) \leq \overline{\varrho}, \ 0 < \underline{\vartheta} \leq \vartheta(t,x) \leq \overline{\vartheta}$$
- $$|\mathbf{u}(t,x)| \leq U$$
- $$\nabla_x \varrho \in L^2((0,T) \times \Omega; R^3)$$

Constitutive equations

Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

$$\mu > 0, \ \eta \geq 0$$

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

$$\kappa > 0$$

Well posedness

Jacques Hadamard, 1865 - 1963

- **Existence.** Given problem is solvable for any choice of (admissible) data
- **Uniqueness.** Solutions are uniquely determined by the data
- **Stability.** Solutions depend continuously on the data

Jacques-Luis Lions, 1928 - 2001

- **Approximations.** Given problem admits an approximation scheme that is solvable analytically and, possibly, numerically
- **Uniform bounds.** Approximate solutions possesses uniform bounds depending solely on the data
- **Stability.** The family of approximate solutions admits a limit representing a (generalized) solution of the given problem

- **A priori bounds.** Natural bounds imposed on *exact solutions* by the data
- **(Weak) sequential stability.** Closedness of the family of solutions bounded by a priori bounds in the framework of *weak formulation*.
- **Consistency.** Qualitative properties of solutions coincide with the expected ones.

Equilibria - static states

$$\mathbf{u}_{\rm static} \equiv 0,\ \vartheta_{\rm static} = \overline{\vartheta} > 0,\ \varrho_{\rm static} = \tilde{\varrho}(x)$$

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F \text{ in } \Omega$$

$$\liminf_{\varrho \to 0} \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0 \text{ for any } \vartheta > 0 \ \Rightarrow \ \inf_{\Omega} \tilde{\varrho} > 0$$

$$F = P(\tilde{\varrho}, \overline{\vartheta}) + {\rm const},\ \frac{\partial P(\varrho, \overline{\vartheta})}{\partial \varrho} = \frac{1}{\varrho} \frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho}$$

Thermodynamic stability hypothesis

- **Positive compressibility:**

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$$

- **Positive specific heat at constant volume:**

$$\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Helmholtz function-ballistic free energy

$$H(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta)$$

$$\frac{\partial^2 H(\varrho, \overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho} \frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho} > 0$$

- $\varrho \mapsto H(\varrho, \overline{\vartheta})$ is strictly convex

$$\frac{\partial H(\varrho, \vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta}(\vartheta - \overline{\vartheta}) \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta}$$

- $\vartheta \mapsto H(\varrho, \vartheta)$ attains its strict local minimum at $\overline{\vartheta}$

Coercivity of Helmholtz function

$$H(\varrho,\vartheta) - \frac{\partial H(\tilde\varrho,\overline{\vartheta})}{\partial \varrho}(\varrho - \tilde\varrho) - H(\tilde\varrho,\overline{\vartheta})$$

$$\geq c(B)\Big(|\varrho - \tilde\varrho|^2 + |\vartheta - \overline{\vartheta}|^2 \Big)$$

provided ϱ,ϑ belong to a compact interval $B \subset (0,\infty)$

$$\geq c(B)\Big(1 + \varrho e(\varrho,\vartheta) + \varrho |s(\varrho,\vartheta)| \Big)$$

otherwise

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as soon as $\tilde\varrho$, $\overline{\vartheta}$ belong to $\mathrm{int}[B]$

Corollary: Principle of maximal entropy

$$\tilde{\varrho},\ \overline{\vartheta} \text{ static state} \Rightarrow \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} = F + \mathrm{const} \Rightarrow$$

$$\int_\Omega \left(H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \right) \ \mathrm{d}x =$$

$$\int_\Omega \left(\left(\varrho e(\varrho, \vartheta) - \varrho F - \tilde{\varrho} e(\tilde{\varrho}, \overline{\vartheta}) + \tilde{\varrho} F \right) - \overline{\vartheta} \varrho s(\varrho, \vartheta) + \overline{\vartheta} \tilde{\varrho} s(\tilde{\varrho}, \overline{\vartheta}) \right) \ \mathrm{d}x$$

as soon as

$$\int_\Omega \varrho \ \mathrm{d}x = \int_\Omega \tilde{\varrho} \ \mathrm{d}x$$

Principle of maximal entropy - conclusion

- Given the total mass and energy, there is a unique static state $\tilde{\varrho}$, $\overline{\vartheta}$
- The static state $\tilde{\varrho}$, $\tilde{\vartheta}$ maximizes the entropy among all admissible states ϱ, ϑ with the same total mass and energy

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho,\vartheta) - \frac{\partial H(\tilde\varrho,\overline{\vartheta})}{\partial\varrho}(\varrho-\tilde\varrho) - H(\tilde\varrho,\overline{\vartheta})\right)\ \mathrm{d}x$$

$$+\overline{\vartheta}\int_\Omega \sigma\ \mathrm{d}x = 0$$

$$\sigma \geq \frac{\mu}{\vartheta}\left|\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right|^2 + \frac{\kappa}{\vartheta^2}|\nabla_x\vartheta|^2$$

Technical hypotheses imposed on constitutive relations

What is needed...

- integrability of all quantities in the weak formulation - hypotheses of coercivity imposed on thermodynamic functions p, e, s
- bounds on the spatial gradients of $\mathbf{u}$, ϑ - the transport coefficients μ, κ depend on the temperature
- compactness of the temperature field on the “vacuum” zones - introducing radiation pressure

Thermodynamic functions

Monoatomic gas:

$$p = \frac{2}{3}\varrho e \ \Rightarrow \ p = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right)$$

Third Law:

$$P(Z) \approx Z^{5/3} \text{ for } Z \to \infty$$

Radiation pressure:

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4$$

Pressure-Energy-Entropy

Pressure:

$$p(\varrho,\vartheta) = p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,$$

Internal energy:

$$e(\varrho,\vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4$$

Entropy:

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4}{3}\frac{a}{\varrho}\vartheta^3$$

Transport coefficients

Shear viscosity:

$$0 < \underline{\mu}(1 + \vartheta^{\alpha}) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^{\alpha}),\ 1/2 \leq \alpha \leq 1$$

Bulk viscosity:

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta^{\alpha})$$

Heat conductivity:

$$0 < \overline{\kappa}(1 + \vartheta^{3}) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^{3})$$

A priori bounds

UNIFORM-IN-TIME L^p–BOUNDS:

$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega;R^3))$$

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega))$$

$$\vartheta \in L^\infty(0,T;L^4(\Omega))$$

Gradient bounds:

$$\mathbf{u} \in L^2(0,T;W^{1,q}(\Omega;R^3)),\ q = \frac{8}{5-\alpha}$$

$$\vartheta \in L^2(0,T;W^{1,2}(\Omega))$$

$$\log(\vartheta) \in L^2(0,T;W^{1,2}(\Omega))$$

Pressure bounds:

$$p(\varrho,\vartheta)\varrho^\beta \in L^1((0,T)\times\Omega) \text{ for a certain } \beta > 0$$

Weak sequential stability

$$\varrho_\varepsilon \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^{5/3}(\Omega))$$

$$\vartheta_\varepsilon \to \vartheta \text{ weakly-(*) in } L^\infty(0,T;L^4(\Omega))$$

$$\text{and weakly in } L^2(0,T;W^{1,2}(\Omega))$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,q}(\Omega;R^3))$$

Div-Curl lemma [F.Murat, L.Tartar, 1975]

Lemma

Let

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } L^p,$$

$$\mathbf{w}_\varepsilon \to \mathbf{w} \text{ weakly in } L^q,$$

with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Let, moreover,

$$\mathrm{div}[\mathbf{v}_\varepsilon],\ \mathrm{curl}[\mathbf{w}_\varepsilon] \text{ be precompact in } W^{-1,s}$$

Then

$$\mathbf{v}_\varepsilon \cdot \mathbf{w}_\varepsilon \to \mathbf{v} \cdot \mathbf{w} \text{ weakly in } L^r.$$

Weak sequential stability of convective terms

$$\mathbf{v}_\varepsilon = [\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon], \ \mathbf{w}_\varepsilon = [u^i_\varepsilon, 0, 0, 0], \ i = 1, 2, 3$$

Aubin-Lions argument (Div-Curl lemma) $\Rightarrow$

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}$$

$$\overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u}$$

$$\overline{\varrho s(\varrho, \vartheta) \vartheta} = \overline{\varrho s(\varrho, \vartheta)} \vartheta$$

Pointwise convergence of temperature, I

GOAL: Use monotonicity of $s(\varrho, \vartheta)$ in ϑ
to show

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \Big) (\vartheta_\varepsilon - \vartheta) \ \mathrm{d}x \ \mathrm{d}t \to 0$$

$$\Rightarrow$$

$$\| \vartheta_\varepsilon - \vartheta \|_{L^4} \to 0$$

STEP 1: Aubin-Lions argument (Div-Curl lemma) $\Rightarrow$

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(\vartheta_\varepsilon - \vartheta) \ \mathrm{d}x \ \mathrm{d}t \to 0$$

Pointwise convergence of temperature, II

STEP 2: Renormalized equation of continuity [DiPerna and P.-L. Lions, 1989]

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho)\Big)\mathrm{div}_x\mathbf{u} = 0$$

STEP 3: Aubin-Lions argument (Div-Curl lemma):

$$\overline{b(\varrho)g(\vartheta)} = \overline{b(\varrho)}\ \overline{g(\vartheta)}$$

Fundamental theorem on Young measures [J.M Ball 1989, P.Pedregal 1997]

Theorem

Let $\mathbf{v}_\varepsilon : Q \subset R^N \to R^M$ *be a sequence of vector fields bounded in* $L^1(Q; R^M)$.
Then there exists a subsequence (not relabeled) and a family of probability measures $\{\nu_y\}_{y\in Q}$ *on* R^M *such that:*
For any Carathéodory function $\Phi = \Phi(y, Z)$, *yinQ*, $Z \in R^M$ *such that*

$$\Phi(\cdot, \mathbf{v}_\varepsilon) \to \overline{\Phi} \text{ weakly in } L^1(Q)$$

we have

$$\overline{\Phi}(y) = \int_{R^M} \Phi(y, Z)\, \mathrm{d}\nu_y(Z) \text{ for a.a. } y \in Q.$$

Pointwise convergence of temperature, III

STEP 4: Since we already know from STEP 3 that

$$\nu[\varrho_\varepsilon \vartheta_\varepsilon] = \nu[\varrho_\varepsilon] \otimes \nu[\vartheta_\varepsilon],$$

Fundamental theorem yields the desired conclusion

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta)(\vartheta_\varepsilon - \vartheta) \, \mathrm{d}x \, \mathrm{d}t \to 0$$

Pointwise convergence of temperature

$$\vartheta_\varepsilon \to \vartheta \text{ a.a. on } (0, T) \times \Omega$$

Pointwise convergence of density,I

STEP 1: Renormalized equation of continuity:

$$\partial_t(\varrho \log(\varrho)) + \mathrm{div}_x(\varrho \log(\varrho)\mathbf{u}) + \varrho \mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t(\overline{\varrho \log(\varrho)}) + \mathrm{div}_x(\overline{\varrho \log(\varrho)}\mathbf{u}) + \overline{\varrho \mathrm{div}_x \mathbf{u}} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right) \, \mathrm{d}x = - \int_\Omega \left(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u} \right) \, \mathrm{d}x$$

POINTWISE CONVERGENCE OF DENSITY,II

STEP 2: Effective viscous pressure [P.-L.Lions, 1998]

$$\overline{p(\varrho,\vartheta)b(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{b(\varrho)} = \overline{[\mathcal{R}:\mathbb{S}]b(\varrho)} - [\mathcal{R}:\mathbb{S}]\overline{b(\varrho)}$$

where

$$\mathcal{R}_{i,j} \equiv \partial_{x_i}\Delta^{-1}\partial_{x_j}$$

$$\mathcal{R}:\mathbb{S} = \mathcal{R}:\mathbb{S} - \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\mathrm{div}_x\mathbf{u} + \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\mathrm{div}_x\mathbf{u}$$

Commutator lemma[in the spirit of Coifman and Meyer]

Lemma

Let $w \in W^{1,r}(R^N)$, $\mathbf{V} \in L^p(R^N; R^N)$ *be given, where*

$$1 < r < N,\ 1 < p < \infty,\ \frac{1}{r} + \frac{1}{p} - \frac{1}{N} < 1.$$

The for any s *satisfying*

$$\frac{1}{r} + \frac{1}{p} - \frac{1}{N} < \frac{1}{s} < 1$$

there exists $\beta > 0$ *such that*

$$\|\mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]\|_{W^{\beta,s}(R^N,R^N)} \leq c\|w\|_{W^{1,r}}\|\mathbf{V}\|_{L^p}.$$

POINTWISE CONVERGENCE OF DENSITY,III

STEP 3: Effective viscous pressure revisited:

$$0 \leq \overline{p(\varrho,\vartheta)\varrho} - \overline{p(\varrho,\vartheta)}\varrho = (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\Big(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u}\Big)$$

yielding

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho)$$

Pointwise convergence of density - general case, I

STEP 1: Renormalized equation of continuity:

$$\partial_t(\varrho L_k(\varrho)) + \mathrm{div}_x(\varrho L_k(\varrho)\mathbf{u}) + T_k(\varrho)\mathrm{div}_x\mathbf{u} = 0$$

$$\partial_t(\overline{\varrho L_k(\varrho)}) + \mathrm{div}_x(\overline{\varrho L_k(\varrho)}\mathbf{u}) + \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} = 0$$

$$T_k(\varrho) = \min\{\varrho, k\}$$

Pointwise convergence of density - general case, II

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho) \right) \mathrm{d}x = \int_\Omega \left(T_k(\varrho)\mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} \right) \mathrm{d}x$$

$$+ \int_\Omega \left(\overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}} \right) \mathrm{d}x$$

STEP 2: Effective viscous flux revisited:

$$\overline{p(\varrho,\vartheta) T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}$$

$$= (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\left(\overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} \right)$$

yielding

Oscillations defect measure

$$\sup_{k \geq 1} \left[\limsup_{\varepsilon \to 0} \int_0^T \int_\Omega | T_k(\varrho_\varepsilon) - T_k(\varrho) |^q \ \mathrm{d}x \ \mathrm{d}t \right] < \infty$$

$$q = 5/3 + 1 = 8/3$$

Boundedness of oscillations defect measure guarantees:

- The limit functions ϱ, $\mathbf{u}$ satisfy the renormalized equation of continuity
- $$\int_{\Omega} \left(T_k(\varrho) \mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} \right) \, \mathrm{d}x \to 0 \text{ for } k \to \infty$$

Conclusion - pointwise convergence of density

$$\overline{\varrho \log(\varrho)} = \lim_{k \to \infty} \overline{\varrho L_k(\varrho)} = \lim_{k \to \infty} \varrho L_k(\varrho) = \varrho \log(\varrho)$$

$$\varrho_\varepsilon \to \varrho \text{ a.a. on } (0, T) \times \Omega$$

Long-time behavior

Conservative driving forces

$$\mathbf{f} = \nabla_x F, \ F = F(x)$$

Conserved quantities

Total mass:

$$M = \int_\Omega \varrho \, \mathrm{d}x$$

Total energy:

$$E = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \, \mathrm{d}x$$

STEP 1: Boundedness of total energy $\Rightarrow$ boundedness of total entropy:

$$S(t) = \int_\Omega \varrho s(\varrho, \vartheta)(t, \cdot) \ \mathrm{d}x \leq S_\infty$$

STEP 2: Boundedness of total entropy $\Rightarrow$ finite integral of the dissipation rate:

$$\int_0^\infty \int_\Omega \left(\frac{\mu}{2\vartheta} \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right|^2 + \frac{\kappa}{\vartheta^2} |\nabla_x \vartheta|^2 \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \sigma[(0, \infty) \times \overline{\Omega}] \ \mathrm{d}t < \infty$$

STEP 3: The velocity field $\mathbf{u}$ as well as the temperature gradient vanish in the asymptotic limit $t \to \infty \Rightarrow$ any solution tends to a uniquely determined *static state*

$$\tilde{\varrho} = \tilde{\varrho}(x), \ \overline{\vartheta} > 0$$

STEP 4: Total dissipation balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho,\vartheta) - \frac{\partial H(\tilde{\varrho},\overline{\vartheta})}{\partial\varrho}(\varrho-\tilde{\varrho}) - H(\tilde{\varrho},\overline{\vartheta})\Big)\ \mathrm{d}x$$

$$+\overline{\vartheta}\int_{\Omega}\sigma\ \mathrm{d}x = 0$$

$$\int_{\Omega}\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho,\vartheta) - \frac{\partial H(\tilde{\varrho},\overline{\vartheta})}{\partial\varrho}(\varrho-\tilde{\varrho}) - H(\tilde{\varrho},\overline{\vartheta})\Big)\ \mathrm{d}x \to 0 \text{ as } t \to \infty$$

Conclusion: Long-time behavior for conservative driving forces

$$\mathbf{f} = \nabla_x F,\ F = F(x)$$

$$\varrho(t,\cdot) \to \tilde{\varrho} \text{ in } L^{5/3}(\Omega) \text{ as } t \to \infty$$

$$\vartheta(t,\cdot) \to \overline{\vartheta} \text{ in } L^{4}(\Omega) \text{ as } t \to \infty$$

$$(\varrho\mathbf{u})(t,\cdot) \to 0 \text{ in } L^{1}(\Omega; R^3) \text{ as } t \to \infty$$

Attractors

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x > M,\ \ t > 0$$

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right)(t,\cdot)\ \mathrm{d}x < E,\ \ t > 0$$

$$\int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)\ \mathrm{d}x > S_0,\ \ t > 0$$

$$\|\varrho(t,\cdot) - \tilde{\varrho}\|_{L^{5/3}(\Omega)} < \varepsilon \text{ for } t > T(\varepsilon)$$

$$\|\vartheta(t,\cdot) - \overline{\vartheta}\|_{L^4(\Omega)} < \varepsilon \text{ for } t > T(\varepsilon)$$

$$\|\varrho\mathbf{u}(t,\cdot)\|_{L^1(\Omega;R^3)} < \varepsilon \text{ for } t > T(\varepsilon)$$

Uniform decay of density oscillations

$$d(t) = \int_\Omega \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right)(t, \cdot) \, \mathrm{d}x$$

$$\partial_t d(t) + \Psi(d(t)) \leq 0$$

General time-dependent driving forces

$$\mathbf{f} = \mathbf{f}(t, x), \ |\mathbf{f}(t, x)| \leq \overline{F}$$

EITHER

$$E(t) \equiv \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) (t, \cdot) \, \mathrm{d}x \to \infty \text{ as } t \to \infty$$

OR

$$|E(t)| \leq E \text{ for a.a. } t > 0$$

In the case $E(t) \leq E$, each sequence of times $\tau_n \to \infty$ contains a subsequence such that

$$\mathbf{f}(\tau_n + \cdot, \cdot) \to \nabla_x F \text{ weakly-(*) in } L^\infty((0,1) \times \Omega),$$

where $F = F(x)$ may depend on $\{\tau_n\}$

STEP 1: Assume that $E(\tau_n) < E$ for certain $\tau_n \to \infty \Rightarrow$ total entropy remains bounded $\Rightarrow$ integral of entropy production bounded

STEP 2: For $\tau_n \to \infty$ we have $\nabla_x p(\varrho, \vartheta) \approx \varrho \mathbf{f}$, $\vartheta \approx \overline{\vartheta}$, meaning, $\mathbf{f} \approx \nabla_x F$

STEP 3: The energy cannot "oscillate" since bounded entropy *static solutions* have bounded total energy

Corollaries:

- $$\mathbf{f} = \mathbf{f}(x) \neq \nabla_x F$$
  $$\Rightarrow$$
  $$E(t) \to \infty$$

- $$\mathbf{f} = \mathbf{f}(t,x) \text{ (almost) periodic in time, } \mathbf{f} \neq \nabla_x F, \ F = F(x)$$
  $$\Rightarrow$$
  $$E(t) \to \infty$$

Rapidly oscillating driving forces

$$\mathbf{f} = \omega(t^\beta)\mathbf{w}(x), \mathbf{w} \in W^{1,\infty}(\Omega; R^3), \ \beta > 2$$

$$\omega \in L^\infty(R), \ \sup_{\tau>0} \left| \int_0^\tau \omega(t) \ \mathrm{d}t \right| < \infty$$

$$(\varrho\mathbf{u})(t,\cdot) \to 0 \text{ in } L^1(\Omega; R^3) \text{ as } t \to \infty$$

$$\varrho(t,\cdot) \to \overline{\varrho} \text{ in } L^{5/3}(\Omega) \text{ as } t \to \infty$$

$$\vartheta(t,\cdot) \to \overline{\vartheta} \text{ in } L^4(\Omega) \text{ as } t \to \infty$$

Motto:

However beautiful the strategy,
you should occasionally look at the results

Sir Winston Churchill, 1874-1965

Singular limits

$$X \approx \frac{X}{X_{\rm char}}$$

$$\text{Mach number } {\rm Ma} = \frac{|\mathbf{u}|_{\rm char}}{\sqrt{p_{\rm char}/\varrho_{\rm char}}}$$

$$\text{Froude number } {\rm Fr} = \frac{|\mathbf{u}|_{\rm char}}{\sqrt{|x|_{\rm char}/|\nabla_x F|_{\rm char}}}$$

Incompressibility: ${\rm Ma} \approx \varepsilon \to 0$

Stratification: ${\rm Fr} \approx \varepsilon^{\alpha/2} \to 0$

Scaled Navier-Stokes-Fourier system:

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2} \nabla_x p = \frac{1}{\mathrm{Re}} \mathrm{div}_x \mathbb{S} + \frac{1}{\mathrm{Fr}^2} \varrho \nabla_x F$$

$$\mathrm{Sr}\ \partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \frac{1}{\mathrm{Pe}} \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma \geq \frac{1}{\vartheta} \left(\frac{\mathrm{Ma}^2}{\mathrm{Re}} \mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}} \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\mathrm{Ma}^2}{2} \varrho |\mathbf{u}|^2 + \varrho e - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \varrho F \right) = 0$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}]_{\mathrm{tan}} = 0$$

Characteristic numbers:

Symbol	Definition	Name
Sr	$\text{length}_{\text{ref}}/(\text{time}_{\text{ref}}\text{velocity}_{\text{ref}})$	Strouhal number
Ma	$\text{velocity}_{\text{ref}}/\sqrt{\text{pressure}_{\text{ref}}/\text{density}_{\text{ref}}}$	Mach number
Re	$\text{density}_{\text{ref}}\text{velocity}_{\text{ref}}\text{length}_{\text{ref}}/\text{viscosity}_{\text{ref}}$	Reynolds number
Fr	$\text{velocity}_{\text{ref}}/\sqrt{\text{length}_{\text{ref}}\text{force}_{\text{ref}}}$	Froude number
Pe	$\text{pressure}_{\text{ref}}\text{length}_{\text{ref}}\text{velocity}_{\text{ref}}/(\text{temperature}_{\text{ref}}\text{heat conductivity}_{\text{ref}})$	Péclet number

Low Mach number limit - weak stratification

$$\mathrm{Ma} = \varepsilon, \ \mathrm{Fr} = \sqrt{\varepsilon}$$

STRATEGY:

1. Existence theory for the primitive Navier-Stokes-Fourier system
2. Uniform bounds independent of the singular parameter
3. Passage to the limit - analysis of acoustic waves
4. Identification of the limit system

Scaled Navier-Stokes-Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \text{ in } (0, T) \times \Omega$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \frac{1}{\varepsilon} \varrho \nabla_x F \text{ in } (0, T) \times \Omega$$

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma \text{ in } (0, T) \times \Omega$$

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varepsilon \varrho F \right) \ \mathrm{d}x = 0$$

$$\sigma \geq \frac{1}{\vartheta} \left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \geq 0$$

Total dissipation balance

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho,\vartheta) - \partial_\varrho H(\tilde\varrho_\varepsilon,\overline{\vartheta})(\varrho - \tilde\varrho_\varepsilon) - H(\tilde\varrho_\varepsilon,\overline{\vartheta}) \right) \right)(\tau,\cdot)\,\mathrm{d}x$$

$$+\frac{\overline{\vartheta}}{\varepsilon^2}\int_0^\tau \int_\Omega \sigma \ \mathrm{d}x \ \mathrm{d}t =$$

$$\int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho_0,\vartheta_0) - \partial_\varrho H(\tilde\varrho_\varepsilon,\overline{\vartheta})(\varrho_0 - \tilde\varrho_\varepsilon) - H(\tilde\varrho_\varepsilon,\overline{\vartheta}) \right) \right) \ \mathrm{d}x$$

$$\nabla_x p(\tilde\varrho_\varepsilon,\overline{\vartheta}) = \varepsilon\tilde\varrho_\varepsilon\nabla_x F, \ \int_\Omega \tilde\varrho_\varepsilon \ \mathrm{d}x = \int_\Omega \varrho_0 \ \mathrm{d}x, \ \tilde\varrho_\varepsilon \approx \overline{\varrho}$$

Ill-prepared initial data

$$\varrho_0 \approx \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^1 \cap L^\infty(\Omega),\ \int_\Omega \varrho^{(1)}_{0,\varepsilon}\ {\rm d}x = 0$$

$$\vartheta_0 \approx \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^1 \cap L^\infty(\Omega),\ \int_\Omega \vartheta^{(1)}_{0,\varepsilon}\ {\rm d}x = 0$$

$$\mathbf{u}_0 \approx \mathbf{u}_{0,\varepsilon},\ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega; R^3)$$

Uniform bounds

$$\left\{ \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T; L^2 \oplus L^q(\Omega)), \ q < 2$$

$$\left\{ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T; L^2 \oplus L^q(\Omega)), \ q < 2$$

$$\left\{ \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \right\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T; L^1(\Omega))$$

$$\left\{ \frac{\sigma_\varepsilon}{\varepsilon^2} \right\}_{\varepsilon>0} \text{ bounded in } \mathcal{M}^+([0,T] \times \overline{\Omega})$$

$$\left\{ \nabla_x \mathbf{u}_\varepsilon \right\}_{\varepsilon>0} \text{ bounded in } L^2((0,T) \times \Omega; R^{3\times 3})$$

$$\left\{ \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right\}_{\varepsilon>0} \text{ bounded in } L^2((0,T) \times \Omega; R^3)$$

Convergence

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } L^\infty(0,T;L^2 \oplus L^q(\Omega))$$

$$\vartheta_\varepsilon \to \overline{\vartheta} \text{ in } L^\infty(0,T;L^2 \oplus L^q(\Omega))$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3))$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega))$$

Oberbeck-Boussinesq system

$$\mathrm{div}_x \mathbf{U} = 0$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U}\otimes\mathbf{U})\Big) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S} + r\nabla_x F \text{ in } (0,T)\times\Omega$$

$$\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ [\mathbb{S}\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0$$

$$\overline{\varrho}c_p\Big(\partial_t\Theta + \mathrm{div}_x(\Theta\mathbf{U})\Big) - \mathrm{div}_x(G\mathbf{U}) - \mathrm{div}_x(\kappa\nabla_x\Theta) = 0 \text{ in } (0,T)\times\Omega$$

$$G = \beta F,\ \nabla_x\Theta\cdot\mathbf{n}|_{\partial\Omega} = 0$$

$$r + \alpha\Theta = 0,\ \alpha > 0$$

Available results

- **Barotropic Navier-Stokes system - weak solutions, large time interval**

P.-L.Lions, N. Masmoudi, J. Math. Pures Appl., 1998

B. Desjardins, E. Grenier, P.-L. Lions, N. Masmoudi, J. Math. Pures Appl., 1999

B. Desjardins, E. Grenier, Royal Soc. London, 1999

- **Navier-Stokes-Fourier system, strong solutions, short time interval**

T. Alazard, Arch. Rational Mech. Anal., SIAM J. Math. Anal., 2006

LIGHTHILL'S ACOUSTIC EQUATION $(F = 0)$

$$\varepsilon \partial_t Z_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}^1_\varepsilon$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x Z_\varepsilon = \varepsilon \left(\mathrm{div}_x \mathbb{F}^2_\varepsilon + \nabla_x F^3_\varepsilon + \frac{A}{\varepsilon^2 \omega} \nabla_x \Sigma_\varepsilon \right)$$

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$Z_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega} \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) + \frac{A}{\varepsilon \omega} \Sigma_\varepsilon, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$< \Sigma_\varepsilon; \varphi > = < \sigma_\varepsilon; I[\varphi] >$$

$$I[\varphi](t,x) = \int_0^t \varphi(z,x) \ \mathrm{d}z \text{ for any } \varphi \in L^1(0,T; C(\overline{\Omega}))$$

Helmholtz decomposition:

$$\mathbf{V}_\varepsilon = \mathbf{w}_\varepsilon + \nabla_x \Phi_\varepsilon,\ \mathrm{div}_x(\mathbf{w}_\varepsilon) = 0,\ \mathbf{w}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Wave equation:

$$\varepsilon \partial_t Z_\varepsilon + \Delta_x \Phi_\varepsilon = \varepsilon G^1_\varepsilon \text{ in } (0,T) \times \Omega$$

$$\varepsilon \partial_t \Phi_\varepsilon + \omega Z_\varepsilon = \varepsilon F^2_\varepsilon \text{ in } (0,T) \times \Omega$$

$$\nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Abstract wave equation:

$$\varepsilon \partial_t r_\varepsilon - A[\Phi_\varepsilon] = \varepsilon h^1_\varepsilon$$
$$\varepsilon \partial_t \Phi_\varepsilon + r_\varepsilon = \varepsilon h^2_\varepsilon$$

$$A[v] = -\omega \Delta_x v, \ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0$$

A is a non-negative self-adjoint operator on the Hilbert space $L^2(\Omega)$

$$h^1_\varepsilon, h^2_\varepsilon \in L^2(0, T; \mathcal{D}(G(A)))$$

Duhamel's formula:

$$\begin{aligned}
\Phi_\varepsilon(t,\cdot) &= \exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{A}\right)\left[\frac{1}{2}\Phi_{0,\varepsilon} + \frac{\mathrm{i}}{2\sqrt{A}}[r_{0,\varepsilon}]\right] \\
&+ \exp\left(-\mathrm{i}\frac{t}{\varepsilon}\sqrt{A}\right)\left[\frac{1}{2}\Phi_{0,\varepsilon} - \frac{\mathrm{i}}{2\sqrt{A}}[r_{0,\varepsilon}]\right] \\
&+ \int_0^t \exp\left(\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{A}\right)\left[\frac{1}{2}h^2_\varepsilon(s) + \frac{\mathrm{i}}{2\sqrt{A}}[h^1_\varepsilon(s)]\right]\,\mathrm{d}s \\
&+ \int_0^t \exp\left(-\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{A}\right)\left[\frac{1}{2}h^2_\varepsilon(s) - \frac{\mathrm{i}}{2\sqrt{A}}[h^1_\varepsilon(s)]\right]\,\mathrm{d}s
\end{aligned}$$

Local (weak) decay of acoustic energy:

$$\left\{ t \mapsto \int_\Omega \Phi_\varepsilon(t,\cdot)\varphi \ \mathrm{d}x \right\} \to 0 \text{ in } L^2(0,T) \text{ as } \varepsilon \to 0$$

$$\left(\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon}\right)[\Psi], \varphi \right\rangle \right|^2 \ \mathrm{d}t \right)^{1/2} \leq \omega(\varepsilon,\varphi) \|\Psi\|_{L^2(\Omega)}$$

for any $\Psi \in L^2(\Omega)$,

$\omega(\varepsilon,\varphi) \to 0$ as $\varepsilon \to 0$ for any fixed φ,

UNIFORM DECAY:

$$\int_0^T \int_0^t \left| \left\langle \exp\left(-\mathrm{i}\sqrt{A}\frac{t-s}{\varepsilon} \right) [G_\varepsilon(s)], \varphi \right\rangle \right|^2 \, \mathrm{d}s \, \mathrm{d}t$$

$$\leq \int_0^T \int_0^T \left| \left\langle \exp\left(-\mathrm{i}\sqrt{A}\frac{t-s}{\varepsilon} \right) [G_\varepsilon(s)], \varphi \right\rangle \right|^2 \, \mathrm{d}t \, \mathrm{d}s$$

$$\leq \omega^2(\varepsilon, \varphi) \int_0^T \left\| \exp\left(\mathrm{i}\sqrt{A}\frac{s}{\varepsilon} \right) [G_\varepsilon(s)] \right\|_{L^2(\Omega)}^2 \, \mathrm{d}s$$

$$= \omega^2(\varepsilon, \varphi) \int_0^T \| G_\varepsilon(s) \|_{L^2(\Omega)}^2 \, \mathrm{d}s.$$

Reformulation via spectral measures:

$$\left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon}\right)[\Psi], \varphi \right\rangle = \int_0^\infty \exp\left(\mathrm{i}\sqrt{\lambda}\frac{t}{\varepsilon}\right) \tilde{\Psi}(\lambda)\ \mathrm{d}\mu_\varphi(\lambda)$$

where μ_φ is the spectral measure associated to the function φ

$$\tilde{\Psi} \in L^2(\Omega; \mathrm{d}\mu_\varphi),\ \|\tilde{\Psi}\|_{L^2_{\mu_\varphi}(\Omega)} \leq \|\Psi\|_{L^2(\Omega)}.$$

Decay via RAGE theorem:

$$\int_0^T \left| \left\langle \exp \left(\mathrm{i} \sqrt{A} \frac{t}{\varepsilon} \right) [\Psi], \varphi \right\rangle \right|^2 \ \mathrm{d}t$$

$$= \int_0^T \int_0^\infty \int_0^\infty \exp \left(\mathrm{i} \left(\sqrt{x} - \sqrt{y} \right) \frac{t}{\varepsilon} \right) \tilde{\Psi}(x) \, \overline{\tilde{\Psi}}(y) \, \mathrm{d}\mu_\varphi(x) \, \mathrm{d}\mu_\varphi(y) \, \mathrm{d}t$$

$$\leq e \int_0^\infty \int_0^\infty \left(\int_{-\infty}^\infty \exp \left(-(t/T)^2 \right) \exp \left(\mathrm{i} \left(\sqrt{x} - \sqrt{y} \right) \frac{t}{\varepsilon} \right) \ \mathrm{d}t \right) \times$$

$$\times \tilde{\Psi}(x) \overline{\tilde{\Psi}(y)} \ \mathrm{d}\mu_\varphi(x) \ \mathrm{d}\mu_\varphi(y)$$

$$\leq eT\sqrt{\pi} \int_0^\infty \int_0^\infty \tilde{\Psi}(x) \, \overline{\tilde{\Psi}(y)} \exp \left(- \frac{T^2 |\sqrt{x} - \sqrt{y}|^2}{4\varepsilon^2} \right) \ \mathrm{d}\mu_\varphi(x) \, \mathrm{d}\mu_\varphi(y).$$

Cauchy-Swartz inequality:

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon} \right) [\Psi], \varphi \right\rangle \right|^2 \, \mathrm{d}t \leq \omega^2(\varepsilon, \varphi) \|\Psi\|^2_{L^2(\Omega)}$$

$$\omega^4(\varepsilon, \varphi) = \sqrt{2} \int_0^\infty \int_0^\infty \exp\left(-\frac{T^2 |\sqrt{x} - \sqrt{y}|^2}{2\varepsilon^2} \right) \, \mathrm{d}\mu_\varphi(x) \, \mathrm{d}\mu_\varphi(y)$$

Decay via Kato's theorem

[Kato, 1965]

Theorem

Let C be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R,\ v \in \mathcal{D}(C^*),\ \|v\|_X = 1} \| C \circ R_H[\lambda] \circ C^*[v] \|_X < \infty.$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \| C \exp(-\mathrm{i} t H)[w] \|_X^2 \ \mathrm{d}t \leq \Gamma^2.$$

Reformulation of the problem:

$$\left(\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon}\right) [\Psi], G(A)[\varphi] \right\rangle \right|^2 \ \mathrm{d}t \right)^{1/2} \leq \omega(\varepsilon, G, \varphi) \|\Psi\|_{L^2(\Omega)}$$

$$\varphi \in C_0^\infty(\Omega), \ G \in C_0^\infty(0,\infty), \ 0 \leq G \leq 1$$

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A}\frac{t}{\varepsilon} \right) [\Psi], G(A)[\varphi] \right\rangle \right|^2 \, \mathrm{d}t$$

$$\leq eT\sqrt{\pi} \int_0^\infty |\Psi(x)|^2 \left(\int_0^\infty \exp\left(-\frac{|\sqrt{x} - \sqrt{y}|^2}{\varepsilon^2} \frac{T^2}{4} \right) \mathrm{d}\mu_\varphi(y) \right) \times$$

$$\times G^2(x) \, \mathrm{d}\mu_\varphi(x)$$

$$\int_0^\infty \exp\left(-\frac{|\sqrt{x}-\sqrt{y}|^2}{\varepsilon^2}\frac{T^2}{4}\right)\mathrm{d}\mu_\varphi(y)$$

$$= \sum_{n=0}^\infty \int_{\varepsilon n \le |\sqrt{y}-\sqrt{x}| < \varepsilon(n+1)} \exp\left(-\frac{|\sqrt{x}-\sqrt{y}|^2}{\varepsilon^2}\frac{T^2}{4}\right)\mathrm{d}\mu_\varphi(y)$$

$$\le \sup_{n\ge 0}\int_{\varepsilon n \le |\sqrt{y}-\sqrt{x}| < \varepsilon(n+1)} 1\mathrm{d}\mu_\varphi(y)\sum_{n=0}^\infty \exp\left(-\frac{n^2T^2}{4}\right).$$

Stone's formula:

$$\mu_\varphi(a,b)$$

$$= \lim_{\delta\to 0+} \lim_{\eta\to 0+} \int_{a+\delta}^{b-\delta} \left\langle \left(\frac{1}{A-\lambda-\mathrm{i}\eta} - \frac{1}{A-\lambda+\mathrm{i}\eta} \right) \varphi, \varphi \right\rangle \, \mathrm{d}\lambda$$

Limiting absorption principle:

$$\left\{ \begin{array}{c} \textit{Operators} \\ \mathcal{V} \circ (A - \lambda \pm \mathrm{i}\eta)^{-1} \circ \mathcal{V} : L^2(\Omega) \to L^2(\Omega), \\ \mathcal{V}[v] = (1 + |x|^2)^{-s/2},\ s > 1 \\ \textit{are bounded uniformly for } \lambda \in [a, b],\ 0 < a < b,\ \eta > 0, \end{array} \right\}$$

$\mu_\varphi[I] \leq c(\delta)|I|$ for any compact interval $I \subset (\delta, 1/\delta),\ \delta > 0$