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Abstract

We obtain new characterizations of Li-Yorke chaos for linear operators on Banach and Fréchet spaces. We also offer conditions under which an operator admits a dense set or linear manifold of irregular vectors. Some of our general results are applied to composition operators and adjoint multipliers on spaces of holomorphic functions.[1]

1 Introduction

Chaos was introduced in linear dynamics by Godefroy and Shapiro [15], who adopted Devaney's definition of (nonlinear) chaos. So, it became usual to say that a continuous linear operator T on a Fréchet space X is *chaotic* if it is hypercyclic (i.e., has a dense orbit) and has a dense set of periodic points. Chaotic operators have been extensively studied by several authors during the last 20 years (see [8, 11, 13, 15, 16, 19, 20, 34, 39], for instance). The recent books [3] and [17] contain a considerably up to date account on linear dynamics, including many results on chaotic operators.

Nevertheless, there are others important and useful notions of chaos, like Li-Yorke chaos, distributional chaos and specification properties. More recently, some authors have started to study these notions of chaos in the context of linear dynamics [2, 5, 12, 14, 21, 22, 23, 25, 26, 28, 38, 40]. In the present work we shall concentrate on Li-Yorke chaos and some of its variants.

Given a metric space M with metric d and a continuous map $f : M \to M$, recall that a pair $(x, y) \in M \times M$ is called a *Li-Yorke pair* for f if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

A *scrambled set* for f is a subset S of M such that (x, y) is a Li-Yorke pair for f whenever x and y are distinct points in S. The map f is said to be *Li-Yorke chaotic* if there exists an uncountable scrambled set for f. This notion of chaos was introduced by Li and Yorke [24] in the context of interval maps. It was the first notion of chaos and became very popular. Since then several variants of this notion have been introduced by several authors. Here we are going to consider four of them. The map f is said to be *densely* (resp. *generically*) *Li-Yorke chaotic* if there exists an uncountable dense (resp. residual) scrambled set for f.

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Moreover, f is said to be *densely* (resp. *generically*) *w-Li-Yorke chaotic* if the set of all Li-Yorke pairs for f is dense (resp. residual) in $M \times M$ (see [30, 36, 37], for instance).

Within the Operator Theory framework, Beauzamy [4] introduced the notion of *irregular vector* $x \in X$ for an operator $T : X \to X$ on a Banach space X, which means $\inf \|T^n x\| = 0$ and, at the same time, $\sup \|T^n x\| = \infty$. Although the origin of this concept was completely independent of Li-Yorke chaos, surprisingly enough both concepts turn out to be equivalent for T.

Bermúdez, Bonilla, Martínez-Giménez and Peris [5] obtained several results concerning Li-Yorke chaos for continuous linear operators on Banach spaces, including a characterization of Li-Yorke chaos in terms of the existence of irregular vectors, a necessary and sufficient criterion for Li-Yorke chaos and a sufficient criterion for the existence of a dense irregular manifold.

In the present work we extend the main results in [5] concerning Li-Yorke chaos from Banach spaces to arbitrary Fréchet spaces. Moreover, we also establish several results that are new even in the Banach space setting.

Among our main results, we introduce the notion of a semi-irregular vector and characterize Li-Yorke chaos, dense Li-Yorke chaos and generic Li-Yorke chaos in terms of the existence of semi-irregular or irregular vectors. Somewhat surprisingly, we prove that dense w-Li-Yorke chaos, generic w-Li-Yorke chaos and dense Li-Yorke chaos are all equivalent for continuous linear operators on separable Fréchet spaces. Also, we present necessary and sufficient criteria for Li-Yorke chaos and for dense Li-Yorke chaos, and a sufficient criterion for the existence of a dense irregular manifold. We apply our general results to unilateral weighted backward shifts on Fréchet sequence spaces and to composition operators and adjoint multipliers on spaces of holomorphic functions. Moreover, we prove that generic Li-Yorke chaos is equivalent to the whole space being a scrambled set for the operator and we give an example of a generically Li-Yorke chaotic operator on Hilbert space that is not completely irregular. We also include some open problems in the final section.

Throughout this paper X denotes an arbitrary infinite-dimensional separable Fréchet space, unless otherwise specified. Moreover, $B(X)$ denotes the set of all continuous linear operators $T : X \to X$, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2 Li-Yorke and dense Li-Yorke chaos

The following concept is a generalization to Fréchet spaces of the one introduced by Beauzamy [4] for Banach spaces.

Definition 1. Given an operator $T \in B(X)$ and a vector $x \in X$, we say that x is an *irregular vector* for T if the sequence $(T^n x)_{n \in \mathbb{N}}$ is unbounded, but it has a subsequence converging to zero.

If the topology of X is given by an increasing sequence $(\|\cdot\|_k)_k$ of seminorms, then x is an irregular vector for T if and only if there are increasing sequences $(n_k)_{k\in\mathbb{N}}$, $(j_k)_{k\in\mathbb{N}}$ of positive integers and $m \in \mathbb{N}$ such that

$$\lim_{k\to\infty} T^{n_k} x = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T^{j_k} x\|_m = \infty.$$

In the case X is a Banach space this is the same as requiring that

$$\inf_{n\in\mathbb{N}}\|T^n x\| = 0 \quad \text{and} \quad \sup_{n\in\mathbb{N}}\|T^n x\| = \infty.$$

We will also need the following weaker notion of irregularity.

Definition 2. Given an operator $T \in B(X)$ and a vector $x \in X$, we say that x is a *semi-irregular vector* for T if the sequence $(T^n x)_{n\in\mathbb{N}}$ does not converge to zero, but it has a subsequence converging to zero.

These notions make sense only for infinite-dimensional spaces, since an easy application of the Jordan form implies that there are no semi-irregular vectors for operators on finite-dimensional spaces.

We will study each condition in the definition of irregular vectors separately.

Proposition 3. Let $T \in B(X)$. The set of all vectors $x \in X$ such that $(T^n x)$ has a subsequence converging to zero is a G_δ-set in X.

Proof. Let $(V_j)_{j\in\mathbb{N}}$ be a countable fundamental system of open neighborhoods of 0 in X. For each $j \in \mathbb{N}$, define

$$A_j := \{x \in X : T^n x \in V_j \text{ for some } n \in \mathbb{N}\}.$$

Clearly each A_j is open in X and the intersection $\bigcap_{j=1}^\infty A_j$ is exactly the set of all vectors $x \in X$ such that $(T^n x)$ has a subsequence converging to zero. □

Corollary 4. Let $T \in B(X)$. If the set of all points $x \in X$ such that $(T^n x)$ has a subsequence converging to zero is dense in X, then it is residual in X.

Proposition 5. Let $T \in B(X)$. If T has a vector with unbounded orbit, then T has a residual set of vectors with unbounded orbits.

Proof. By hypothesis, there exists a vector $u \in X$ whose T-orbit

$$\mathrm{Orb}(u,T) := \{u, Tu, T^2u, \ldots\}$$

is unbounded. Hence, there is an absolutely convex closed neighborhood V of 0 in X such that

$$\mathrm{Orb}(u,T) \not\subset tV \quad \text{for all } t > 0.$$

For each $j \in \mathbb{N}$, define

$$A_j := \{x \in X : \mathrm{Orb}(x,T) \not\subset jV\}.$$

Clearly A_j is open in X. We claim that it is also dense. Indeed, if $y \in X \setminus A_j$ then $\mathrm{Orb}(y,T) \subset jV$ and $y + \varepsilon u \in A_j$ for each $\varepsilon > 0$. Thus, $\bigcap_{j=1}^\infty A_j$ is a residual set in X consisting entirely of vectors with unbounded orbits. □

Corollary 6. Let $T \in B(X)$. If the set of all irregular vectors for T is dense in X, then it is residual in X.

The following result will be fundamental in this section.

Lemma 7. Let $T \in B(X)$ and suppose that $x \in X$ is a semi-irregular vector for T which is not irregular for T. Then there exists a series $\sum x_j$ of non-zero vectors in X such that

$$\alpha x + \sum_{j=1}^{\infty} \beta_j x_j$$

is an irregular vector for T, whenever α is a scalar and (β_j) is a sequence of scalars that takes only finitely many values and has infinitely many non-zero coordinates.

Proof. Since the sequence $(T^n x)$ does not converge to 0, there exists an absolutely convex closed neighborhood V of 0 in X such that

$$T^n x \notin V \quad \text{for infinitely many values of } n. \tag{1}$$

Let $(V_j)_{j\in\mathbb{N}_0}$ be a countable fundamental system of neighborhoods of 0 in X such that each V_j is absolutely convex and closed,

$$V_0 = V, \quad V_j + V_j \subset V_{j-1} \quad \text{and} \quad T(V_j) \subset V_{j-1} \quad \text{for every } j \in \mathbb{N}. \tag{2}$$

Note that

$$V_0 \supset V_1 \supset V_2 \supset V_3 \supset \cdots, \tag{3}$$

$$T^n(V_j) \subset V_{j-n} \quad \text{whenever } n \leq j, \text{ and} \tag{4}$$

$$V_p + V_{p+1} + \cdots + V_q \subset V_{p-1} \quad \text{whenever } 1 \leq p < q. \tag{5}$$

Since x is not an irregular vector for T, the sequence $(T^n x)$ must be bounded. Hence, there exists $r \in \mathbb{N}$ such that

$$T^n x \in rV \quad \text{for every } n \in \mathbb{N}. \tag{6}$$

We define recursively an increasing sequence $(c_k)_{k\in\mathbb{N}_0}$ of non-negative integers by

$$c_0 = 0 \quad \text{and} \quad c_k = k^2(2 + r(c_0 + \cdots + c_{k-1})) \text{ for } k \geq 1.$$

Now, we shall construct inductively sequences $n_1 < m_1 < n_2 < m_2 < \cdots$ and $p_1 < p_2 < \cdots$ of positive integers so that the following properties hold for every $k \in \mathbb{N}$:

(a) $T^{n_k} x \in c_k^{-1} V_{m_{k-1}+p_{k-1}}$,

(b) $T^{m_k} x \notin V$,

(c) $T^{p_k} x \in V_k$, and

(d) $T^{p_k}\big(\sum_{j=1}^{k} \lambda_j c_j T^{n_j} x\big) \in V_k$ whenever $|\lambda_j| \leq k$ for every j,

where $m_0 = p_0 = 0$. By (1) and the fact that $(T^n x)$ has a subsequence converging to zero, we may choose $n_1 \in \mathbb{N}$ such that $T^{n_1} x \in c_1^{-1} V_0$, $m_1 > n_1$ such that $T^{m_1} x \notin V$, and $p_1 \in \mathbb{N}$ such that $T^{p_1} x \in V_1$ and $T^{p_1}(c_1 T^{n_1} x) = c_1 T^{n_1} T^{p_1} x \in V_1$. If $s \geq 2$ and n_k, m_k, p_k have already been chosen for $1 \leq k \leq s-1$, then it is enough to choose $n_s > m_{s-1}$ such that $T^{n_s} x \in c_s^{-1} V_{m_{s-1}+p_{s-1}}$, $m_s > n_s$ such that $T^{m_s} x \notin V$, and $p_s > p_{s-1}$ such that $T^{p_s} x$ is so close to zero that (c) and (d) hold with s in place of k.

For each $j \in \mathbb{N}$, let

$$x_j := c_j T^{n_j} x.$$

We shall prove that the series $\sum x_j$ has the desired properties. For this purpose, let us fix a sequence (β_j) of scalars that takes only finitely many values and has infinitely many non-zero coordinates. Let $\gamma \in \mathbb{N}$ be such that

$$\min\{|\beta_j| : j \in \mathbb{N} \text{ and } \beta_j \neq 0\} \geq \frac{1}{\gamma} \quad \text{and} \quad \max\{|\beta_j| : j \in \mathbb{N}\} \leq \gamma.$$

By (a), (3) and (5),

$$\sum_{j=p}^{q} \beta_j x_j \in \sum_{j=p}^{q} \beta_j V_{m_{j-1}+p_{j-1}} \subset \gamma(V_p + V_{p+1} + \cdots + V_q) \subset \gamma V_{p-1},$$

whenever $2 \leq p < q$. Hence, the partial sums of the series $\sum \beta_j x_j$ form a Cauchy sequence in X, and so we may define

$$y := \sum_{j=1}^{\infty} \beta_j x_j \in X.$$

We shall prove that y is an irregular vector for T. Fix $k \geq 2$. If $j \geq k+1$ then $x_j \in V_{m_{j-1}+p_{j-1}} \subset V_{j+p_k}$, and so $T^{p_k} x_j \in V_j$ by (4). Thus,

$$\sum_{j=k+1}^{q} T^{p_k}(\beta_j x_j) \in \gamma(V_{k+1} + V_{k+2} + \cdots + V_q) \subset \gamma V_k,$$

for every $q > k+1$. Since V_k is closed, by letting $q \to \infty$ we obtain

$$\sum_{j=k+1}^{\infty} T^{p_k}(\beta_j x_j) \in \gamma V_k.$$

By (d), we conclude that

$$T^{p_k} y = T^{p_k}\Big(\sum_{j=1}^{k} \beta_j x_j\Big) + \sum_{j=k+1}^{\infty} T^{p_k}(\beta_j x_j) \in V_k + \gamma V_k$$

whenever $k \geq \gamma$. This proves that

$$T^{p_k} y \to 0 \quad \text{as } k \to \infty.$$

Now, fix $k \geq \gamma$ such that $\beta_k \neq 0$. Note that

$$T^{m_k - n_k} y = \sum_{j=1}^{k-1} T^{m_k - n_k}(\beta_j x_j) + T^{m_k}(\beta_k c_k x) + \sum_{j=k+1}^{\infty} T^{m_k - n_k}(\beta_j x_j). \tag{7}$$

By (b), $T^{m_k}(\beta_k c_k x) \notin \beta_k c_k V = \beta_k k^2 (2 + r(c_1 + \cdots + c_{k-1}))V$, which implies that

$$T^{m_k}(\beta_k c_k x) \notin k(2 + r(c_1 + \cdots + c_{k-1}))V, \tag{8}$$

because $|\beta_k k| \geq 1$. By (6),

$$\sum_{j=1}^{k-1} T^{m_k-n_k}(\beta_j x_j) = \sum_{j=1}^{k-1} \beta_j c_j T^{m_k-n_k+n_j} x \in \sum_{j=1}^{k-1} (c_j \gamma r V) \subset kr(c_1 + \cdots + c_{k-1})V. \qquad (9)$$

If $j \geq k+1$ then $x_j \in V_{m_{j-1}+p_{j-1}} \subset V_{m_k-n_k+j-1}$, and so $T^{m_k-n_k} x_j \in V_{j-1}$. Thus,

$$\sum_{j=k+1}^{q} T^{m_k-n_k}(\beta_j x_j) \in \gamma(V_k + V_{k+1} + \cdots + V_{q-1}) \subset \gamma V_{k-1} \subset kV,$$

for every $q > k+1$, which gives

$$\sum_{j=k+1}^{\infty} T^{m_k-n_k}(\beta_j x_j) \in kV. \qquad (10)$$

¿From (7)–(10), we conclude that

$$T^{m_k-n_k} y \notin kV.$$

This shows that the sequence $(T^n y)$ is unbounded, and so y is an irregular vector for T. Now, for each scalar α, the sequence $(T^n(\alpha x + y))$ is unbounded (because $(T^n x)$ is bounded) and $T^{p_k}(\alpha x + y) \to 0$ (by (c)). Therefore, $\alpha x + y$ is an irregular vector for T for every scalar α, which completes the proof. □

As a first application of the previous lemma, we have the following result.

Theorem 8. If $T \in B(X)$ then every neighborhood of a semi-irregular vector for T contains an irregular vector for T.

Proof. Suppose that x is a semi-irregular vector for T. If x is irregular for T, we are done. If this is not the case, then Lemma 7 implies the existence of a vector y such that $x + \beta y$ is irregular for T for every $\beta \neq 0$, which proves the theorem. □

Now we shall apply the above theorem in order to establish some interesting characterizations of Li-Yorke chaos and dense Li-Yorke chaos for operators. The first of these results extends Theorem 5 of [5] from Banach spaces to Fréchet spaces.

Theorem 9. If $T \in B(X)$ then the following assertions are equivalent:

(i) T is Li-Yorke chaotic;

(ii) T admits a Li-Yorke pair;

(iii) T admits a semi-irregular vector;

(iv) T admits an irregular vector.

Proof. (i) ⇒ (ii): Obvious.
(ii) ⇒ (iii): Let (a, b) be a Li-Yorke pair for T. By definition,

$$\liminf_{n\to\infty} d(T^n a, T^n b) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n a, T^n b) > 0.$$

Thus,

$$\liminf_{n\to\infty} d(T^n(a-b),0) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n(a-b),0) > 0,$$

which shows that the vector $x := a - b$ is semi-irregular for T.
(iii) $\Rightarrow$ (iv): Follows from Theorem 8.
(iv) $\Rightarrow$ (i): If $x \in X$ is an irregular vector for T, then it is clear that span$\{x\}$ is a scrambled set for T. □

Theorem 10. If $T \in B(X)$ then the following assertions are equivalent:

(i) T is densely Li-Yorke chaotic;

(ii) T is densely w-Li-Yorke chaotic;

(iii) T is generically w-Li-Yorke chaotic;

(iv) T admits a dense set of semi-irregular vectors;

(v) T admits a dense set of irregular vectors;

(vi) T admits a residual set of irregular vectors.

Proof. (i) $\Rightarrow$ (ii): Obvious.
(ii) $\Rightarrow$ (iv): Fix $x \in X$ and a neighborhood V of 0 in X. Let U be a balanced neighborhood of 0 in X such that $U + U \subset V$. By hypothesis, there is a Li-Yorke pair (a, b) for T such that $(a, b) \in (x, 0) + (U \times U)$. Hence, $y := a - b$ is a semi-irregular vector for T that lies in the neighborhood $x + V$ of x in X.
(iv) $\Rightarrow$ (v): Follows from Theorem 8.
(v) $\Rightarrow$ (vi): Follows from Corollary 6.
(vi) $\Rightarrow$ (iii): By hypothesis, there is a sequence (A_j) of dense open sets in X such that the intersection $\bigcap A_j$ consists of irregular vectors for T. For each $j \in \mathbb{N}$, let

$$B_j := \{(a, b) \in X \times X : a - b \in A_j\}.$$

Then, each B_j is dense and open in $X \times X$, and $\bigcap B_j$ consists of Li-Yorke pairs for T.
(iii) $\Rightarrow$ (ii): Obvious.
(vi) $\Rightarrow$ (i): By hypothesis, there is a residual set R in X consisting entirely of irregular vectors for T. Let $D := \mathbb{Q}$ or $\mathbb{Q} + i\mathbb{Q}$, depending on whether the scalar field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, respectively. Let (y_j) be a dense sequence in X. We choose inductively vectors $x_1, x_2, x_3, \ldots \in X$ in the following fashion:

$$\begin{aligned}
&x_1 \in B(y_1; 1) \cap R,\\
&x_2 \in B(y_2; \frac{1}{2}) \cap \bigcap_{\alpha_1 \in D} (\alpha_1 x_1 + R),\\
&x_3 \in B(y_3; \frac{1}{3}) \cap \bigcap_{(\alpha_1,\alpha_2) \in D^2} (\alpha_1 x_1 + \alpha_2 x_2 + R),
\end{aligned}$$

and so on. Moreover, we may make the choices so that the sequence (x_j) is linearly independent. In this way,

$$M := \{\alpha_1 x_1 + \cdots + \alpha_m x_m : m \geq 1 \text{ and } \alpha_1, \ldots, \alpha_m \in D\}$$

is a dense D-vector subspace of X consisting (up to 0) of irregular vectors for T. In particular, it is a dense scrambled set for T. However, this dense scrambled set is countable. We shall enlarge M in order to obtain an uncountable dense scrambled set for T. For this purpose, we shall need the following fact:

$(*)$ If $y, z \in X$ and the set

$$A := \{\lambda \in \mathbb{K} : y - \lambda z \text{ is semi-irregular for } T\}$$

is dense in $\mathbb{K}$, then it is residual in $\mathbb{K}$.

Indeed, let B be the set of all $\lambda \in \mathbb{K}$ such that $(T^n(y - \lambda z))$ has a subsequence converging to zero. By arguing as in the proof of Proposition 3, we see that B is a G_δ-set in $\mathbb{K}$. Since $B \supset A$, B is residual in $\mathbb{K}$.

If $A \supset \mathbb{K}\backslash\{0\}$ then we are done. So, assume that there exists $\lambda_0 \in \mathbb{K}\backslash\{0\}$ such that $y - \lambda_0 z$ is not semi-irregular for T. Then we have two possibilities:

(1) $T^n(y - \lambda_0 z) \to 0$.

Since A is nonempty, we may fix a scalar $\gamma \in A$. Then $T^n(y - \gamma z) \not\to 0$. Since $T^n(y - \lambda_0 z) \to 0$, it follows that $T^n z \not\to 0$. Hence, if $\lambda \neq \lambda_0$ then $T^n(y - \lambda z) \not\to 0$, which proves that $B\backslash\{\lambda_0\} \subset A$. Thus A is residual in $\mathbb{K}$.

(2) No subsequence of $(T^n(y - \lambda_0 z))$ converges to zero.

Fix a scalar $\gamma \in A\backslash\{0\}$. Some subsequence $(T^{n_j}(y - \gamma z))$ must converge to zero, which implies that $T^{n_j} z \not\to 0$. Hence, if $\lambda \neq \gamma$ then $T^{n_j}(y - \lambda z) \not\to 0$. As before, $B\backslash\{\gamma\} \subset A$ and A is residual in $\mathbb{K}$.

Now, let

$$N := \{\alpha_2 x_2 + \cdots + \alpha_m x_m : m \geq 2 \text{ and } \alpha_2, \ldots, \alpha_m \in D\}.$$

For each $y \in N\backslash\{0\}$, let

$$A_y := \{\lambda \in \mathbb{K} : y - \lambda x_1 \text{ is semi-irregular for } T\}.$$

Since A_y contains D, A_y is dense in $\mathbb{K}$. Hence, by $(*)$, A_y is residual in $\mathbb{K}$. Let

$$A := \bigcap_{y \in N\backslash\{0\}} A_y,$$

which is also a residual set in $\mathbb{K}$ containing D. A simple application of Zorn's Lemma shows that there exists a maximal D-vector subspace H of $\mathbb{K}$ subjected to the condition

$$D \subset H \subset A.$$

We claim that H is uncountable. Indeed, suppose that H is countable. Then

$$\bigcap_{\beta \in D\backslash\{0\}} \bigcap_{\alpha \in H} \beta(\alpha + A)$$

is a residual set in $\mathbb{K}$. So we may take a scalar γ that belongs to this intersection and does not belong to H. Then

$$H' := H + \{\beta\gamma : \beta \in D\}$$

is a D-vector subspace of $\mathbb{K}$ satisfying $D \subset H' \subset A$ and $H \subsetneq H'$. This contradicts the maximality of H and prove that H is uncountable. Finally,

$$M' := \{\alpha x_1 : \alpha \in H\} + N$$

is a scrambled set for T with the property that $M' \cap V$ is uncountable for every nonempty open set V in X. Note also that M' is a D-vector subspace of X. $\square$

It is easy to construct a continuous linear operator that has an irregular vector but not a dense set of irregular vectors (see Remark 2.3 of [31]). In view of Theorems 9 and 10, such an operator is Li-Yorke chaotic but not densely Li-Yorke chaotic.

Proposition 11. If $T \in B(X)$ and T^* has an eigenvalue λ with $|\lambda| \geq 1$, then T is not densely Li-Yorke chaotic.

Proof. Assume that T is densely Li-Yorke chaotic and that λ is an eigenvalue of T^*. Let $\phi \in X^* \backslash \{0\}$ be such that $T^*\phi = \lambda\phi$. Then

$$\phi(T^n x) = ((T^*)^n \phi)(x) = \lambda^n \phi(x) \qquad (x \in X, n \in \mathbb{N}_0).$$

By Theorem 10, T has a dense set of irregular vectors. In particular, there exists $x \in X$ such that $\phi(x) \neq 0$ and some subsequence $(T^{n_k} x)_{k \in \mathbb{N}}$ converges to zero. Hence

$$\lim_{k \to \infty} \lambda^{n_k} \phi(x) = \lim_{k \to \infty} \phi(T^{n_k} x) = 0.$$

This implies that $|\lambda| < 1$. $\square$

Remark 12. The unilateral weighted forward shift on ℓ_2 of weights

$$2,\ 0.5,\ 0.5,\ 2,\ 2,\ 2,\ 0.5,\ 0.5,\ 0.5,\ 0.5, \ldots$$

has a dense set of irregular vectors (Proposition 3.9 in [31]), hence it is densely Li-Yorke chaotic, and T^* has eigenvalues λ with $|\lambda| < 1$.

Let us now establish the following auxiliary result.

Lemma 13. Let $T \in B(X)$ and suppose that there exists a subset X_0 of X with the following properties:

- $T^n x \to 0$ for every $x \in X_0$.
- There is a bounded sequence (a_n) in $Y := \overline{\text{span}(X_0)}$ such that the sequence $(T^n a_n)$ is unbounded.

Then there exists a series $\sum x_j$ of non-zero vectors in X such that

$$\sum_{j=1}^{\infty} \beta_j x_j$$

is an irregular vector for T, whenever (β_j) is a sequence of scalars that takes only finitely many values and has infinitely many non-zero coordinates.

Proof. If there is a semi-irregular vector for T which is not irregular for T, then the result follows from Lemma 7. So, let us assume that every semi-irregular vector for T is irregular for T. Since $(T^n a_n)$ is unbounded, there are a subsequence $(T^{q_k} a_{q_k})$ of $(T^n a_n)$ and a sequence (t_k) of positive real numbers such that $t_k \to 0$ but the sequence $(t_k T^{q_k} a_{q_k})$ does not converge to zero. Since (a_n) is bounded, $t_k a_{q_k} \to 0$. By the density of span(X_0) in Y, we may choose vectors y_{q_k} in span(X_0) $(k \in \mathbb{N})$ so that

$$\lim_{k\to\infty} (y_{q_k} - t_k a_{q_k}) = 0 \quad \text{and} \quad \lim_{k\to\infty} T^{q_k}(y_{q_k} - t_k a_{q_k}) = 0.$$

By putting $y_n = 0$ whenever $n \neq q_k$ for every $k \in \mathbb{N}$, we obtain a sequence (y_n) in span(X_0) such that

$$\lim_{n\to\infty} y_n = 0 \tag{11}$$

but $(T^n y_n)$ does not converge to zero. Let us fix a balanced closed neighborhood V of 0 in X such that

$$T^n y_n \notin V \quad \text{for infinitely many values of } n. \tag{12}$$

By hypothesis,

$$T^n x \to 0 \quad \text{for every } x \in \text{span}(X_0). \tag{13}$$

Let $(V_j)_{j\in\mathbb{N}_0}$ be a countable fundamental system of neighborhoods of 0 in X such that each V_j is balanced and closed,

$$V_0 = V \quad \text{and} \quad V_j + V_j \subset V_{j-1} \ \text{ for every } j \in \mathbb{N}.$$

Then

$$V_p + V_{p+1} + \cdots + V_q \subset V_{p-1} \quad \text{whenever } 1 \leq p < q. \tag{14}$$

It follows from (11)–(13) that there are increasing sequences $m_1 < m_2 < \cdots$ and $p_1 < p_2 < \cdots$ of positive integers such that the following properties hold for every $k \in \mathbb{N}$:

(a) $T^{m_k} y_{m_k} \notin V$,

(b) $y_{m_k} \in V_k$,

(c) $T^{m_j} y_{m_k} \in k^{-2} V_k$ for $j = 1, \ldots, k-1$,

(d) $T^{p_j} y_{m_k} \in V_k$ for $j = 1, \ldots, k-1$,

(e) $T^{m_k} y_{m_j} \in k^{-2} V_{j+1}$ for $j = 1, \ldots, k-1$, and

(f) $T^{p_k} y_{m_j} \in V_{k+j}$ for $j = 1, \ldots, k$.

For each $j \in \mathbb{N}$, let

$$x_j := y_{m_j}.$$

We shall prove that the series $\sum x_j$ has the desired properties. Fix a sequence (β_j) of scalars that takes only finitely many values and has infinitely many non-zero coordinates. Let $\gamma \in \mathbb{N}$ be such that

$$\min\{|\beta_j| : j \in \mathbb{N} \text{ and } \beta_j \neq 0\} \geq \frac{1}{\gamma} \quad \text{and} \quad \max\{|\beta_j| : j \in \mathbb{N}\} \leq \gamma.$$

It follows from (b) and (14) that the vector

$$y := \sum_{j=1}^{\infty} \beta_j x_j$$

is well defined since the series is convergent. Fix $k \geq 2$. By (d) and (f), for every $q > k+1$,

$$\sum_{j=1}^{q} T^{p_k}(\beta_j x_j) \in \sum_{j=1}^{k} \gamma V_{k+j} + \sum_{j=k+1}^{q} \gamma V_j \subset \gamma(V_k + V_k) \subset \gamma V_{k-1}.$$

Thus,

$$T^{p_k} y = \sum_{j=1}^{\infty} T^{p_k}(\beta_j x_j) \in \gamma V_{k-1}.$$

This proves that

$$T^{p_k} y \to 0 \quad \text{as } k \to \infty.$$

Now, fix $k \geq \gamma$ such that $\beta_k \neq 0$. By (c) and (e), for every $q > k+1$,

$$\sum_{j=1}^{q} T^{m_k}(\beta_j x_j) \in \sum_{j=1}^{k-1} \gamma k^{-2} V_{j+1} + \beta_k T^{m_k} y_{m_k} + \sum_{j=k+1}^{q} \gamma j^{-2} V_j \subset \beta_k T^{m_k} y_{m_k} + \gamma^{-1} V_1.$$

Hence,

$$T^{m_k} y = \sum_{j=1}^{\infty} T^{m_k}(\beta_j x_j) \in \beta_k T^{m_k} y_{m_k} + \gamma^{-1} V_1.$$

Since $T^{m_k} y_{m_k} \notin V$ by (a), it follows that

$$T^{m_k} y \notin \gamma^{-1} V_1.$$

Thus, the sequence $(T^n y)$ does not converge to zero, which shows that y is a semi-irregular vector for T. Since we are assuming that every semi-irregular vector for T is irregular for T, the proof is complete. □

Definition 14. Let $T \in B(X)$. We say that T satisfies the *Li-Yorke Chaos Criterion* if there exists a subset X_0 of X with the following properties:

(a) $(T^n x)$ has a subsequence converging to zero, for every $x \in X_0$.

(b) There is a bounded sequence (a_n) in $\overline{\text{span}(X_0)}$ such that the sequence $(T^n a_n)$ is unbounded.

Let us observe that the above criterion is slightly different from the Li-Yorke Chaos Criterion defined in [5] in the context of Banach spaces. Our property (b) is equivalent to the corresponding property (b) in Definition 7 of [5], but our property (a) is weaker than property (a) in that definition. In fact, in [5] it is required the existence of an increasing sequence (n_k) of positive integers such that

$$T^{n_k} x \to 0 \quad \text{for every } x \in X_0.$$

In our definition, the sequence (n_k) may depend on x.

We shall now show that the Li-Yorke Chaos Criterion characterizes Li-Yorke chaos, thereby generalizing Theorem 8 of [5] from Banach spaces to Fréchet spaces.

Theorem 15. If $T \in B(X)$ then the following assertions are equivalent:

(i) T is Li-Yorke chaotic;

(ii) T satisfies the Li-Yorke Chaos Criterion.

Proof. (i) $\Rightarrow$ (ii): By Theorem 9, T admits an irregular vector $x \in X$. By setting $X_0 = \{x\}$, it is clear that T verifies the Li-Yorke Chaos Criterion.
(ii) $\Rightarrow$ (i): Let X_0 be as in the definiton of the Li-Yorke Chaos Criterion. If some vector in X_0 is semi-irregular for T, then we are done by Theorem 9. So, let us assume that this is not the case. For each $x \in X_0$, since $(T^n x)$ has a subsequence converging to zero by hypothesis, it follows that $(T^n x)$ must converge to zero. Therefore, Lemma 13 assures that there is an irregular vector for T. Again by Theorem 9, we conclude that T is Li-Yorke chaotic. □

Definition 16. Let $T \in B(X)$. We say that T satisfies the *Dense Li-Yorke Chaos Criterion* if there exists a *dense* subset X_0 of X with properties (a) and (b) of Definition 14.

Let us now prove that the Dense Li-Yorke Chaos Criterion characterizes dense Li-Yorke chaos.

Theorem 17. If $T \in B(X)$ then the following assertions are equivalent:

(i) T is densely Li-Yorke chaotic;

(ii) T satisfies the Dense Li-Yorke Chaos Criterion.

Proof. (i) $\Rightarrow$ (ii): Let X_0 be the set of all irregular vectors for T. By Theorem 10, X_0 is dense in X. Moreover, it is easy to see that X_0 has properties (a) and (b) of Definition 14.
(ii) $\Rightarrow$ (i): Since T satisfies the Li-Yorke Chaos Criterion, T is Li-Yorke chaotic (Theorem 15). Thus, T admits a semi-irregular vector y (Theorem 9). Since X_0 is dense in X, it follows from Theorem 10 that it is enough to prove that arbitrarily close to any point of X_0 there is a semi-irregular vector for T. So, let us fix a point $x \in X_0$. If x is semi-irregular for T, then we are done. Assume that this is not the case. Since $(T^n x)$ has a subsequence converging to zero (because of property (a) in Definition 14), we conclude that

$$\lim_{n \to \infty} T^n x = 0.$$

Therefore, $x + \delta y$ is a semi-irregular vector for T for every scalar $\delta \neq 0$, which completes the proof. □

It was proved in [22] and [31] that a compact operator on a complex Hilbert space cannot be Li-Yorke chaotic. As observed in [5], this result holds on arbitrary Banach spaces. It is natural to ask if there is any compact operator on a non-normable Fréchet space which is Li-Yorke chaotic. The answer is no. The argument is similar to the one given in Proposition 8 of [9].

Proposition 18. No compact operator on X can be Li-Yorke chaotic.

Proof. Suppose that $T : X \to X$ is a compact and Li-Yorke chaotic operator. Let U be an absolutely convex neighbourhood of 0 in X such that $T(U)$ is relatively compact, and let p_U be the Minkowski functional of U. By Theorem 9, T admits an irregular vector $x \in X$. Without loss of generality, we may assume that the sequence $(p_U(T^n x))$ is unbounded.

We set X_U as the local Banach space that is the completion of $X/\ker p_U$ with the norm induced by p_U. Let $\Phi_U : X \to X_U$ be the natural map, and $T_U : X_U \to X_U$ the operator induced by T that satisfies $T_U \circ \Phi_U = \Phi_U \circ T$. For $y := \Phi_U(x)$ we have that it is an irregular vector for T_U. On the other hand, we easily have that T_U is compact too, which contradicts the fact that no compact operator on a Banach space is Li-Yorke chaotic. □

If $x \in X$ is a hypercyclic vector for an operator $T \in B(X)$ and p is a nonzero polynomial, then $p(T)x$ is also hypercyclic. This enables a simple proof of the fact that each hypercyclic operator has a dense linear manifold consisting (up to 0) of hypercyclic vectors.

For irregular vectors this approach does not work. If $x \in X$ is irregular for $T \in B(X)$ and p is a nonzero polynomial, then there is an increasing sequence (n_k) in $\mathbb{N}$ such that $T^{n_k}x \to 0$, and so $T^{n_k}p(T)x = p(T)T^{n_k}x \to 0$. However, in general it is not true that the sequence $(T^n p(T)x)$ is unbounded, even in the context of Banach spaces. We give an example that this may happen.

Example 19. Consider the vector space Y_0 formed by all finite linear combinations of the basis elements $e_0, e_1, e_2, \dots$, and let

$$D := \{e_{j+1} - e_j \ (j \geq 0), 2^k e_{2^k}, 2^{k-1} e_{2^k+1}, \dots, 2e_{2^k+k-1}, e_{2^k+k} \ (k \in \mathbb{N})\}.$$

We define a norm in Y_0 by

$$\|y\| := \inf\Big\{\sum_{d\in D} |\alpha_d| : y = \sum_{d \in D} \alpha_d d\Big\}$$

(all sums are finite). Equivalently, the absolutely convex hull of D is the unit ball in $(Y_0, \|\cdot\|)$. Moreover, we define a linear mapping $T : Y_0 \to Y_0$ by $Te_j := e_{j+1}$ $(j \geq 0)$. Note that $\|Td\| \leq 2$ for each $d \in D$ (the only nontrivial estimate is $\|Te_{2^k+k}\| = \|e_{2^k+k+1}\| \leq \|e_{2^k+k+1} - e_{2^k+k}\| + \|e_{2^k+k}\| \leq 2$). So $\|T\| \leq 2$ and T can be uniquely extended to a bounded linear operator (denoted by the same symbol T) on the completion Y of Y_0.

Consider the vector e_0. We have

$$\inf_n \|T^n e_0\| = \inf_n \|e_n\| \leq \inf_k \|e_{2^k}\| = \inf_k 2^{-k} = 0.$$

It is an easy exercise that

$$\begin{aligned}\|e_{2^k-1}\| &= \|e_{2^k-1} - e_{2^k-2}\| + \|e_{2^k-2} - e_{2^k-3}\| + \cdots + \|e_{2^{k-1}+k+1} - e_{2^{k-1}+k}\| + \|e_{2^{k-1}+k}\| \\ &= 2^{k-1} - k.\end{aligned}$$

So $\sup_n \|T^n e_0\| = \infty$ and e_0 is an irregular vector for T. Now, let $p(z) = z - 1$. Then

$$\sup_n \|T^n p(T) e_0\| = \sup_n \|e_{n+1} - e_n\| \leq 1.$$

So $p(T)e_0$ is not irregular for T.

3 Existence of dense irregular manifolds and some special classes of operators

A vector subspace Y of X is said to be an *irregular manifold* for $T \in B(X)$ if every vector $y \in Y\backslash\{0\}$ is irregular for T [5]. Clearly, an irregular manifold for T is a scrambled set for T.

The next theorem gives us a useful sufficient condition for the existence of a dense irregular manifold. It extends Theorem 25 of [5] from Banach spaces to Fréchet spaces.

Theorem 20. Suppose that $T \in B(X)$ satisfies the following conditions:

(A) There is a dense subset X_0 of X such that $T^n x \to 0$ for all $x \in X_0$.

(B) There is a bounded sequence (a_n) in X such that the sequence $(T^n a_n)$ is unbounded.

Then T admits a dense irregular manifold.

Proof. By Lemma 13, there exists a series $\sum x_j$ of non-zero vectors in X such that

$$\sum_{j=1}^{\infty} \beta_j x_j$$

is an irregular vector for T, whenever (β_j) is a sequence of scalars that takes only finitely many values and has infinitely many non-zero coordinates. Moreover, it follows from the construction in Lemma 13 (and in Lemma 7) that the set of the sums of the subseries of $\sum x_j$ is bounded in X. By following the ideas from [5], let $N_1, N_2, N_3, \dots$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$ and, for each $m \in \mathbb{N}$, consider the sequence $\beta^{(m)} := (\beta_j^{(m)})_{j \in \mathbb{N}}$ where $\beta_j^{(m)} = 1$ if $j \in N_m$ and $\beta_j^{(m)} = 0$ otherwise. Put

$$y_m := \sum_{j=1}^{\infty} \beta_j^{(m)} x_j \qquad (m \in \mathbb{N}).$$

Let (v_m) be a dense sequence in X_0 and put

$$z_m := v_m + \frac{1}{m} y_m \qquad (m \in \mathbb{N}).$$

Since the sequence (y_m) is bounded in X, the sequence (z_m) is dense in X. Thus, $Y := \text{span}(\{z_m : m \in \mathbb{N}\})$ is a dense subspace of X. If $y \in Y\backslash\{0\}$ then we can write

$$y = v + \sum_{j=1}^{\infty} \beta_j x_j,$$

where $v \in \text{span}(X_0)$ and the sequence (β_j) takes only finitely many values and has infinitely many non-zero coordinates. Hence, $T^n v \to 0$ (by (A)) and $\sum_{j=1}^{\infty} \beta_j x_j$ is an irregular vector for T, which shows that y is an irregular vector for T. □

Note that condition (A) in the above theorem is automatically satisfied by any operator $T : X \to X$ whose generalized kernel $\bigcup_{n=1}^{\infty} \ker(T^n)$ is dense in X.

By combining Theorems 9 and 20, we obtain the following result.

Corollary 21. Let $T \in B(X)$ and suppose that

$$T^n x \to 0 \quad \text{for all } x \in X_0,$$

where X_0 is a dense subset of X. Then the following assertions are equivalent:

(i) T is Li-Yorke chaotic;

(ii) T admits a dense irregular manifold;

(iii) T admits an unbounded orbit.

Proof. (iii) $\Rightarrow$ (ii): By hypothesis, there exists a vector $y \in X$ such that $\mathrm{Orb}(y,T)$ is unbounded. Hence, by putting $a_n := y$ for every $n \in \mathbb{N}$, we see that condition (B) in Theorem 20 is satisfied. Since condition (A) is also satisfied, Theorem 20 ensures that (ii) holds.
(ii) $\Rightarrow$ (i): Obvious.
(i) $\Rightarrow$ (iii): By Theorem 9, T admits an irregular vector $y \in X$. Since the sequence $(T^n y)$ is unbounded by definition, the proof is complete. □

We shall now study the notion Li-Yorke chaos for some important special classes of operators. We begin with unilateral weighted backward shifts on Fréchet sequence spaces.

Theorem 22. Let Z be a Fréchet sequence space in which (e_n) is a basis (see [17], Section 4.1). Suppose that the unilateral weighted backward shift

$$B_w(x_1, x_2, x_3, \ldots) := (w_2 x_2, w_3 x_3, w_4 x_4, \ldots)$$

is an operator on Z. Then the following assertions are equivalent:

(i) B_w is Li-Yorke chaotic;

(ii) B_w admits a dense irregular manifold;

(iii) B_w admits an unbounded orbit.

Proof. This is just a special case of the previous corollary, since B_w has dense generalized kernel. □

Remark 23. A unilateral weighted backward shift (even on ℓ_2) can be Li-Yorke chaotic without being hypercyclic.

Indeed, let $Z := \ell_p$ $(1 \leq p < \infty)$ or $Z := c_0$, and consider a unilateral weighted backward shift B_w on Z. It was proved in [5] that B_w is Li-Yorke chaotic if and only if

$$\sup\{|w_n \cdots w_m| : n \in \mathbb{N}, m > n\} = \infty.$$

On the other hand, it is well-known that B_w is hypercyclic if and only if

$$\sup\{|w_1 \cdots w_n| : n \in \mathbb{N}\} = \infty$$

(Example 4.9(a) in [17]). From these characterizations it is easy to construct a weight sequence w such that B_w is Li-Yorke chaotic (hence admits a dense irregular manifold) but is not hypercyclic.

Remark 24. Every hypercyclic operator on X admits a dense irregular manifold.

Indeed, this is an immediate consequence of the Herrero-Bourdon theorem [10, 18], which asserts that every hypercyclic operator admits a dense invariant subspace with all non-zero vectors hypercyclic.

We now consider composition operators on spaces of holomorphic functions. Given a domain (= nonempty connected open set) Ω in the complex plane $\mathbb{C}$, we denote by $H(\Omega)$ the Fréchet space of all holomorphic functions $f : \Omega \to \mathbb{C}$ endowed with the compact-open topology. Recall that an *automorphism* of Ω is a bijective holomorphic function $\varphi : \Omega \to \Omega$. For each automorphism φ of Ω, the corresponding *composition operator* $C_\varphi : H(\Omega) \to H(\Omega)$ is defined by

$$C_\varphi f := f \circ \varphi.$$

It is clear that C_φ is a continuous linear operator on $H(\Omega)$.

Theorem 25. Let φ be an automorphism of a domain Ω in $\mathbb{C}$. For the composition operator C_φ on $H(\Omega)$, the following assertions are equivalent:

(i) C_φ is Li-Yorke chaotic;

(ii) C_φ admits a dense irregular manifold;

(iii) C_φ is hypercyclic.

Proof. We begin by proving that if C_φ is Li-Yorke chaotic, then (φ^n) is a run-away sequence, that is, for each compact set $K \subset \Omega$ there exists $n \in \mathbb{N}$ with

$$\varphi^n(K) \cap K = \emptyset.$$

Indeed, suppose that C_φ is Li-Yorke chaotic but (φ^n) is not a run-away sequence. Then, C_φ admits an irregular vector $f \in H(\Omega)$ (Theorem 9) and there is a compact set $K_1 \subset \Omega$ such that $\varphi^n(K_1) \cap K_1 \neq \emptyset$ for all $n \in \mathbb{N}$. Since the sequence $((C_\varphi)^n f)$ is unbounded, there is a compact set $K_2 \subset \Omega$ such that the sequence $(f \circ \varphi^n)$ is not uniformly bounded on K_2. Let K be a compact *connected* subset of Ω containing K_1 and K_2. Then,

$$\varphi^n(K) \cap K \neq \emptyset \ \text{ for all } n \in \mathbb{N}, \tag{15}$$

and

$$(f \circ \varphi^n) \ \text{ is not uniformly bounded on } K. \tag{16}$$

Since f is not identically zero, f has at most finitely many zeros in K. On the other hand, if f were zero-free on K, then it would follow from (15) that no subsequence of $(f \circ \varphi^n)$ could converge uniformly to 0 on K. This would contradict the fact that f is an irregular vector for C_φ. Thus, f must have at least one zero in K. So, let $z_1, \ldots, z_r$ be the zeros of f in K. By enlarging K a little bit, if necessary, we may also assume that

$$z_1, \ldots, z_r \in \overset{\circ}{K}\ .$$

Now we choose pairwise disjoint open sets $V_1, \ldots, V_r$ such that $z_j \in V_j \subset K$ for every $j \in \{1, \ldots, r\}$. Let $\epsilon := \min\{|f(z)| : z \in K \backslash (V_1 \cup \ldots \cup V_r)\} > 0$. There exists $m \in \mathbb{N}$ such that $\sup_{z \in K} |f(\varphi^m(z))| < \epsilon$. Therefore, $\varphi^m(K) \cap K \subset V_1 \cup \ldots \cup V_r$, which implies that $\varphi^m(K)$ intersects some V_j (because of (15)) and

$$\varphi^m(K) \subset V_1 \cup \ldots \cup V_r \cup (\Omega \backslash K).$$

By the connectedness of $\varphi^m(K)$, we conclude that there exists $i \in \{1, \dots, r\}$ such that $\varphi^m(K) \subset V_i$. In particular, $\varphi^m(K) \subset K$, and so

$$\varphi^n(K) \subset K \cup \varphi(K) \cup \ldots \cup \varphi^{m-1}(K) \quad \text{for all } n \in \mathbb{N}_0.$$

Since f is necessarily bounded on the compact set $K \cup \varphi(K) \cup \ldots \cup \varphi^{m-1}(K)$, this contradicts (16).

If Ω is simply connected (i.e., $\widehat{\mathbb{C}}\backslash\Omega$ is connected) or infinitely connected (i.e., $\widehat{\mathbb{C}}\backslash\Omega$ has infinitely many connected components), it was proved in [7] that C_φ is hypercyclic if and only if (φ^n) is a run-away sequence. Hence, in this case, C_φ Li-Yorke chaotic implies C_φ hypercyclic by what we have seen above.

It remains to consider the case where Ω is finitely connected (i.e., $\widehat{\mathbb{C}}\backslash\Omega$ has only finitely many connected components) but is not simply connected. In this case, we shall prove that there is no Li-Yorke chaotic composition operator on $H(\Omega)$. Indeed, it is known that there is no run-away sequence in Ω unless Ω is conformally equivalent to $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$ [7]. Hence, by what we have seen in the first paragraph, it is enough to consider the case $\Omega = \mathbb{C}^*$. Then φ is necessarily of the form $\varphi(z) = az$ with $|a| \neq 1$ [7]. Let $f \in H(\Omega)$ and suppose that the sequence $((C_\varphi)^n f)$ has a subsequence converging to 0 in $H(\Omega)$. Then there is an increasing sequence (n_j) of positive integers such that

$$f(a^{n_j} z) \to 0 \;\text{ uniformly on the unit circle } \mathbb{T}.$$

Therefore, it follows from the Maximum Modulus Theorem that

$$f(z) \to 0 \;\text{ as } |z| \to \infty \quad \text{ if } |a| > 1,$$

and

$$f(z) \to 0 \;\text{ as } |z| \to 0 \quad \text{ if } |a| < 1.$$

In both cases, we see that

$$f(a^n z) \to 0 \;\text{ uniformly on } K,$$

for every compact set $K \subset \Omega$, which means that $(C_\varphi)^n f \to 0$ in $H(\Omega)$. This proves that C_φ has no semi-irregular vector. Hence, C_φ is not Li-Yorke chaotic. □

Let us now consider multiplication operators on Hilbert spaces of holomorphic functions and their adjoints. We begin by recalling some terminology and some results. Let Ω be a domain in $\mathbb{C}^N$. Assume that $H \neq \{0\}$ is a Hilbert space of holomorphic functions on Ω such that each point evaluation $f \to f(z)$, $z \in \Omega$, is a continuous linear functional on H. A complex valued continuous function φ on Ω is called a *multiplier* of H if $\varphi f \in H$ whenever $f \in H$, where φf denotes the pointwise product of φ by f. Each multiplier φ of H determines a *multiplication operator* $M_\varphi : H \to H$ by the formula

$$M_\varphi f := \varphi f.$$

It follows from the Closed Graph Theorem that M_φ is a continuous linear operator on H. Every multiplier φ of H is a bounded holomorphic function on Ω and

$$\|\varphi\|_\infty := \sup_{z\in\Omega} |\varphi(z)| \leq \|M_\varphi\|$$

(Proposition 4.4 in [15]). The (Hilbert space) adjoint M_φ^* of M_φ is called an *adjoint multiplication operator*. Godefroy and Shapiro proved that

$$M_\varphi^* \text{ is hypercyclic whenever } \varphi \text{ is nonconstant and } \varphi(\Omega) \cap \mathbb{T} \neq \emptyset$$

(Theorem 4.5 in [15]). Moreover, they also proved that the converse is true if H satisfies the following additional hypothesis:

(P) Every bounded holomorphic function φ on Ω is a multiplier of H and $\|\varphi\|_\infty = \|M_\varphi\|$.

We observe that many Hilbert spaces of holomorphic functions satisfy this additional hypothesis, like the Bergman space of a bounded domain and the Hardy space H^2 of either the unit ball (Chap. 5 in [33]) or the unit polydisc (Chap. 3 in [32]).

Theorem 26. Assume H satisfies property (P). Then:

(a) No multiplication operator M_φ on H is Li-Yorke chaotic.

(b) For an adjoint multiplication operator M_φ^* on H, the following assertions are equivalent:

  (i) M_φ^* is Li-Yorke chaotic;

  (ii) M_φ^* admits a dense irregular manifold;

  (iii) M_φ^* is hypercyclic.

Proof. (a) Suppose that a multiplication operator M_φ on H is Li-Yorke chaotic. By Theorem 9, M_φ admits an irregular vector $f \in H$. Let (n_j) be an increasing sequence of positive integers such that $(M_\varphi)^{n_j} f \to 0$ in H. Then, by the continuity of point evaluations,

$$\varphi(z)^{n_j} f(z) \to 0 \quad \text{for all } z \in \Omega.$$

Hence, $|\varphi(z)| < 1$ for all $z \in \Omega \backslash Z_f$, where Z_f denotes the zero-set of f. Since f is not identically zero, Z_f is a discrete subset of Ω. Thus,

$$|\varphi(z)| \leq 1 \quad \text{for all } z \in \Omega.$$

This implies that

$$\|(M_\varphi)^n f\| \leq \|M_\varphi\|^n \|f\| = \|\varphi\|_\infty^n \|f\| \leq \|f\| \quad \text{for all } n \in \mathbb{N},$$

which contradicts the fact that f is an irregular vector for M_φ.

(b) Suppose that an adjoint multiplication operator M_φ^* on H is Li-Yorke chaotic. We shall prove that it is hypercyclic. In view of the above-mentioned result from [15], it is enough to show that $\varphi(\Omega)$ intersects the unit circle. Suppose that this is not the case. Then, either $\varphi(\Omega) \subset \mathbb{D}$ or $\varphi(\Omega) \subset \mathbb{C} \backslash \overline{\mathbb{D}}$. If $\varphi(\Omega) \subset \mathbb{D}$ then

$$\|M_\varphi^*\| = \|M_\varphi\| = \|\varphi\|_\infty \leq 1,$$

and so M_φ^* does not admit an irregular vector. Assume $\varphi(\Omega) \subset \mathbb{C} \backslash \overline{\mathbb{D}}$. Then $\psi := 1/\varphi$ defines a holomorphic function on Ω with $\psi(\Omega) \subset \mathbb{D}$, so that $\|M_\psi^*\| \leq 1$. Hence,

$$\|f\| = \|(M_\psi^*)^n (M_\varphi^*)^n f\| \leq \|(M_\varphi^*)^n f\| \quad (f \in H, n \in \mathbb{N}),$$

which also shows that M_φ^* does not admit an irregular vector. By Theorem 9, in both cases we have a contradiction. □

4 Existence of dense irregular manifolds in the case of Banach spaces

Our goal in the present section is to establish a sufficient criterion for the existence of a dense irregular manifold that improves Theorem 20 in the case X is a complex Banach space. Before, we need some preliminary results.

Throughout this section we assume that X is a complex Banach space.

The following result is a direct consequence of the Banach-Steinhaus theorem.

Proposition 27. Let $T \in B(X)$. The following statements are equivalent:

(i) $\sup_n \|T^n\| = \infty$;

(ii) there exists $x \in X$ such that $\sup_n \|T^n x\| = \infty$;

(iii) the set of all $x \in X$ satisfying $\sup_n \|T^n x\| = \infty$ is residual in X.

Given $T \in B(X)$, note that the set of all vectors with bounded orbits under T is a linear manifold.

Lemma 28. Let $T \in B(X)$ and $M := \{x \in X : \sup_n \|T^n x\| < \infty\}$. If $\operatorname{codim} M < \infty$ then M is closed.

Proof. Define a new norm $|||\cdot|||$ on M by $|||x||| := \sup_n \|T^n x\|$. Clearly $|||x||| \geq \|x\|$ for all $x \in M$.

We show that $(M, |||\cdot|||)$ is complete. Let $(x_k) \subset M$ be a $|||\cdot|||$-Cauchy sequence. So it is bounded and there exists a constant $K > 0$ such that $|||x_k||| \leq K$ for all k, that is, $\|T^n x_k\| \leq K$ for all k, n. The sequence (x_k) is Cauchy in the norm $\|\cdot\|$, so there exists $x \in X$ such that $\|x_k - x\| \to 0$. For each n we have $\lim_{k\to\infty} \|T^n x_k - T^n x\| = 0$, so $\|T^n x\| \leq K$ for all n. Hence $x \in M$.

Since (x_k) is $|||\cdot|||$-Cauchy, for each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\sup_n \|T^n x_k - T^n x_j\| \leq \varepsilon$$

for all $k, j \geq k_0$. Hence

$$\sup_n \|T^n x_k - T^n x\| \leq \varepsilon$$

for all $k \geq k_0$, and so $|||x_k - x||| \to 0$. Thus $(M, |||\cdot|||)$ is a Banach space and the identical mapping $(M, |||\cdot|||) \to (X, \|\cdot\|)$ is continuous. Hence M is an operator range. Since $\operatorname{codim} M < \infty$, it is a closed subspace. □

Lemma 29. Let $T \in B(X)$ satisfy $\sup_n \|T^n\| = \infty$ and suppose that there exists a dense subset X_0 of X such that $\inf_n \|T^n x\| = 0$ for each $x \in X_0$. Then

$$\operatorname{codim}\{x \in X : \sup_n \|T^n x\| < \infty\} = \infty.$$

Proof. Write $M := \{x \in X : \sup_n \|T^n x\| < \infty\}$. Suppose on the contrary that $\operatorname{codim} M < \infty$. Then M is closed and $T|M$ is power bounded. Let $K := \sup_n \|T^n|M\|$. Let $F \subset X$ be a finite-dimensional subspace such that $X = M \oplus F$. Let Q be the projection on F with $\ker Q = M$. In the decomposition $X = M \oplus F$ we can write

$$T = \begin{pmatrix} T_M & S \\ 0 & T_F \end{pmatrix}.$$

For each $u \in X_0$, we have $\inf_n \|T_F^n Qu\| = \inf_n \|QT^n u\| = 0$. Since $\overline{QX_0} = F$, we have $r(T_F) < 1$, and so $\sum_n \|T_F^n\| < \infty$. Moreover, for $f \in F$ we have

$$\begin{aligned} \sup_n \|T^n f\| &= \sup_n \left\| T_F^n f + \sum_{k=0}^{n-1} T_M^{n-k-1} S T_F^k f \right\| \\ &\le \sup_n \left(\|T_F^n f\| + K\|S\| \cdot \|f\| \sum_{k=0}^{n-1} \|T_F^k\| \right) < \infty. \end{aligned}$$

Hence $F \subset M$ and $M = X$, a contradiction with the assumption that T is not power bounded. □

Lemma 30. Let $T \in B(X)$ satisfy $\sup_n \|T^n\| = \infty$ and suppose that there exists a dense subset X_0 of X and a sequence (n_s) in $\mathbb{N}$ with $\lim_{s\to\infty} \|T^{n_s} x\| = 0$ for all $x \in X_0$. Let $k \in \mathbb{N}$, $u_1, \dots, u_k \in X$, $\varepsilon > 0$, $K > 0$. Then there exist $m \in \mathbb{N}$ and $w_1, \dots, w_k \in X$ such that $\|w_i - u_i\| < \varepsilon$ $(i = 1, \dots, k)$, and for every $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ with $\sum_{i=1}^k |\alpha_i| = 1$,

$$\inf_{1\le j\le m} \left\| T^j \sum_{i=1}^k \alpha_i w_i \right\| < \frac{1}{K} \qquad \text{and} \qquad \sup_{1\le j\le m} \left\| T^j \sum_{i=1}^k \alpha_i w_i \right\| > K.$$

Proof. For $i = 1, \dots, k$ find $v_i \in X_0$ such that $\|v_i - u_i\| < \varepsilon/2$. Since $\lim_{s\to\infty} \|T^{n_s} x\| = 0$ for all $x \in X_0$, there exists $t \in \mathbb{N}$ such that $\|T^{n_t} v_i\| < \frac{1}{2K}$ for $i = 1, \dots, k$.

As above, let $M := \{x \in X : \sup_n \|T^n x\| < \infty\}$. Let M_0 be the linear manifold spanned by $M \cup \{v_1, \dots, v_k\}$. Since $\operatorname{codim} M_0 = \infty$, there are elements $z_1, \dots, z_k \in X$ which are linearly independent modulo M_0. We may assume that $\|z_i\| < \min\{\frac{\varepsilon}{2}, \frac{1}{2K\|T\|^{n_t}}\}$. Let $w_i := v_i + z_i$. Then $\|w_i - u_i\| \le \|w_i - v_i\| + \|v_i - u_i\| < \varepsilon$.

If $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ and $\sum_{i=1}^k |\alpha_i| = 1$, then

$$\begin{aligned} \left\| T^{n_t} \sum_{i=1}^k \alpha_i w_i \right\| &\le \left(\sum_{i=1}^k \|\alpha_i T^{n_t} v_i\| + \sum_{i=1}^k \|\alpha_i T^{n_t} z_i\| \right) \\ &< \frac{1}{2K} + \|T\|^{n_t} \max\{\|z_i\| : i = 1, \dots, k\} \le \frac{1}{K}. \end{aligned}$$

Moreover, for each n-tuple $(\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$ with $\sum_{i=1}^k |\alpha_i| = 1$, we have $\sum_{i=1}^k \alpha_i w_i \notin M$, so there exists $n_\alpha \in \mathbb{N}$ such that $\|T^{n_\alpha} \sum_{i=1}^k \alpha_i w_i\| > K$. By a compactness argument, there exists $m > n_t$ such that

$$\sup_{1\le j\le m} \left\| T^j \sum_{i=1}^k \alpha_i w_i \right\| > K$$

for all $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ with $\sum_{i=1}^k |\alpha_i| = 1$. □

Theorem 31. Suppose that $T \in B(X)$ satisfies the following conditions:

(A) There are a dense subset X_0 of X and a sequence (n_s) in $\mathbb{N}$ with $\lim_{s\to\infty} \|T^{n_s} x\| = 0$ for all $x \in X_0$.

(B) $\sup_n \|T^n\| = \infty$.

Then T admits a dense irregular manifold.

Proof. Fix a countable dense subset $\{y_i : i \in \mathbb{N}\}$ of X. We construct an increasing sequence $(m_k) \subset \mathbb{N}$ and vectors $v_{i,k} \in X$, $i \le k$, such that

$$v_{i,i} = y_i,$$

$$\|v_{i,k+1} - v_{i,k}\| < \frac{1}{2^{k+2}\|T\|^{m_k}},$$

$$\inf_{1\le j\le m_k} \left\| T^j \sum_{i=1}^{k-1} \alpha_i v_{i,k} \right\| < \frac{1}{2^{k+1}},$$

$$\sup_{1\le j\le m_k} \left\| T^j \sum_{i=1}^{k-1} \alpha_i v_{i,k} \right\| > 2^{k+1},$$

for all $\alpha_1, \dots, \alpha_{k-1} \in \mathbb{C}$ with $\sum_{i=1}^{k-1} |\alpha_i| = 1$.

Let $k \ge 1$ and suppose that the numbers $m_1 < m_2 < \cdots < m_k$ and the vectors $v_{i,k}$ for $i \le k$ have already been constructed. By the previous lemma, there exist $m_{k+1} > m_k$ and vectors $v_{1,k+1}, \dots, v_{k,k+1} \in X$ such that

$$\|v_{i,k+1} - v_{i,k}\| < \frac{1}{2^{k+2}\|T\|^{m_k}},$$

and

$$\inf_{1\le j\le m_{k+1}} \left\| T^j \sum_{i=1}^{k} \alpha_i v_{i,k+1} \right\| < \frac{1}{2^{k+2}},$$

$$\sup_{1\le j\le m_{k+1}} \left\| T^j \sum_{i=1}^{k} \alpha_i v_{i,k+1} \right\| > 2^{k+2},$$

for every $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ with $\sum_{i=1}^{k} |\alpha_i| = 1$.

Suppose we have constructed the numbers $m_1 < m_2 < \cdots$ and the vectors $v_{i,k}$ in this way. For each $i \in \mathbb{N}$, let $v_i := \lim_{k\to\infty} v_{i,k}$. For each $k \ge i$, we have

$$\|v_{i,k} - v_i\| \le \sum_{j=k}^{\infty} \|v_{i,j} - v_{i,j+1}\| \le \sum_{j=k}^{\infty} \frac{1}{2^{j+2}\|T\|^{m_j}} \le \frac{1}{2^{k+1}\|T\|^{m_k}}$$

(note that $\|T\| > 1$). In particular, $\|y_i - v_i\| = \|v_{i,i} - v_i\| \le \frac{1}{2^{i+1}\|T\|^{m_i}} \le 2^{-i-1}$. Hence (v_i) is a dense sequence in X.

Let $k \in \mathbb{N}$, $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, $\sum_{i=1}^{k} |\alpha_i| = 1$. For $s > k$ we have

$$\sup_{1\le j\le m_s} \left\| T^j \sum_{i=1}^{k} \alpha_i v_i \right\| \ge \sup_{1\le j\le m_s} \left\| T^j \sum_{i=1}^{k} \alpha_i v_{i,s} \right\| - \|T\|^{m_s} \max_{1\le i\le k} \|v_i - v_{i,s}\| \ge 2^{s+1} - 1 > 2^s.$$

Similarly we have

$$\inf_{1\le j\le m_s} \left\| T^j \sum_{i=1}^{k} \alpha_i v_i \right\| \le \inf_{1\le j\le m_s} \left\| T^j \sum_{i=1}^{k} \alpha_i v_{i,s} \right\| + \|T\|^{m_s} \max_{1\le i\le k} \|v_i - v_{i,s}\| \le 2^{-s-1} + 2^{-s-1} = 2^{-s}.$$

So $\sum_{i=1}^{k} \alpha_i v_i$ is an irregular vector for T. Hence the linear manifold generated by the vectors $v_1, v_2, \dots$ consists (up to 0) of irregular vectors for T. $\square$

Corollary 32. If $T \in B(X)$ satisfies $\sup_n \|T^n\| = \infty$ and there exists a cyclic vector x for T with $\inf_n \|T^n x\| = 0$, then T admits a dense irregular manifold.

Corollary 33. Let $T \in B(X)$ be such that there exist a dense subset X_0 of X and a sequence (n_s) in $\mathbb{N}$ with $\lim_{s\to\infty} \|T^{n_s}x\| = 0$ for all $x \in X_0$. Then the following assertions are equivalent:

(i) T is Li-Yorke chaotic;

(ii) T admits a dense irregular manifold;

(iii) $\sup_n \|T^n\| = \infty$.

5 Generic Li-Yorke chaos

We now establish some characterizations of generic Li-Yorke chaos.

Theorem 34. If $T \in B(X)$ then the following assertions are equivalent:

(i) T is generically Li-Yorke chaotic;

(ii) Every non-zero vector is semi-irregular for T;

(iii) X is a scrambled set for T.

Proof. Clearly, (ii) $\Leftrightarrow$ (iii) and (iii) $\Rightarrow$ (i). So, it remains to prove that (i) $\Rightarrow$ (ii). For this purpose, fix a non-zero vector $x \in X$. By hypothesis, there is a residual scrambled set S for T. Since both S and $x + S$ are residual sets in X, their intersection $S \cap (x + S)$ is also a residual set in X and, in particular, is nonempty. Hence, there are vectors $a, b \in S$ such that $a = x + b$. Since $x \neq 0$, we have that $a \neq b$. Thus, (a, b) is a Li-Yorke pair for T, and so the vector $x = a - b$ is semi-irregular for T. □

Corollary 35. The following classes of operators contain no generically Li-Yorke chaotic operator: unilateral weighted backward shifts on Fréchet sequence spaces, composition operators on the Fréchet spaces $H(\Omega)$ (Ω a domain in $\mathbb{C}$), adjoint multiplication operators on Hilbert spaces of holomorphic functions.

Also, for generically Li-Yorke chaotic operators on Banach spaces, some conditions that involve the spectrum of the operator must be satisfied.

Remark 36. As consequence of Theorem 34, it is not difficult to show that any generically Li-Yorke chaotic operator $T : X \to X$ on a Banach space X is so that every component of its spectrum must intersect the unit circle, the spectral radius is 1, and it does not have eigenvalues (the proof mimics [31, Section 4]).

A hypercyclic operator need not be generically Li-Yorke chaotic. In fact, there are many ways to see this. First, there are examples of hypercyclic operators (like certain unilateral weighted backward shifts) that admit non-zero vectors with orbits tending to zero. Second, every infinite-dimensional separable Fréchet space supports a hypercyclic operator with a non-trivial fixed point. And third, there exist Devaney chaotic operators. All these operators fail condition (ii) of Theorem 34, and therefore are not generically Li-Yorke chaotic.

By modifying an example due to Beauzamy [4], Prăjitură [31] obtained an example of a continuous linear operator on ℓ_2 with all non-zero vectors irregular, but none of them hypercyclic. In particular, this operator is generically Li-Yorke chaotic (by Theorem 34) but not hypercyclic. By slightly modifying this example, we shall show that we cannot include the phrase "Every non-zero vector is irregular for T" in Theorem 34.

Theorem 37. There is a generically Li-Yorke chaotic operator $S : \ell_2 \to \ell_2$ which admits a dense set of non-irregular vectors.

Proof. In the proof of Theorem 3.13 of [31], it was constructed a sequence $(w_j)_{j\in\mathbb{N}}$ of weights such that the unilateral weighted forward shift

$$T : (a_1, a_2, a_3, \dots) \in \ell_2 \mapsto (0, w_1 a_1, w_2 a_2, w_3 a_3, \dots) \in \ell_2$$

has the property that all non-zero vectors $x \in \ell_2$ are irregular for T. Moreover, the weights w_j satisfy

$$\frac{1}{2} \leq w_j \leq 2 \quad \text{for all } j \in \mathbb{N}$$

and

$$\limsup_{n\to\infty} \prod_{j=1}^{n} w_j = \infty. \tag{17}$$

We shall construct a new sequence $(w'_j)_{j\in\mathbb{N}}$ of weights in the following way. By (17), there is a smallest positive integer r_1 such that

$$\prod_{j=1}^{r_1} w_j > 2.$$

Put $w'_j := w_j$ for $1 \leq j < r_1$ and $w'_{r_1} := w_{r_1}/(\prod_{j=1}^{r_1} w_j)$. Now, again by (17), there is a smallest positive integer $r_2 > r_1$ such that

$$\prod_{j=r_1+1}^{r_2} w_j > 2.$$

Put $w'_j := w_j$ for $r_1 < j < r_2$ and $w'_{r_2} := w_{r_2}/(\prod_{j=r_1+1}^{r_2} w_j)$. By continuing in this way, we obtain our sequence (w'_j) of weights together with an increasing sequence (r_n) of positive integers so that

$$w'_j \leq w_j \quad \text{for every } j \in \mathbb{N}, \tag{18}$$

$$\prod_{j=1}^{n} w'_j \leq 2 \quad \text{for every } n \in \mathbb{N}, \text{ and} \tag{19}$$

$$\prod_{j=1}^{r_n} w'_j = 1 \quad \text{for every } n \in \mathbb{N}. \tag{20}$$

Let $S : \ell_2 \to \ell_2$ be the unilateral weighted forward shift of weights (w'_j). It follows from (18) that $\|S^n x\| \leq \|T^n x\|$ for all $x \in \ell_2$ and all $n \in \mathbb{N}$. Hence,

$$\liminf_{n\to\infty} \|S^n x\| = 0 \quad \text{for every } x \in \ell_2.$$

On the other hand, by (19) and (20),

$$1 \leq \limsup_{n\to\infty} \|S^n e_1\| \leq 2, \tag{21}$$

which implies that

$$\limsup_{n\to\infty} \|S^n x\| > 0 \quad \text{for every } x \in \ell_2\backslash\{0\}.$$

Hence, every non-zero vector is semi-irregular for S. In view of Theorem 34, this means that S is generically Li-Yorke chaotic. Finally, it follows from (21) that every finitely supported element of ℓ_2 is not irregular for S. □

The next theorem extends Proposition 2.4 of [31] to Fréchet spaces.

Theorem 38. If $T \in B(X)$ and $x \in X$, then the following assertions are equivalent:

(i) x is a semi-irregular (resp. an irregular) vector for T;

(ii) x is a semi-irregular (resp. an irregular) vector for T^p for some $p \in \mathbb{N}$;

(iii) x is a semi-irregular (resp. an irregular) vector for T^p for every $p \in \mathbb{N}$.

Proof. Obviously, (iii) implies (ii) and (ii) implies (i). Let us prove that (i) implies (iii). Fix $p \in \mathbb{N}$. Since

$$\{T^n x : n \in \mathbb{N}\} = \bigcup_{r=0}^{p-1} T^r(\{T^{pq}x : q \in \mathbb{N}\})$$

and $(T^n x)$ does not converge to zero (resp. is unbounded), it follows that $((T^p)^n x)$ does not converge to zero (resp. is unbounded). On the other hand, there is a subsequence $(T^{n_k}x)$ which converges to zero. Write $n_k = pq_k + r_k$ with $q_k \in \mathbb{N}_0$ and $r \in \{0, \ldots, p-1\}$. By passing to a subsequence, if necessary, we may assume $r_k = r$ for every $k \in \mathbb{N}$. Hence,

$$(T^p)^{(q_k+1)}x = T^{p-r}T^{n_k}x \to 0,$$

which completes the proof. □

As a consequence we have the following result.

Corollary 39. If $T \in B(X)$ then the following assertions are equivalent:

(i) T is chaotic;

(ii) T^p is chaotic for some $p \in \mathbb{N}$;

(iii) T^p is chaotic for every $p \in \mathbb{N}$;

where by chaotic we mean any of the following notions of chaos: Li-Yorke chaos, dense Li-Yorke chaos and generic Li-Yorke chaos.

6 Final comments and open problems

We finish the paper with a brief summary of some of the results presented here and with some open problems. Consider the following groups of properties concerning a continuous linear operator T on X:

(P1) Every non-zero vector is irregular for T.

(P2) T is generically Li-Yorke chaotic.
Every non-zero vector is semi-irregular for T.
X is a scrambled set for T.

(P3) T admits a dense irregular manifold.

(P4) T is densely Li-Yorke chaotic.
T is densely w-Li-Yorke chaotic.
T is generically w-Li-Yorke chaotic.
T admits a dense set of semi-irregular vectors.
T admits a dense set of irregular vectors.
T admits a residual set of irregular vectors.

(P5) T is Li-Yorke chaotic.
T admits a Li-Yorke pair.
T admits a semi-irregular vector.
T admits an irregular vector.

(HC) T is hypercyclic.

Operators satisfying (P1) are called *completely irregular* [31]. We have seen that all sentences in (P2) (resp. in (P4), in (P5)) are equivalent to each other. Moreover,

$$\text{(P1)} \Rightarrow \text{(P2)} \Rightarrow \text{(P3)} \Rightarrow \text{(P4)} \Rightarrow \text{(P5)} \quad \text{and} \quad \text{(HC)} \Rightarrow \text{(P3)}.$$

We don't know if (P4) implies (P3), but we know that all the others reverse implications are false. Finally, (P1) does not imply (HC) (so that none of the properties (P1)–(P5) implies (HC)) and (HC) does not imply (P2) (so that it also does not imply (P1)). In order to complete the picture here, it remains to answer the following basic open problem.

Problem 1. Does dense Li-Yorke chaos imply the existence of a dense irregular manifold for operators on Fréchet (or Banach) spaces?

It is well-known that every infinite-dimensional separable Fréchet space supports a hypercyclic operator [1, 6, 9]. In particular, it supports an operator with a dense irregular manifold (Remark 24). This suggests the following question.

Problem 2. Does every infinite-dimensional separable Fréchet (or Banach) space support a generically Li-Yorke chaotic operator?

We have presented several characterizations of Li-Yorke chaos, dense Li-Yorke chaos and generic Li-Yorke chaos (Theorems 9, 10, 15, 17 and 34), but only two sufficient (yet useful) conditions for the existence of a dense irregular manifold (Theorems 20 and 31). This suggests the following problem.

Problem 3. To find useful characterizations for the existence of a dense irregular manifold.

We close the paper by remarking that the separability assumption on X was used only in the proof of (vi) $\Rightarrow$ (i) in Theorem 10, in Theorems 20 and 31, and in Corollaries 21, 32 and 33. All the remaining results are true for non-separable Fréchet spaces, but we should replace "dense Li-Yorke chaos" by "dense w-Li-Yorke chaos" in Proposition 11 and in Theorem 17. In particular, Theorem 17 becomes a Dense w-Li-Yorke Chaos Criterion.

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