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“LARGE” WEAK ORBITS OF C_0-SEMIGROUPS

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Abstract. We show the existence of ”large” weak orbits of C_0-semigroups with generators satisfying natural spectral assumptions. We give also certain applications of our results to harmonic analysis and discuss related results.

1. Introduction and preliminaries

It is a well-known fact that the Fourier transform of an integrable function may decay arbitrarily slowly at infinity. More precisely if $f : \mathbb{R} \to [0, \infty)$ is any function going to zero at infinity, then there exists $f \in L^1(\mathbb{R})$ such that its Fourier transform $\widehat{g}$ satisfies

$$|\widehat{g}(\xi)| \geqslant g(\xi), \qquad \xi \in \mathbb{R}. \tag{1.1}$$

Thus, loosely speaking, Fourier transforms of integrable functions are order dense in $C_0(\mathbb{R})$, and the only restriction on the size of the Fourier transforms is imposed by the Riemann-Lebesgue Lemma. This fact underlines the heuristic principle that the Fourier transform is as large as it is allowed to be by very basic constraints. For other illustrations of the principle see e.g. the survey paper [5]. Note that if μ is a nonatomic finite measure on $\mathbb{R}$, then by classical Wiener's theorem (see e.g. [23], and [22, Theorem 5.4] for a simple proof)

$$\frac{1}{T}\int_0^T |\widehat{\mu}(\xi)|^2 \, d\xi \to 0, \qquad T \to \infty,$$

which is equivalent to

$$\widehat{\mu}(\xi) \to 0 \quad \text{as} \quad |\xi| \to \infty \tag{1.2}$$

in density, that is (1.2) holds for ξ from a subset of $\mathbb{R}$ of density one. In this situation we are not aware of results similar to (1.1) and we show below that one can derive certain analogues of (1.1) for equivalence classes of a fixed measure in the sense of density. (An interplay between Wiener's theorem and semigroup theory was studied in [3], see also [4],[12].)

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The above issues of decay of Fourier transforms can be put in a more general setting of orbits of C_0-semigroups. It suffices to note that if $(M(t))_{t\geqslant 0}$ is a multiplication C_0-semigroup on $H = L^2(d\mu)$ given by

$$(M(t)g)(s) = e^{-its}g(s), \qquad t \geqslant 0, s \in \mathbb{R}, \tag{1.3}$$

and $\langle\cdot,\cdot\rangle$ is the inner product on H, then, for any $f, g \in H$, the weak orbit $\langle M(t)f, g\rangle$ of the semigroup M is the Fourier transform of the function $f\overline{g} \in L^1(d\mu)$.

In this paper, we treat weak orbits of C_0-semigroups $(T(t))_{t\geqslant 0}$ on Banach and Hilbert spaces. In particular, in the Hilbert space case, we show that if T converges to zero in the weak operator topology then, under natural spectral assumptions, its weak orbits are as large as it is possible, that is any function going to zero at infinity can be dominated by a weak orbit of the semigroup. If $(T(t))_{t\geqslant 0}$ is merely bounded then its weak orbits are large when taken along large subsets of $\mathbb{R}_+$, e.g. those with density one. We give also a version of these results for semigroups acting on reflexive Banach spaces. Moreover, in most cases we are able to find weak orbits starting at comparatively small set of smooth vectors of the semigroup generator. Thus, in particular if $(T(t))_{t\geqslant 0}$ is weakly (or strongly) stable and the peripheral spectrum of the generator is non-empty, then decay of the semigroup to zero is far from being uniform in several natural senses. This fact is of value for the study of decay properties of solutions to abstract Cauchy problems.

Similar results in the setting of discrete semigroups $(T^n)_{n\in N}$ were obtained in [2]. However, the case of C_0-semigroups contains several additional difficulties with respect to [2]. One of them is that our argument depends on the semigroup spectral structure and the spectral mapping theorem for C_0-semigroups does not, in general, hold. Thus, the study of the spectrum of a semigroup in terms of the spectrum of its generator requires new tools. Moreover, we consider large weak orbits in the sense of density, and this issue was not addressed in [2]. Finally, we also treat smooth orbits and the notion of smooth vectors seems to be meaningless in the case of discrete semigroups.

Note that the results close in spirit were obtained in [17], [18], [20] and [21]. The paper [21] is the closest to our considerations and shows the existence of large weak semigroup orbits under assumptions similar to those in this paper, see e.g. [21, Theorem 1]. However, the notions of an orbit being large in [21] and in the present article differ. While we are interested in domination of a fixed function by a weak semigroup orbit, [21] claims only the existence of a large set where the weak orbit is large. Thus, in several cases, our results give much more precise information on weak orbits. On the other hand, the main result of [21] cannot be deduced from our results (nor our results can be obtained from [21]) but it rather can be reproved by our arguments.

Our main results are contained in Section 5 and Section 6. Several straightforward applications of our semigroup results to the study of Fourier transforms are given in Section 7. Finally, possible generalizations and related matters are discussed in Section 8.

Our technique stems from [15]. It has been developed subsequently in [2], [13]-[15]. For its detailed exposition see [16, Chapter V].

Several facts from the theory of C_0-semigroups will be crucial for us. Let X be a Banach space. For a C_0-semigroup $(T(t))_{t\geqslant 0}$ on X with generator A define the semigroup exponential growth bound by

$$\omega_0(T) := \lim_{t\to\infty} \frac{\ln\|T(t)\|}{t},$$

the spectral bound by

$$s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\},$$

and pseudo-spectral bound (or abscissa of uniform boundedness of the resolvent of A) by

$$s_0(A) := \inf\{\omega > s(A) : R(\lambda, A) \text{ is uniformly bounded for } \operatorname{Re}\lambda \geqslant \omega\}.$$

It is well known (and easy to prove) that

$$s(A) \leqslant s_0(A) \leqslant \omega_0(T). \tag{1.4}$$

There are various examples of C_0-semigroups such that the above inequalities are strict, see e.g. [1, Chapter 5.1], [10, Chapter 5.1] or [19, Chapter 1]. However, if X is a Hilbert space, then the Gearhart-Herbst-Pruess theorem guarantees that

$$\omega_0(T) = s_0(A), \tag{1.5}$$

thus the resolvent of A determines the exponential growth of T. For a discussion of relations between the three bounds see e.g. [1, Chapters 5.1-5.3] or [19, Chapters 1-4].

Another useful fact concerns weak convergence of bounded C_0-semigroups. Recall that if X is a separable reflexive Banach space, and $(T(t))_{t\geqslant 0}$ is a bounded C_0-semigroup on X, with generator A such that the point spectrum of A on $i\mathbb{R}$ is empty, then T almost weakly converges to zero in the sense that for any $x \in X$ and $x^* \in X^*$

$$\langle T(t)x, x^*\rangle \to 0, \qquad t \to \infty, \tag{1.6}$$

for t from a subset of $\mathbb{R}_+$ of density 1, see e.g. [8] and [9] for more information. This property will help us to deal with semigroups which do not necessarily converge in any sense.

2. Notations

All Banach and Hilbert spaces in this paper will be *complex*. For a linear operator A we denote by $\sigma(A), \partial\sigma(A), \sigma_\pi(A), \sigma_p(A)$, and $\rho(A)$ the spectrum, the topological boundary of the spectrum, the approximate spectrum, the point spectrum, and the resolvent set of A, respectively. Moreover, $D(A), \operatorname{Im} A$, and $\operatorname{Ker} A$ will stand for the domain, the range and the kernel of A. If X is a Banach space, then by $\mathcal{L}(X)$ we denote the Banach algebra of all bounded linear operators on X. Let also $C_0(\mathbb{R})$ stand for the spaces of continuous functions on $\mathbb{R}$ vanishing at $\pm\infty$, and denote by $C^\infty(\mathbb{R})$ the set of smooth functions on $\mathbb{R}$.

Let m stand for the Lebesgue measure on the real line.

3. Smooth approximate eigenvectors for closed operators

First we need several spectral properties of semigroup generators restricted to their sets of smooth vectors. We put our presentation in a little bit more general framework of closed linear operators on Banach spaces since we think the results will also be of use in other instances as well.

Let X be a Banach space. Let $A : D(A) \subset X \to X$ be a densely defined linear operator with $\rho(A) \neq \emptyset$. Note that A is then automatically closed. For $k \in \mathbb{N}$ consider the norm $\|\cdot\|_k$ defined on $D(A^k)$ by

$$\|x\|_k := \|x\| + \|Ax\| + \cdots + \|A^k x\|.$$

Then $(D(A^k), \|\cdot\|_k)$ is a Banach space. Define

$$C^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n).$$

The (Frechet) space $C^\infty(A)$ is called the set of smooth (or infinitely differentiable) vectors of A. If A generates a C_0-semigroup $(T(t))_{t\geqslant 0}$, then $C^\infty(A)$ is precisely the set of elements $x \in X$ such that the orbit $T(\cdot)x$ is infinitely differentiable on $[0,\infty)$.

The following lemma is well-known and can be found e.g. in [24, Corollary 3.3].

Lemma 3.1. *For every $k \in \mathbb{N}$ the set $C^\infty(A)$ is dense in $(D(A^k), \|\cdot\|_k)$.*

We start with obtaining a $C^\infty(A)$-version of a well-known result on approximate eigenvectors of linear operators. To this aim we need several auxiliary facts.

Lemma 3.2. *Let $\lambda \in \partial\sigma(A)$. Then $A - \lambda$ is not onto.*

Proof. Without loss of generality we may assume that $\lambda = 0$. Fix $\mu \in \rho(A)$ and consider $T := A(A-\mu)^{-1}$. Then $T = I + \mu(A-\mu)^{-1}$ is a bounded linear operator. By the spectral mapping theorem for resolvents (see e.g. [10, Chapter IV.1.13]), we have $\sigma(T) \setminus \{1\} = \{1 + \frac{\mu}{z-\mu} : z \in \sigma(A)\}$. Thus $0 \in \partial\sigma(T)$, and so T is not onto. Since $\operatorname{Im} T = \operatorname{Im} A$, the operator A is not onto as well. □

Lemma 3.3. *Let $\lambda \in \partial\sigma(A)$ and let $\{\lambda_n : n \geqslant 1\} \subset \rho(A)$ be such that $\lambda_n \to \lambda, n \to \infty$. For a fixed $k \in \mathbb{N}$ consider the operators $(A-\lambda_n)^{-1}, n \in \mathbb{N}$, acting on the space $(D(A^k), \|\cdot\|_k)$. Then*

$$\lim_{n\to\infty} \|(A-\lambda_n)^{-1}\|_k = \infty.$$

Proof. Without loss of generality we may assume that $\lambda = 0$. Suppose on the contrary (passing to a subsequence if necessary) that

$$\sup_n \|(A-\lambda_n)^{-1}\|_k < \infty.$$

Then

$$A(A-\lambda_n)^{-1} = I + \lambda_n(A-\lambda_n)^{-1} \to I, \qquad n \to \infty,$$

in $\mathcal{L}(D(A^k), \|\cdot\|_k)$. So, for n large enough, $A(A-\lambda_n)^{-1}$ is invertible in $\mathcal{L}\left(D(A^k), \|\cdot\|_k\right)$ and then

$$\big(A(A-\lambda_n)^{-1}\big)^k D(A^k) = D(A^k).$$

Let $x \in X$. Then $(A-\lambda_n)^{-k}x \in D(A^k)$, and so there exists $u \in D(A^k)$ such that

$$(A-\lambda_n)^{-k}A^k u = \big(A(A-\lambda_n)^{-1}\big)^k u = (A-\lambda_n)^{-k}x.$$

Hence $A^k u = x$. Thus $A^k D(A^k) = X$. In particular, A is surjective, which is a contradiction with Lemma 3.2. □

Lemma 3.4. *Let $\lambda \in \partial\sigma(A)$ and let $\{\lambda_n : n \geqslant 1\} \subset \rho(A)$ be such that $\lambda_n \to \lambda, n \to \infty$. Then for every $k \in \mathbb{N}$ there exists $x \in D(A^k)$ such that $\sup_n \|(A-\lambda_n)^{-1}x\| = \infty$.*

Proof. Let $k \in \mathbb{N}$ be fixed. Without loss of generality we may assume that $\lambda = 0$. Suppose on the contrary that $\sup_n \|(A-\lambda_n)^{-1}x\| < \infty$ for each $x \in D(A^k)$.

We show by induction on j, $1 \leqslant j \leqslant k$, that $\sup_n \|A^j(A-\lambda_n)^{-1}x\| < \infty$. We have

$$A^j(A-\lambda_n)^{-1}x = A^{j-1}x + \lambda_n A^{j-1}(A-\lambda_n)^{-1}x,$$

and so $\sup_n \|A^j(A-\lambda_n)^{-1}x\| < \infty$ by the induction assumption. Hence

$$\begin{aligned}\sup_n \|(A-\lambda_n)^{-1}x\|_k &= \sup_n\big(\|(A-\lambda_n)^{-1}x\| + \|A(A-\lambda_n)^{-1}x\| + \cdots \\ &+ \|A^k(A-\lambda_n)^{-1}x\|\big) \\ &< \infty.\end{aligned}$$

By the Banach-Steinhaus theorem, $\sup_n \|(A-\lambda_n)^{-1}\|_k < \infty$, a contradiction. □

The next statement ensures the existence of an approximate eigenvector for A consisting of smooth vectors.

Proposition 3.5. *Let $\lambda \in \partial\sigma(A)$, let $k \in \mathbb{N}$ and $\varepsilon \in (0,1)$. Then there exists $x \in C^\infty(A)$ such that $\|x\| = 1$ and $\|(A-\lambda)^j x\| < \varepsilon$ for every $j \in \mathbb{N}, 1 \leqslant j \leqslant k$.*

Proof. Without loss of generality we my assume that $\lambda = 0$. Choose $\lambda_n \in \rho(A)$, $n \in \mathbb{N}$, such that $\lambda_n \to 0, n \to \infty$. By Lemma 3.4, there exists $u \in D(A^k)$ such that $\sup_n \|(A-\lambda_n)^{-1}u\| = \infty$. By passing to a subsequence if necessary we may assume that $\|(A-\lambda_n)^{-1}u\| \to \infty, n \to \infty$.

For $n \in \mathbb{N}$ set $x_n = \frac{(A-\lambda_n)^{-1}u}{\|(A-\lambda_n)^{-1}u\|}$. Then $\|x_n\| = 1$. We show by induction on $j, 1 \leqslant j \leqslant k$, that $\lim_{n\to\infty} \|A^j x_n\| = 0$. We have

$$Ax_n = \frac{u}{\|(A-\lambda_n)^{-1}u\|} + \lambda_n x_n$$

and $\|Ax_n\| \to 0, n \to \infty$.

Similarly, for $j = 1, \dots, k$ we have

$$A^j x_n = \frac{A^{j-1}u}{\|(A-\lambda_n)^{-1}u\|} + \lambda_n A^{j-1} x_n$$

and $\|A^j x_n\| \to 0$ by the induction assumption.

For n sufficiently large, we have $x_n \in D(A^k)$, $\|x_n\| = 1$ and $\|A^j x_n\| < \varepsilon/2 \quad (j = 1, \dots, k)$. Since $C^\infty(A)$ is dense in $(D(A^k), \|\cdot\|_k)$, there exists $x' \in C^\infty(A)$ such that $\|x' - x_n\|_k < \varepsilon/4$.

Set $x = \frac{x'}{\|x'\|}$. Then $x \in C^\infty(A)$ and $\|x\| = 1$. Moreover for every j, $1 \leqslant j \leqslant k$, we have

$$\|A^j x\| = \frac{\|A^j x'\|}{\|x'\|} \leqslant \frac{\|A^j x_n\| + \|A^j(x' - x_n)\|}{\|x'\|} \leqslant \frac{3\varepsilon/4}{1 - \varepsilon/4} < \varepsilon.$$

□

Now we show that if λ is an approximate eigenvalue for A which is not an eigenvalue, then we can find a "smooth" approximate eigenvector corresponding to λ in any given subspace of finite codimension.

Proposition 3.6. *Let $\lambda \in \sigma_\pi(A) \setminus \sigma_p(A)$. Let $M \subset X$ be a closed subspace of finite codimension, let $k \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $x \in M \cap C^\infty(A)$ such that $\|x\| = 1$ and $\|(A-\lambda)^j x\| < \varepsilon$ for $j = 1, \dots, k$.*

Proof. Without loss of generality we may assume that $\lambda = 0$. Write $l = \operatorname{codim} M$ and let $M_0 = M \cap C^\infty(A)$. If $u_1, \dots, u_{l+1} \in C^\infty(A)$ are linearly independent then there exists a nontrivial linear combination $u = \sum_{j=1}^{l+1} \alpha_j u_j$ such that $u \in M$, and so $u \in M_0$. Thus $\dim (C^\infty(A)/M_0) \leqslant l$.

Let $F \subset C^\infty(A)$ be a subspace such that $F \cap M_0 = \{0\}$ and $F + M_0 = C^\infty(A)$. Then $\dim F \leqslant l$. We have $F + M \supset \overline{F + M_0} = \overline{C^\infty(A)} = X$. Since $\dim F \leqslant l = \operatorname{codim} M$, we have $F \cap M = \{0\}$ and $X = F \oplus M$. Let P be the projection satisfying $\operatorname{Ker} P = M$ and $PX = F$.

By Proposition 3.5, there exists a sequence $\{x_n : n \geqslant 1\} \subset C^\infty(A)$ such that $\|x_n\| = 1$ for all n and $\|A^j x_n\| \to 0, j = 1, \dots, k$. Write

$$x_n = m_n + f_n, \qquad n \in \mathbb{N},$$

where $f_n = Px_n \in F$ and $m_n = (I - P)x_n \in M \cap C^\infty(A) = M_0$. Since $\|f_n\| \leqslant \|P\|$ for all n, we may assume (passing to a subsequence if necessary) that $f_n \to f, n \to \infty$, and $f \in F$. Since $A\restriction_F$ is bounded, we have

$$Af_n \to Af \quad \text{and} \quad Am_n = Ax_n - Af_n \to -Af, \quad n \to \infty.$$

Similarly, for every $j, 1 \leqslant j \leqslant k$,

$$A^j f_n \to A^j f \quad \text{and} \quad A^j m_n \to -A^j f, \quad n \to \infty.$$

Suppose on the contrary that there exists $\varepsilon > 0$ such that $\sum_{j=1}^{k} \|A^j u\| \geqslant \varepsilon \|u\|$ for all $u \in M \cap C^\infty(A)$. Since $\sum_{j=1}^{k} \|A^j(m_n - m_{n'})\| \to 0$ as $n, n' \to \infty$, the sequence $\{m_n : n \geqslant 1\}$ is Cauchy and hence convergent. Denote its limit by m.

We have $m_n \in D(A)$, $m_n \to m$ and $Am_n \to -Af, n \to \infty$, and, since A is a closed operator, $m \in D(A)$ and $Am = -Af$. Thus $A(m + f) = 0$. Moreover,

$$\|m + f\| = \lim_{n\to\infty} \|m_n + f_n\| = \lim_{n\to\infty} \|x_n\| = 1.$$

This contradicts the assumption that $0 \notin \sigma_p(A)$. □

4. Smooth approximate eigenvectors for semigroups

In this section we show that smooth approximate eigenvectors of semigroup generators are approximate eigenvectors for semigroup as well, and moreover, under natural spectral assumptions, any subspace of finite codimension contains smooth approximate eigenvectors for semigroup.

Proposition 4.1. *Let $(T(t)_{t\geqslant 0}$ be a C_0-semigroup on a Banach space X, with generator A. Let $\lambda \in \sigma_\pi(A) \setminus \sigma_p(A)$, let $M \subset X$ be a subspace of finite codimension and $\varepsilon > 0$. Then for any $t_0 > 0$ and $n_0 \in \mathbb{N}$ there exists $x \in M \cap C^\infty(A), \|x\| = 1$, such that*

$$\|T(t)x - e^{\lambda t} x\| < \varepsilon, \qquad t \in [0, t_0],$$

and

$$\|(A - \lambda)^j x\| < \varepsilon, \quad 1 \leqslant j \leqslant n_0.$$

Proof. Let $\varepsilon_0 \in (0, \varepsilon)$ satisfy

$$\varepsilon_0 t_0 (\max\{\|T(t)\| : 0 \leqslant t \leqslant t_0\} \cdot \max\{e^{t_0 \mathrm{Re}\lambda}, 1\} < \varepsilon.$$

By Proposition 3.6, there exists $x \in M \cap C^\infty(A)$ such that $\|x\| = 1$ and

$$\|(A - \lambda)^j x\| < \varepsilon_0, \qquad 1 \leqslant j \leqslant j_0.$$

Let $0 \leqslant t \leqslant t_0$. Then

$$\begin{aligned} \|T(t)x - e^{\lambda t} x\| &= \left\| \int_0^t e^{\lambda(t-s)} T(s)(\lambda - A)x ds \right\| \\ &\leqslant t \max\{1, e^{\mathrm{Re}\lambda t_0}\} \cdot \max\{\|T(s)\| : 0 \leqslant s \leqslant t_0\} \cdot \|(\lambda - A)x\| \\ &< \varepsilon. \end{aligned}$$

□

We will also need a generalization of Proposition 4.1 to the situation when an approximate eigenvector for semigroup is not "induced" by the spectrum of the generator. This generalization will rely on a property of approximate eigenvectors, which is of independent interest.

Proposition 4.2. *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on a Banach space X, with generator A. Let $\sigma(A)\cap(s_0(A)+i\mathbb{R})=\emptyset$. Then there exist sequences $\{\mu_n : n\geqslant 1\}\subset\mathbb{C}$ and $\{u_n : n\geqslant 1\}\subset D(A)$ such that*

a) $\operatorname{Re}\mu_n\to s_0(A),\ n\to\infty,\quad \operatorname{Re}\mu_n<s_0(A)$,

b) $\|u_n\|=1$ *for all* $n\in\mathbb{N}$, $\|(\mu_n-A)u_n\|\to 0, n\to\infty$, *and for every* $y^*\in D(A^*)$,
$$\langle u_n, y^*\rangle\to 0,\quad n\to\infty.$$
In particular, $u_n\to 0, n\to\infty$, *weakly if* X *is reflexive.*

Proof. If $s(A)=s_0(A)$ then there exists a sequence $\{\mu_n : n\geqslant 1\}\subset\sigma(A)$ such that $\operatorname{Re}\mu_n\to s_0(A), n\to\infty$. Then $|\operatorname{Im}\mu_n|\to\infty$ since otherwise there is a limit point μ of $\{\mu_n : n\geqslant 1\}$, which belongs to $\sigma(A)$ and satisfies $\operatorname{Re}\mu=s_0(A)$, a contradiction.

If $s(A)<s_0(A)$, then there exists a sequence $\{\mu_n : n\geqslant 1\}\subset\rho(A)$ such that $\operatorname{Re}\mu_n\to s_0(A)$, and $\|(\mu_n-A)^{-1}\|\to\infty, n\to\infty$. Again $|\operatorname{Im}\mu_n|\to\infty$ since otherwise there is a limit point μ of $\{\mu_n : n\geqslant 1\}$ with $\operatorname{Re}\mu=s_0(A)$. Necessarily $\mu\in\sigma(A)$, a contradiction.

In both cases there exists a sequence $\{\mu_n : n\geqslant 1\}$ with $\operatorname{Re}\mu_n\to s_0(A)$, $|\mu_n|\to\infty$ and unit vectors $u_n\in D(A), n\geqslant 1$, such that $\|(\mu_n-A)u_n\|\to 0, n\to\infty$.

We show that
$$\langle u_n, y^*\rangle\to 0,\quad n\to\infty,$$
for each $y^*\in D(A^*)$. Let $y^*\in D(A^*)\subset X^*$, $\|y^*\|=1$. Find $y\in D(A)$ with $\langle y, y^*\rangle>\frac{1}{2}$. Let $M=\operatorname{Ker}y^*$. Write $u_n=m_n+\alpha_n y$ for some $m_n\in M$, $\alpha_n\in\mathbb{C}$. Then the sequences $\{m_n : n\geqslant 1\}$ and $\{\alpha_n : n\geqslant 1\}$ are bounded. Furthermore,
$$\langle(\mu_n-A)u_n, y^*\rangle\to 0,\qquad n\to\infty,$$
and
$$\begin{aligned}\langle(\mu_n-A)u_n, y^*\rangle &= \langle(\mu_n-A)m_n, y^*\rangle+\alpha_n\langle(\mu_n-A)y, y^*\rangle\\ &= \alpha_n\mu_n\langle y, y^*\rangle-\langle m_n, A^*y^*\rangle-\alpha_n\langle Ay, y^*\rangle.\end{aligned}$$
Since the last two terms are uniformly bounded and $|\mu_n|\to\infty, n\to\infty$, we have $\alpha_n\to 0, n\to\infty$. So $\langle u_n, y^*\rangle\to 0, n\to\infty$.

If X is reflexive then $D(A^*)$ is dense in X^*, and we have $u_n\to 0, n\to\infty$, weakly. ☐

Remark 4.3. Note that if X is not reflexive then it is not true in general that a sequence $\{u_n : n\geqslant 1\}$ as above converges weakly to zero. For example if $X=l_1$ then weak and strong convergences in X coincide, hence $\{u_n : n\geqslant 1\}$ does not converge weakly in X since $\|u_n\|=1, n\geqslant 1$, and the set $D(A^*)$ separates elements of X.

Now we are ready to show that, under natural spectral assumptions, the semigroup possesses an approximate eigenvalue with real part equal to the pseudo-spectrum bound. Moreover, the corresponding approximate eigenvectors can be found in any subspace of finite codimension.

Proposition 4.4. *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on a reflexive Banach space X, with generator A. Suppose that $\sigma_p(A)\cap(s_0(A)+i\mathbb{R})=\emptyset$. Let $M\subset X$ be a subspace of finite codimension. Then for any $t_0>0$ and $\varepsilon>0$ there exist $\mu\in\mathbb{C}$ with $\operatorname{Re}\mu=s_0(A)$ and $x\in M$, $\|x\|=1$, such that*

$$\|T(t)x-e^{\mu t}x\|<\varepsilon,\qquad 0\leqslant t\leqslant t_0.$$

Proof. Suppose first that there exists $\mu\in\sigma(A)$ with $\operatorname{Re}\mu=s_0(A)$. Then $\mu\in\partial\sigma(A)\setminus\sigma_p(A)\subset\sigma_\pi(A)\setminus\sigma_p(A)$.

By Lemma 3.6, there exists $x\in D(A)\cap M$ such that $\|x\|=1$ and

$$\|T(t)x-e^{\mu t}x\|=\left\|\int_0^t e^{\mu(t-s)}T(s)(\mu-A)x\,ds\right\|<\varepsilon$$

for all t, $0\leqslant t\leqslant t_0$.

Suppose that $\sigma(A)\cap(s_0(A)+i\mathbb{R})=\emptyset$. By Proposition 4.2, there exist sequences $\{\mu_n:n\geqslant 1\}\subset\mathbb{C}$ and $\{u_n:n\geqslant 1\}\subset D(A)$ such that

$$\begin{array}{rc} \operatorname{Re}\mu_n\to s_0(A), & \operatorname{Re}\mu_n<s_0(A),\\ \|u_n\|=1,\ n\in\mathbb{N}, & u_n\to 0, n\to\infty \text{ weakly},\\ \|(\mu_n-A)u_n\|\to 0, & n\to\infty. \end{array}$$

Let $\varepsilon>0$ and $t_0>0$ be fixed and $K:=\sup\{\|T(t)\|:0\leqslant t\leqslant t_0\}$. As in Lemma 3.6, there exists a finite-dimensional subspace $F\subset D(A)$ such that $X=M\oplus F$. Let P be the projection onto F with $\operatorname{Ker}P=M$. Then $\|Pu_n\|\to 0$, so $\|(I-P_n)u_n\|\to 1, n\to\infty$, and

$$\left\|u_n-\frac{u_n-Pu_n}{\|u_n-Pu_n\|}\right\|\to 0,\qquad n\to\infty.$$

Choose $n_0\in\mathbb{N}$ such that

$$\begin{aligned} \left\|u_{n_0}-\frac{u_{n_0}-Pu_{n_0}}{\|u_{n_0}-Pu_{n_0}\|}\right\| &\leqslant \min\left\{\frac{\varepsilon}{4K},\frac{\varepsilon}{4e^{s_0(A)t_0}}\right\},\\ e^{t_0(s_0(A)-\operatorname{Re}\mu_{n_0})} &< \frac{\varepsilon}{4}, \end{aligned}$$

and

$$\|(\mu_{n_0}-A)u_{n_0}\|<\frac{\varepsilon}{4\max\{\|T(t)\|:0\leqslant t\leqslant t_0\}\cdot t_0\cdot\max\{1,e^{t_0s_0(A)}\}}.$$

Set

$$\mu=s_0(A)+i\operatorname{Im}\mu_{n_0}\quad\text{and}\quad x=\frac{u_{n_0}-Pu_{n_0}}{\|u_{n_0}-Pu_{n_0}\|}.$$

Let $0\leqslant t\leqslant t_0$. We have

$$\|T(t)u_{n_0}-e^{\mu_{n_0}t}u_{n_0}\|=\left\|\int_0^t e^{\mu_n(t-s)}T(s)(\mu_{n_0}-A)u_{n_0}ds\right\|<\varepsilon/4$$

and

$$\begin{aligned} \|T(t)x-e^{\mu t}x\| &\leqslant \|T(t)x-T(t)u_{n_0}\|+\|T(t)u_{n_0}-e^{\mu_{n_0}t}u_{n_0}\|\\ &+ \|e^{\mu_{n_0}t}u_{n_0}-e^{\mu t}u_{n_0}\|+\|e^{\mu t}u_{n_0}-e^{\mu t}x\| \end{aligned}$$

$$\begin{aligned} &\leqslant K\|x-u_{n_0}\|+\varepsilon/4+e^{t(s_0(A)-\mathrm{Re}\mu_{n_0})}+e^{ts_0(A)}\|x-u_{n_0}\| \\ &< \varepsilon. \end{aligned}$$

□

5. Lower bounds for weak orbits

In this section, we first prove that if the spectrum of the generator meets the imaginary axis then any function tending to zero at infinity can be dominated by a certain smooth weak orbit of the corresponding semigroup. For shorthand, we say that a C_0-semigroup $(T(t))_{t\geqslant 0}$ is *weakly stable* if it converges to zero as $t\to\infty$ in the weak operator topology.

Theorem 5.1. *Let $(T(t)_{t\geqslant 0}$ be a weakly stable C_0-semigroup on a Hilbert space H, with generator A. Suppose that $0\in\sigma(A)$. Let $f:[0,\infty)\to(0,\infty)$ be a bounded function such that $\lim_{t\to\infty}f(t)=0$ and let $\varepsilon>0$. Then there exists $x\in C^\infty(A)$ such that $\|x\|<\sup\{f(t):t\geqslant 0\}+\varepsilon$ and*

$$\mathrm{Re}\langle T(t)x,x\rangle > f(t)$$

for all $t\geqslant 0$.

Proof. Without loss of generality we can assume that f is non-increasing. Indeed, we may replace f by $\tilde f$ defined by $\tilde f(t)=\sup\{f(s):s\geqslant t\}$. We may also assume that $f(0)=1-\varepsilon$, and to show that there exists $x\in H$ with $\|x\|=1$ satisfying the required property.

Since $(T(t))_{t\geqslant 0}$ is weakly stable, it is bounded by the uniform boundedness principle. Let $K=\sup\{\|T(t)\|:0\leqslant t<\infty\}$.

If $0\in\sigma_p(A)$, then there exists $x\in D(A)$ with $\|x\|=1$ and $Ax=0$. Then $x\in C^\infty(A)$, $T(t)x=x$ and $\mathrm{Re}\langle T(t)x,x\rangle=1>f(t)$ for all t.

Thus we can assume that $0\in\sigma_\pi(A)\setminus\sigma_p(A)$. By [15, Lemma 1] (or by [16, Lemma 11, p. 355]), there exist positive numbers $c_k, k\in\mathbb{N}$, such that $\sum_{k=1}^\infty c_k^2=1$ and

$$\sum_{k=j+1}^{\infty} c_k^2 > 3Kc_j$$

for all $j\geqslant 1$.

Choose positive numbers $\delta_j, j\geqslant 1$, such that

$$\delta_j<\frac{1-f(0)}{2^j}\quad\text{and}\quad \delta_j<\frac{K}{j^2 2^{j+2}}\cdot\min\{c_k:k=1,2,\dots,j+1\}. \tag{5.1}$$

Find t_0 such that $f(t_0)<\sum_{j=2}^\infty c_j^2-3Kc_1$. Choose $x_1\in C^\infty(A)$ such that

$$\|T(t)x_1-x_1\|<\delta_1,\qquad 0\leqslant t\leqslant t_0.$$

Find $t_1>t_0$ such that $f(t_1)<\sum_{j=3}^\infty c_j^2-3Kc_2$.

We construct inductively an increasing sequence $\{t_k:k\geqslant 1\}$ of positive numbers and unit vectors $\{x_k:k\geqslant 1\}\subset C^\infty(A)$ in the following way. Let $k\geqslant 2$ and suppose that $x_j\in C^\infty(A)$, $t_j>0$, have already been constructed for $j = 1, \dots, k-1$. Using Proposition 4.1, we can find $x_k \in C^\infty(A)$ such that $\|x_k\| = 1$ and

$$\|T(t)x_k - x_k\| < \delta_k, \qquad 0 \leqslant t \leqslant t_{k-1}; \tag{5.2}$$

$$\|A^j x_k\| \leqslant \delta_k \qquad j = 1, \dots, k; \tag{5.3}$$

$$x_k \perp x_j, \qquad j = 1, \dots, k-1. \tag{5.4}$$

Moreover, since the set $L := \{T(t)x_j : 0 \leqslant t \leqslant t_{k-1}, 1 \leqslant j \leqslant k-1\}$ is compact, there exists a finite $\delta_k/2$-net $L_0 \subset L$.

As the subspace $M := L_0^\perp$ has a finite codimension, by choosing x_k in addition to (5.2)–(5.4) such that $x_k \in M$ we can also assume

$$|\langle T(t)x_j, x_k\rangle| < \delta_k, \qquad t \leqslant t_{k-1}, 1 \leqslant j \leqslant k-1.$$

Using the weak stability of $(T(t))_{t\geqslant 0}$, find $t_k > \max(t_{k-1}, k)$ such that

$$|\langle T(t)x_j, x_s\rangle| < \delta_k \qquad t \geqslant t_k, 1 \leqslant j, s \leqslant k,$$

and

$$f(t_k) < \sum_{j=k+2}^{\infty} c_j^2 - 3Kc_{k+1}.$$

Suppose that the vectors $x_j \in H$ and positive numbers t_j have been constructed in the above described way. Set

$$x = \sum_{k=1}^{\infty} c_k x_k.$$

Then $\|x\| = (\sum_{k=1}^\infty c_k^2)^{1/2} = 1$. Fix an arbitrary $j \in \mathbb{N}$. We have $\|A^j c_k x_k\| \leqslant \delta_k$ for $k \geqslant j$, so the series $\sum_{k=1}^\infty c_k A^j x_k$ is convergent by (5.1). Since A^j is a closed operator, we have $x \in D(A^j)$. Hence, since j was arbitrary, we have $x \in C^\infty(A)$.

For $0 \leqslant t \leqslant t_0$ we obtain

$$\begin{aligned}
\operatorname{Re}\langle T(t)x, x\rangle &= \operatorname{Re}\sum_{s=1}^{\infty}\langle T(t)c_s x_s, x\rangle \\
&= \sum_{s=1}^{\infty} c_s \operatorname{Re}\Big(\langle x_s, x\rangle - \langle x_s - T(t)x_s, x\rangle\Big) \\
&\geqslant \sum_{s=1}^{\infty} c_s^2 - \sum_{s=1}^{\infty} c_s\delta_s \geqslant 1 - \sum_{s=1}^{\infty}\delta_s > f(0) \geqslant f(t).
\end{aligned}$$

Let $k \geqslant 1$, $t_{k-1} < t \leqslant t_k$. Then

$$\begin{aligned}
\operatorname{Re}\langle T(t)x, x\rangle &= \operatorname{Re}\Big\langle \sum_{s=1}^{k} c_s T(t)x_s, x\Big\rangle + \operatorname{Re}\Big\langle \sum_{s=k+1}^{\infty} c_s T(t)x_s, x\Big\rangle \\
&= \operatorname{Re}\Big\langle \sum_{s=1}^{k-1} c_s T(t)x_s, \sum_{j=1}^{k-1} c_j x_j\Big\rangle + \operatorname{Re}\Big\langle \sum_{s=1}^{k-1} c_s T(t)x_s, c_k x_k\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \mathrm{Re}\Big\langle c_kT(t)x_k, \sum_{j=1}^{k} c_jx_j\Big\rangle + \mathrm{Re}\Big\langle \sum_{s=1}^{k} c_sT(t)x_s, \sum_{j=k+1}^{\infty} c_jx_j\Big\rangle \\
&+ \quad \mathrm{Re}\Big\langle \sum_{s=k+1}^{\infty} c_sx_s, x\Big\rangle - \mathrm{Re}\Big\langle \sum_{s=k+1}^{\infty} c_s(x_s - T(t)x_s), x\Big\rangle \\
&\geqslant \quad -\sum_{j=1}^{k-1}\sum_{s=1}^{k-1} c_sc_j\delta_{k-1} - c_kK\Big\|\sum_{s=1}^{k-1} c_sx_s\Big\| - Kc_k\Big\|\sum_{j=1}^{k} c_jx_j\Big\| \\
&- \quad \sum_{s=1}^{k}\sum_{j=k+1}^{\infty} c_sc_j\delta_j + \sum_{s=k+1}^{\infty} c_s^2 - \sum_{s=k+1}^{\infty} c_s\delta_s \\
&\geqslant \quad -\Big(\sum_{s=1}^{k-1} c_s\Big)^2 \delta_{k-1} - 2Kc_k - k\sum_{j=k+1}^{\infty}\delta_j + \sum_{s=k+1}^{\infty} c_s^2 - \sum_{s=k+1}^{\infty}\delta_s \\
&\geqslant \quad \sum_{s=k+1}^{\infty} c_s^2 - 2Kc_k - (k-1)^2\delta_{k-1} - 2k\sum_{j=k+1}^{\infty}\delta_j \\
&\geqslant \quad \sum_{s=k+1}^{\infty} c_s^2 - 3Kc_k > f(t_{k-1}) \geqslant f(t).
\end{aligned}
$$

□

Corollary 5.2. *Let $(T(t)_{t\geqslant 0}$ be a weakly stable C_0-semigroup on a Hilbert space H, with generator A. Suppose that $\sigma(A)\cap i\mathbb{R}\neq\emptyset$. Let $f:[0,\infty)\to(0,\infty)$ be a bounded function such that $\lim_{t\to\infty} f(t)=0$ and let $\varepsilon>0$. Then there exists $x\in C^\infty(A)$ such that $\|x\|<\sup\{f(t): t\geqslant 0\}+\varepsilon$ and*

$$|\langle T(t)x, x\rangle| > f(t)$$

for all $t\geqslant 0$.

Proof. There exists $\lambda\in\sigma(A)$ such that $\mathrm{Re}\,\lambda=0$. Consider the rescaled semigroup $(e^{-\lambda t}T(t))_{t\geqslant 0}$ and apply the previous theorem.

□

The statement of Corollary (5.2) remains true if we replace the assumption $\sigma(A)\cap i\mathbb{R}\neq\emptyset$ by a weaker assumption that $\omega_0(T)=0$. However, we have no additional information about the "smoothness" of vector x then.

Theorem 5.3. *Let $(T(t)_{t\geqslant 0}$ be a weakly stable C_0-semigroup on a Hilbert space H, with generator A. Suppose that $\omega_0(T)=0$. Let $f:[0,\infty)\to(0,\infty)$ be a bounded function such that $\lim_{t\to\infty} f(t)=0$ and let $\varepsilon>0$. Then there exists $x\in H$ such that $\|x\|<\sup\{f(s): s\geqslant 0\}+\varepsilon$ and*

$$|\langle T(t)x, x\rangle| > f(t) \tag{5.5}$$

for all $t\geqslant 0$.

Proof. The proof is similar to that of Theorem 5.1. We use the fact that $\omega_0 = s_0(A)$ and Proposition 4.4 instead of Proposition 4.1. (In this case we cannot guarantee the existence of a *smooth* approximate eigenvector of $(T(t))_{t\geqslant 0}$ corresponding to an approximate eigenvalue with real part equal to $s_0(A)$.) The only difference is that the vectors x_k and x do not have to be taken from $C^\infty(A)$ now and we do not have to take care of terms like $A^j x_k$. □

Remark 5.4. To show that it is indeed not possible, in general, to formulate Theorem 5.3 for smooth elements, consider the following example. Let

$$\Omega := \left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant -|\operatorname{Im}\lambda|^{-1}\right\}.$$

Let $X = L^2(\Omega, d\mu)$ with two-dimensional Lebesgue measure μ, and let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on X defined by

$$(T(t)f)(s) = (e^{\lambda t}f)(s), \qquad t \geqslant 0, f \in L^2(\Omega, d\mu).$$

Its generator A is given by

$$(Af)(z) = zf(z), \qquad z \in \Omega,$$

with maximal domain. Then $s(A) = \omega_0(T) = 0$. However, it is easy to show that $\|T(t)A^{-1}\| = \mathrm{O}(1/t)$ as $t \to \infty$, and so the domination properties like (5.5) for smooth vectors cannot be true.

6. Lower bounds for weak orbits on large sets

In this section we drop the assumption of weak stability of the semigroup and obtain certain counterparts of the results in the previous section in this more general setting.

Let $B \subset [0, \infty)$ be a measurable set. Denote by $\operatorname{Dens} B$ and $\overline{\operatorname{Dens}}\, B$ the density and upper density of B, respectively, defined by

$$\operatorname{Dens} B := \lim_{t\to\infty} t^{-1} m(B \cap [0, t])$$

(if the limit exists) and

$$\overline{\operatorname{Dens}}\, B := \limsup_{t\to\infty} t^{-1} m(B \cap [0, t]),$$

where m stands for Lebesgue measure.

The next lemma, which will be instrumental in Theorem 6.2 below, can be found e.g. in [8, Theorem 2.5] and [9, Theorem 3.2 and Remark 1, p. 388-389].

Lemma 6.1. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup on a separable Hilbert space H, with generator A. Suppose that $\sigma_p(A) \cap i\mathbb{R} = \{\emptyset\}$. Then there exists a subset $B \subset \mathbb{R}$ with* $\operatorname{dens}(B) = 1$ *such that T converges in the weak operator topology to zero as $t \in B, t \to \infty$.*

Now we are able to formulate our first result for domination properties of not necessarily weakly stable semigroups.

Theorem 6.2. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A. Let $f : [0,\infty) \to (0,\infty)$ be a function such that $\lim_{t\to\infty} f(t) = 0$.*

(i) *If $\sigma(A)\cap i\mathbb{R} \neq \emptyset$, then there exist $x \in C^\infty(A), \|x\| = 1$, and $B \subset \mathbb{R}$ with* $\operatorname{Dens} B = 1$ *such that*
$$|\langle T(t)x, x\rangle| > f(t)$$
for all $t \in B$.

(ii) *If $\omega_0(T) = 0$, then there exist $x \in H$, $\|x\| = 1$, and $B \subset \mathbb{R}$ with* $\operatorname{Dens} B = 1$ *such that*
$$|\langle T(t)x, x\rangle| > f(t)$$
for all $t \in B$.

Proof. Without loss of generality we may assume that $\sup\{f(t) : t \geqslant 0\} < 1$ (otherwise replace f by $\tilde f$ defined by $\tilde f(t) = \max\{f(t), 1/2\}$, which differ from f only on a subset of density zero).

We may also assume that H is separable. Indeed, in case (i), let $\lambda \in \sigma(A)\cap i\mathbb{R}$. Then $\lambda \in \sigma_\pi(A)$ and there exist a sequence $\{u_n : n \geqslant 1\}$ of unit vectors in H such that $\|(A-\lambda)u_n\| \to 0$. Set $H_0 = \bigvee_{n,t\geqslant 0} T(t)u_n$. Since $(T(t))_{t\geqslant 0}$ is strongly continuous, H_0 is a separable subspace of H invariant with respect to T. Write $T_0 = T|H_0$. Clearly $(T_0(t))_{t\geqslant 0}$ is a C_0-semigroup on H_0. Let A_0 be the generator of the semigroup $T_0(t)$. Clearly $\lambda \in \sigma(A_0)$.

In case (ii), for each rational positive number r and each $k \in \mathbb{N}$ there exists a unit vector $u_{r,k} \in H$ such that $\|T(r)u_{r,k}\| \geqslant \|T(r)\| \cdot \frac{k-1}{k}$. Let $H_0 = \bigvee_{r,k,t\geqslant 0} T(t)u_{r,k}$. Again H_0 is a separable subspace of H invariant with respect to T. The restriction $T_0 = T|H_0$ satisfies $\omega_0(T_0) = \omega_0(T) = 0$.

So in both cases we may assume that the semigroup is acting on a separable Hilbert space.

If $\sigma_p(A)\cap i\mathbb{R} \neq \emptyset$, then there exist $\lambda \in i\mathbb{R}$ and a unit vector $x \in H_0$ with $Ax = \lambda x$. Then $|\langle T(t)x, x\rangle| = |e^{\lambda t}| = 1 > f(t)$ for all $t \geqslant 0$.

So we may assume that $\sigma_p(A_0)\cap i\mathbb{R} = \emptyset$. By Lemma 6.1, there exists a subset $B \subset \mathbb{R}$ with $\operatorname{Dens} B = 1$ such that
$$\lim_{t\to\infty, t\in B} \langle T(t)x, y\rangle = 0$$
for all $x, y \in H_0$.

Now the statements (i) and (ii) can be proved as Theorem 5.3 and Corollary 5.2 (that is Theorem 5.1), respectively, with the only difference that we consider only the values t from the set B.

□

We proceed now with proving counterparts of Corollary 5.2 and Theorem 5.3 for not necessarily weakly stable semigroups on *Banach spaces*. To this aim, we will need a very useful lemma providing a substitute for “orthogonal complements” in general Banach spaces. Since [15] it became a more or less standard tool in construction of specific weak orbits, for its proof see e.g. [16, Lemma 37.6].

Lemma 6.3. *Let E be a finite-dimensional subspace of a Banach space X, and let $\varepsilon > 0$. Then there exists a closed subspace $Y \subset X$ of finite codimension such that*

$$\|e + y\| \geqslant \frac{(1-\varepsilon)}{2} \max\{\|e\|, \|y\|\}$$

for all $e \in E$ and $y \in Y$.

Theorem 6.4. *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on a Banach space X with generator A, let $\sigma(A) \cap \{z : \operatorname{Re} z \geqslant 0\} \neq \emptyset$. Let $f : (0, \infty) \to (0, \infty)$ be a function such that $\lim_{t\to\infty} f(t) = 0$. Then there exist $x \in C^\infty(A)$, $x^* \in X^*$ and a set $B \subset \mathbb{R}$ such that $\|x\| \leqslant 1, \|x^*\| \leqslant 1$, $\overline{\operatorname{Dens}} B = 1$ and*

$$|\langle T(t)x, x^*\rangle| > f(t)$$

for all $t \in B$.

Proof. As usual, we may assume that f is non-increasing and $f(0) < 1$.

Since $\sigma(A) \cap \{z : \operatorname{Re} z \geqslant 0\} \neq \emptyset$, there exists $\lambda \in \partial\sigma(A)$ with $\operatorname{Re}\lambda \geqslant 0$. Without loss of generality we may assume that $\lambda = 0$ (if $\lambda \neq 0$ then we can consider the semigroup $e^{-\lambda t}T(t)$).

If $0 \in \sigma_p(A)$ then there exists $x \in X$ such that $\|x\| = 1$ and $Ax = x$. Then $x \in C^\infty(A)$. Choose $x^* \in X^*$ such that $\|x^*\| = 1 = \langle x, x^*\rangle$. Then $T(t)x = x$ for all t and $|\langle T(t)x, x^*\rangle| = 1 > f(t)$ for all t.

So we may also assume that $0 \in \sigma_\pi(A) \setminus \sigma_p(A)$.

For $k \in \mathbb{N}$ set

$$\beta_k = \frac{1}{9 \cdot 8^k \cdot k!}\,.$$

Choose an increasing sequence $\{t_k : k \geqslant 1\} \subset (0, \infty)$ such that $t_k > kt_{k-1}, k \geqslant 2$, and

$$f(t_k) < \frac{\beta_{k+1}}{28(k+1)}.$$

Set

$$\varepsilon_k = \frac{\beta_k}{14k}, \qquad k \in \mathbb{N}.$$

Choose $x_1 \in C^\infty(A)$, $\|x_1\| = 1$ such that

$$\|T(t)x_1 - x_1\| < \frac{1}{42}, \qquad 0 \leqslant t \leqslant t_1,$$

$\|Ax_1\| \leqslant \frac{1}{2}$ and $x_1^* \in X^*$, $\|x_1^*\| = 1$ such that $\langle x_1, x_1^*\rangle = 1$.

We construct inductively vectors $x_k \in C^\infty(A)$, $x_k^* \in X^*$ in the following way: suppose that $k \geqslant 2$ and that the vectors $x_1, \dots, x_{k-1} \in C^\infty(A)$, $x_1^*, \dots, x_{k-1}^* \in X^*$ have already been constructed. Let $L_k := \{T(t)x_j : 1 \leqslant j \leqslant k-1, 0 \leqslant t \leqslant t_k\}$. The set L_k is compact, so there exists a finite

$\varepsilon_k/3$-net $L_k^0 \subset L_k$. Let $F_k = \bigvee L_k^0$. Then $\dim F_k < \infty$ and by Lemma 6.3 there exists a subspace $M_k \subset X$ with $\operatorname{codim} M_k < \infty$ such that

$$\|f + m\| \geqslant \frac{\|m\|}{3}$$

for all $f \in F_k$ and $m \in M_k$.

Similarly, the set $S_k = \{T(t)^* x_j^* : 1 \leqslant j \leqslant k-1, 0 \leqslant t \leqslant t_k\}$ is compact, so there exists a finite ε_k-net $S_k^0 \subset S_k$. Then $M_k \cap \bigcap_{y^* \in S_k^0} \ker y^*$ is a subspace of finite codimension. By Proposition 4.1, there exists

$$x_k \in C^\infty(A) \cap M_k \cap \bigcap_{y^* \in S_k^0} \ker y^*$$

such that $\|x_k\| = 1$,

$$\|T(t)x_k - x_k\| < \frac{1}{21k \cdot 2^k}, \qquad t \leqslant t_k, \tag{6.1}$$

and

$$\|A^j x_k\| \leqslant \varepsilon_k, \qquad 1 \leqslant j \leqslant k. \tag{6.2}$$

Moreover,

$$|\langle T(t)x_k, x_l^* \rangle| \leqslant \varepsilon_k, \qquad 1 \leqslant l \leqslant k-1, t \leqslant t_k. \tag{6.3}$$

For $f \in F_k$ and $\alpha \in \mathbb{C}$ we have $\|f + \alpha x_k\| \geqslant \frac{|\alpha|}{3}$. Define $x_k^* \in (F_k \vee \{x_k\})^*$ by $x_k^*|_{F_k} = 0$ and $\langle x_k, x_k^* \rangle = 1$. So $\|x_k^*\| \leqslant 3$ and by the Hahn-Banach theorem we can extend x_k^* to a functional (denoted by the same symbol x_k^*) with the same norm on the whole space X. So $\|x_k^*\| \leqslant 3$ and $\langle x_k, x_k^* \rangle = 1$.

By the construction,

$$|\langle T(t)x_l, x_k^* \rangle| \leqslant \varepsilon_k \qquad 1 \leqslant l \leqslant k-1, 1 \leqslant t \leqslant t_k.$$

The vectors $x \in X$ and $x^* \in X^*$ will be constructed as linear combinations of the the vectors x_k and x_k^*, respectively.

Let $\alpha_1 = \beta_1^{1/2}$. We construct positive numbers $\alpha_2, \alpha_3, \dots$ inductively. Let $k \geqslant 2$ and suppose that $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ have already been constructed. Set $u_k = \sum_{i=1}^{k-1} \alpha_i x_i$ and $v_k^* = \sum_{i=1}^{k-1} \alpha_i x_i^*$. For $s = k, k+1, \dots, 2k$ let

$$C_{k,s} = \Big\{ t : t_{k-1} < t \leqslant t_k, \big| \langle T(t)u_k, v_k^* \rangle + \frac{\beta_k s}{2k} \big| < \frac{\beta_k}{4k} \Big\}.$$

Clearly the sets $C_{k,k}, C_{k,k+1}, \dots, C_{k,2k}$ are disjoint, so there exists s_0, $k \leqslant s_0 \leqslant 2k$ such that $m(C_{k,s_0}) \leqslant \frac{t_k - t_{k-1}}{k+1}$. Let $B_k = (t_{k-1}, t_k] \setminus C_{k,s_0}$. Then

$$m(B_k) \geqslant \frac{(t_k - t_{k-1})k}{k+1}.$$

Let

$$\alpha_k = \left(\frac{\beta_k s_0}{2k} \right)^{1/2}.$$

Then $\frac{\beta_k}{2} \leqslant \alpha_k^2 \leqslant \beta_k$.

Suppose that the numbers α_k, $k \in \mathbb{N}$, have been constructed in the way described above. Set

$$x = \sum_{k=1}^{\infty} \alpha_k x_k, \quad x^* = \sum_{k=1}^{\infty} \alpha_k x_k^* \quad \text{and} \quad B = \bigcup_{k=2}^{\infty} B_k.$$

We have

$$\|x\| \leqslant \sum_{k=1}^{\infty} \alpha_k \|x_k\| \leqslant \sum_{k=1}^{\infty} \beta_k^{1/2} < 1$$

and similarly,

$$\|x^*\| \leqslant \sum_{k=1}^{\infty} \alpha_k \|x_k^*\| \leqslant 3 \sum_{k=1}^{\infty} \beta_k^{1/2} < 1.$$

We have

$$\begin{aligned} \overline{\operatorname{Dens}}\, B &\geqslant \limsup_{k\to\infty} \frac{\sum_{j=2}^{k} m(B_j)}{t_k} \geqslant \limsup_{k\to\infty} \frac{m(B_k)}{t_k} \\ &\geqslant \frac{t_k - t_{k-1}}{t_k} \cdot \frac{k}{k+1} \geqslant \limsup_{k\to\infty} (1 - k^{-1}) \frac{k}{k+1} = 1. \end{aligned}$$

Let $k \geqslant 2$ and $t_{k-1} < t \leqslant t_k$ and $t \in B_k$. Then, by (6.1)–(6.3),

$$\begin{aligned} (6.4)\quad |\langle T(t)x, x^*\rangle| &\geqslant \left| \langle T(t)u_k, v_k^*\rangle + \left\langle \sum_{j=k}^{\infty} \alpha_j^2 T(t)x_j, x_j^* \right\rangle \right| \\ &- \sum_{j\neq s, \max\{j,s\}\geqslant k} \alpha_j \alpha_s |\langle T(t)x_j, x_s^*\rangle| \\ &\geqslant \left| \langle T(t)u_k, v_k^*\rangle + \alpha_k^2 \right| - \sum_{j=k}^{\infty} \alpha_j^2 |\langle T(t)x_j - x_j, x_j^*\rangle| \\ &- \sum_{j\neq s, \max\{j,s\}\geqslant k} \alpha_j \alpha_s \varepsilon_k. \end{aligned}$$

Since $t \in B_k$, we have $\left|\langle T(t)u_k, v_k^*\rangle + \alpha_k^2\right| \geqslant \frac{\beta_k}{4k}$. So (6.4) yields

$$\begin{aligned} |\langle T(t)x, x^*\rangle| &\geqslant \frac{\beta_k}{4k} - 3\sum_{j=k}^{\infty} \frac{\beta_j}{21j2^j} - \varepsilon_k \\ &\geqslant \frac{\beta_k}{k}\left(\frac{1}{4} - \frac{1}{7} - \frac{1}{14}\right) = \frac{\beta_k}{28k} \\ &> f(t_{k-1}) \geqslant f(t). \end{aligned}$$

It remains to show that $x \in C^\infty(A)$. For each $j \in \mathbb{N}$ the sum $\sum_{k=1}^{\infty} \alpha_k A^j x_k$ converges by the choice of α_k and (6.2). Since A^k is closed, we have $x \in D(A^j)$. Since $j \in \mathbb{N}$ was arbitrary, we have $x \in C^\infty(A)$. $\square$

If X is reflexive then it is sufficient to assume $s_0(A) \geqslant 0$. However, we are not able to choose a smooth "dominating" weak orbit then.

Theorem 6.5. *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on a reflexive Banach space X, with generator A. Let $f:[0,\infty)\to(0,\infty)$ be a function such that $\lim_{t\to\infty} f(t)=0$. If $s_0(A)\geqslant 0$ then there exist $x\in X$, $x^*\in X^*$, $\|x\|\leqslant 1$, $\|x^*\|\leqslant 1$, and a set $B\subset\mathbb{R}$ with $\overline{\operatorname{Dens}}\,B=1$ such that*

$$|\langle T(t)x,x^*\rangle|>f(t)$$

for all $t\in B$.

Proof. The proof is similar to the proof of Theorem 6.4 where as in the proof of Theorem 5.3 we use Proposition 4.4 instead of Proposition 4.1. ☐

Results similar to Theorems 6.2 and 6.4 are true for power bounded operators with essentially the same proof up to a change of notation. We restrict ourselves to formulating a discrete analogue of Theorem 6.2 and leave its proof as well as formulations of analogues of Theorem 6.4 to the interested reader.

As in the case of the real semiaxis, the density of a set $B\subset\mathbb{N}$ is defined as

$$\operatorname{Dens}B:=\lim_{n\to\infty}\frac{\operatorname{card}\{x\in B: x\leqslant n\}}{n},$$

if the limit exists.

Theorem 6.6. *Let T be a power bounded operator on a Hilbert space H with spectral radius equal to 1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $a_n\to 0, n\to\infty$. Then there exist $x\in H$, $\|x\|\leqslant 1$ and $B\subset\mathbb{N}$ with* $\operatorname{Dens}B=1$ *such that*

$$|\langle T^n x,x\rangle|>a_n$$

for all $n\in B$.

7. Applications to Fourier transforms

As an illustration we show how theorems proved in the previous sections can be applied to the study of decay of Fourier transforms.

If μ is a positive Borel measure on $\mathbb{R}$ then for $f\in L^1(d\mu)$ we define its Fourier transform by

$$\widehat{f}(t):=\int_{\mathbb{R}} e^{-ist}f(s)\,d\mu(s). \tag{7.1}$$

If μ is finite, then the Fourier transform $\widehat{\mu}$ of μ is defined as the Fourier transform of the function $f\equiv 1$.

We say that a positive Borel measure μ is *Laplace transformable* if there exists $\omega\in\mathbb{R}$ such that

$$\int_0^\infty e^{-\omega s}d\mu(s)<\infty. \tag{7.2}$$

Let ω_μ be the infimum of those ω for which (7.2) holds. So if μ is finite then it is Laplace transformable and $\omega_\mu\leqslant 0$. If μ is Laplace transformable then its Laplace transform $\mathcal{L}\mu$ given by

$$\mathcal{L}\mu(\lambda) := \int_0^\infty e^{-\lambda s} d\mu(s) \tag{7.3}$$

is well defined for λ with $\operatorname{Re}\lambda > \omega_\mu$, and moreover $\mathcal{L}\mu$ is analytic in $\mathbb{C}_\mu := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_\mu\}$. Measures μ such that

$$\mathcal{L}\mu(\omega + i\beta) \to 0, \qquad |\beta| \to \infty, \tag{7.4}$$

for some $\omega > \omega_\mu$, will play an important role.

Remark 7.1. Note that if (7.4) holds for some $\omega_0 > \omega_\mu$ then it holds for all $\omega > \omega_\mu$. Indeed, for a fixed $\omega_0 > \omega_\mu$ choose $a, b > \omega_\mu$ such that $b > \omega_0 > a$. If

$$S := \{\lambda \in \mathbb{C} : a < \operatorname{Re}\lambda < b, \ |\operatorname{Im}\lambda| < 1\},$$

then the sequence of analytic functions $(\mathcal{L}_n\mu)_{n\in\mathbb{Z}}$ defined by

$$\mathcal{L}_n\mu(\lambda) := \mathcal{L}\mu(\lambda + in), \qquad n \in \mathbb{Z}, \quad \lambda \in S,$$

is uniformly bounded in S and converge to zero on $\{\omega_0 + is : |s| < 1\} \subset S$, as $|n| \to \infty$. Hence, by Vitali's theorem, $(\mathcal{L}_n\mu)_{n\in\mathbb{Z}}$ converges to zero as $|n| \to \infty$ uniformly on compact subsets of S. Therefore, $\mathcal{L}\mu(\omega + i\beta) \to 0, |\beta| \to \infty$, for every $\omega \in (a, b)$. Since the choice of $a \in (\omega_\mu, \omega_0)$ and $b \in (\omega_0, \infty)$ was arbitrary, our claim follows. For similar statement and argument, see [7, Theorem 2.4].

The following lemma is probably well-known, however we are short of a precise reference. For $\omega \in \mathbb{R}$ denote $e_\omega(s) = e^{-\omega s}, s \geqslant 0$.

Lemma 7.2. *The set $\Omega = \{e_\omega : \omega > \omega_\mu\}$ is total in $L^p(\mathbb{R}, d\mu), p \geqslant 1$.*

Proof. Assume that Ω is not total in $L^p(\mathbb{R}, d\mu)$. If q is the conjugate exponent, then there exists $g \in L^q(\mathbb{R}, d\mu), g \neq 0$, such that

$$F(\lambda) = \int_0^\infty e^{-\lambda s}\overline{g(s)}\, d\mu(s) = 0, \qquad \lambda > \omega_\mu. \tag{7.5}$$

Since F is analytic in $\mathbb{C}_\mu$, F is identically zero in $\mathbb{C}_\mu$. In particular, if $\omega > \omega_\mu$ is fixed, then the finite (complex) measure $e_\omega g \cdot \mu$ satisfies

$$\widehat{e_\omega \bar{g} \cdot \mu} = \widehat{\varphi \cdot \mu} = 0, \tag{7.6}$$

where $\varphi = e_\omega \bar{g}$. So $\varphi \cdot \mu = 0$ by the uniqueness theorem for Fourier transforms of bounded measures. Assume that μ is non-zero. We can suppose that that φ is real and fix any function from the equivalence class of φ denoted by the same symbol. By assumption $\varphi \neq 0$ so that $\mu(\{s : \varphi(s) \neq 0\}) > 0$. Let without loss of generality $\mu(\{s : \varphi(s) > 0\}) > 0$. Then there exists $n \in \mathbb{N}$ such that $\mu(\{s : \varphi(s) > 1/n\}) > 0$. But then

$$0 = \int_{\{s:\varphi(s)>1/n\}} \varphi(s) d\mu(s) \geqslant (1/n)\mu(\{s : \varphi(s) > 1/n\}),$$

a contradiction. □

Theorem 7.3. *Let μ be a Laplace transformable (in particular, finite) positive Borel measure on $\mathbb{R}$.*

(i) *If $\mathcal{L}\mu(\omega + \cdot) \in C_0(\mathbb{R})$ for some/all $\omega > \omega_\mu$, then for every bounded function $f : \mathbb{R} \to [0,\infty)$ satisfying $\lim_{|t|\to\infty} f(t) = 0$ there exists a positive $g \in L^1(\mathbb{R}, d\mu)$ such that $\widehat{g} \in C^\infty(\mathbb{R})$ and*

$$|\widehat{g}(t)| > f(t), \qquad t \in \mathbb{R}. \tag{7.7}$$

(ii) *in the general case, for every function $f : \mathbb{R} \to [0,\infty)$ satisfying $\lim_{|t|\to\infty} f(t) = 0$ there exist a positive $g \in L^1(\mathbb{R}, d\mu)$ and $B \subset \mathbb{R}, \mathrm{Dens}\,(B) = 1$, such that $\widehat{g} \in C^\infty(\mathbb{R})$ and*

$$|\widehat{g}(t)| > f(t), \qquad t \in B. \tag{7.8}$$

Proof. Without loss of generality we can assume that that $f(t) = f(-t), t > 0$. (Otherwise, replace f by $\tilde{f}$ defined by $\tilde{f}(t) = \max(f(t), f(-t)), t > 0$.)

Let $H = L^2(\mathbb{R}, d\mu)$. Consider the multiplication, unitary C_0-semigroup $(M(t))_{t\geqslant 0}$ on H defined by

$$(M(t)g)(s) = e^{-its}g(s), \qquad g \in H,\, t \geqslant 0.$$

The generator G of $(M(t))_{t\geqslant 0}$ is given by

$$(Gg)(s) = -isg(s), \qquad s \in \mathbb{R},$$

with maximal domain. Moreover,

$$\begin{aligned} C^\infty(G) =& \big\{h \in L^2(\mathbb{R}, d\mu) : s^k h \in L^2(\mathbb{R}, d\mu) \text{ for every } k \in \mathbb{N}\cup\{0\}\big\} \\ =& \big\{h \in L^2(\mathbb{R}, d\mu) : s^k |h|^2 \in L^1(\mathbb{R}, d\mu) \text{ for every } k \in \mathbb{N}\cup\{0\}\big\}. \end{aligned} \tag{7.9}$$

To prove (i) note that by our assumption and Remark 7.1, $\widehat{\mu}(\omega + i\cdot) \in C_0(\mathbb{R})$ for all $\omega > \omega_\mu$. Hence $\langle M(t)g_0, g_0\rangle \to 0, t \to \infty$, for g_0 from $\Omega = \{e_\omega : \omega > \omega_\mu\}$. Since by Lemma 7.2 the set Ω is total in $L^2(\mathbb{R}, d\mu)$, this implies that M converges to zero in the weak operator topology. Moreover, $\sigma(G) \subset i\mathbb{R}$, and is nonempty. By Corollary 5.2 there exists $g \in C^\infty(G)$ such that

$$\langle M(t)g, g\rangle = \widehat{|g|^2}(t) > f(t), \qquad t \geqslant 0. \tag{7.10}$$

By (7.9), $\widehat{|g|^2}$ is smooth. Since $(M(t))_{t\geqslant 0}$ is unitary, (7.10) holds also for negative t.

The proof of (ii) is similar but one uses Theorem 6.2, (i) instead of Corollary 5.2. □

Remark 7.4. If $0 \in \mathrm{supp}\,\mu$ then using Theorem 5.1 one can show that given $f : \mathbb{R} \to [0,\infty)$ with $\lim_{|t|\to\infty} f(t) = 0$ there exists a positive $g \in L^1(\mathbb{R}, d\mu)$ with smooth Fourier transform such that

$$\mathrm{Re}\,\widehat{g}(t) > f(t), \qquad t \in \mathbb{R}.$$

Remark 7.5. Theorem 7.3 implies, in particular, that given a Borel set $E \subset \mathbb{R}$ of finite and positive Lebesgue measure and a function f on $\mathbb{R}$ vanishing at $\pm\infty$ there exists $g \in L^1(E)$ such that (7.7) holds. Therefore, we are able to find $g \in L^1(\mathbb{R})$ with smooth Fourier transform dominating f and such that g has arbitrarily "small" essential support.

Remark 7.6. Note that if μ is a finite positive Borel measure then the assumption $\mathcal{L}\mu(\omega_0 + \cdot) \in C_0(\mathbb{R})$ for some $\omega_0 > 0$ is equivalent to $\widehat{\mu} \in C_0(\mathbb{R})$. Indeed, if $\widehat{\mu} \in C_0(\mathbb{R})$, then, since simple functions are dense in $L^1(\mathbb{R}, d\mu)$, $\widehat{f} \in C_0(\mathbb{R})$ for every $f \in L^1(\mathbb{R}, d\mu)$. In particular, this holds for $f = e_{\omega_0}$, hence $\widehat{e}_{\omega_0} = \mathcal{L}\mu(\omega_0 + \cdot) \in C_0(\mathbb{R})$. On the other hand, if $\mathcal{L}\mu(\omega_0 + \cdot) \in C_0(\mathbb{R})$ for some $\omega_0 > 0$, then $\mathcal{L}\mu(\omega + \cdot) \in C_0(\mathbb{R})$ for all $\omega > 0$ by Remark 7.1. Since the set $\{e_\omega : \omega > 0\}$ is total in $L^1(\mathbb{R}, d\mu)$ by Lemma 7.2 and $1 \in L^1(\mathbb{R}, d\mu)$, one has $\widehat{\mu} \in C_0(\mathbb{R})$.

Recall that, as it was shown in [2, Example 8], there exists a measure ν on the unit circle with $\widehat{\nu}(n) \not\to 0, |n| \to \infty$, and a sequence $\{a_n : n \geqslant 0\} \subset \mathbb{R}_+$ such that there is no function $f \in L^1(d\mu)$ with $|\hat{f}(n)| \geqslant a_n, n \geqslant 0$. By adjusting [2, Example 8] to the real line case, one can show that if μ is finite positive Borel measure on $\mathbb{R}$ then the assumption $\mathcal{L}\mu(\omega + \cdot) \in C_0(\mathbb{R})$, that is $\widehat{\mu} \in C_0(\mathbb{R})$ by the above, cannot in general be omitted in Theorem 7.3,(i).

Now we consider a little bit different situation when our auxiliary multiplication semigroup is isometric but not unitary. Let $\mathbb{C}^+$ denote the upper halfplane, and let $H^2(\mathbb{C}^+)$ be the Hardy space on $\mathbb{C}^+$.

Theorem 7.7. *For every bounded function $f : \mathbb{R} \to [0, \infty)$ such that $\lim_{|t|\to\infty} f(t) = 0$ and every $\varepsilon > 0$ there exists a positive $g \in L^1(\mathbb{R})$ satisfying the following conditions:*

(i) $\operatorname{Re} \widehat{g}(t) > f(t), \qquad t \in \mathbb{R};$
(ii) $\displaystyle\int_{\mathbb{R}} \frac{\ln|g(s)|}{1 + s^2}\, ds > -\infty;$
(iii) $\widehat{g} \in C^\infty(\mathbb{R})$.

Proof. As in the proof of Theorem 7.3 we can assume that $f(t) = f(-t), t > 0$.

Let $H = H^2(\mathbb{C}^+)$. We identify elements from H with their boundary values. Consider the C_0-semigroup $(M(t))_{t\geqslant 0}$ on H defined by

$$(M(t)h)(s) = e^{its}h(s), \qquad h \in H^2(\mathbb{C}^+),\ t \geqslant 0,\ s \in \mathbb{R}.$$

Let G be the generator of M. Since

$$\begin{aligned} D(G) &= \{h \in H^2(\mathbb{C}^+) : ish \in H^2(\mathbb{C}^+)\}, \\ (Gf)(s) &= ish(s), \quad s \in \mathbb{R}, \end{aligned}$$

we have

$$C^\infty(G) = \big\{h \in H^2(\mathbb{C}^+) : s^k h \in H^2(\mathbb{C}^+) \text{ for every } k \in \mathbb{N} \cup \{0\}\big\} \tag{7.11}$$

$$\subset \big\{h \in H^2(\mathbb{C}^+) : s^k |h|^2 \in L^1(\mathbb{R}) \text{ for every } k \in \mathbb{N} \cup \{0\}\big\}.$$

Clearly, $(M(t))_{t\geqslant 0}$ is weakly stable, isometric and

$$\sigma(G) = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leqslant 0\}.$$

By applying Theorem 5.1 to the semigroup $(M(t))_{t\geqslant 0}$ we get $h \in C^\infty(G)$ such that

$$\mathrm{Re}\,\langle M(t)h, h\rangle = \mathrm{Re}\,\widehat{|h|^2}(-t) \geqslant f(t), \qquad t \geqslant 0.$$

Thus if $g(s) := |h|^2(-s), s \in \mathbb{R}$, then (i) is satisfied for $t \geqslant 0$. Since g is positive, $\mathrm{Re}\,\widehat{g}(t) = \mathrm{Re}\,\widehat{g}(-t), t \geqslant 0$, so that (i) holds for all $t \in \mathbb{R}$. Since $\sqrt{g(-s)} = |h|(s), s \in \mathbb{R}$, is modulus of a $H^2(\mathbb{C}^+)$-function, the condition (ii) holds as well by [11, Theorem 4.4.4]. Finally, by (7.11), $\widehat{g}$ is smooth, i.e. (iii) is true. □

8. Final remarks and problems

In this section we would like to draw one's attention to another property dealing with large weak orbits. Given a family $\{T_n : n \in \mathbb{N}\}$ of bounded linear operators on a complex Hilbert space H we are looking for elements x and y from H such that the weak orbit $\langle T_n x, y\rangle, n \in \mathbb{N}$, almost matches the sequence of norms $\|T_n\|, n \in \mathbb{N}$. Recall the following two results on the existence of large orbits in this sense, see e.g. [16, Theorem 37.17] and [16, Theorem 39.8] for their proofs.

Theorem 8.1. *Let H be a Hilbert space and $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(H)$. Then*

(i) *if $\{\alpha_n : n \geqslant 1\} \subset (0,\infty)$ satisfy $\sum_n \alpha_n^2 < \infty$, and if $\varepsilon > 0$, then there exist $x \in H$, $\|x\| \leqslant (\sum_n \alpha_n^2)^{1/2} + \varepsilon$ such that*

$$\|T_n x\| \geqslant a_n \|T_n\|, \qquad n \in \mathbb{N};$$

(ii) *if $\{\alpha_n : n \geqslant 1\} \subset (0,\infty)$ satisfy $\sum_n \alpha_n < \infty$, then there exist $x, y \in H$ such that*

$$|\langle T_n x, y\rangle| \geqslant \alpha_n \|T_n\|, \qquad n \in \mathbb{N}.$$

The above result is corollary of a deep result on a so-called "plank-problem" due to K. Ball [6] concerning points where the sequence of unit norms functionals is large.

Theorem 8.2. *Let H be a (complex) Hilbert space, and let $\{f_n : n \geqslant 1\} \subset H$ be a sequence of unit vectors. Let $\{a_n : n \geqslant 1\} \subset (0,\infty)$ be such that $\sum_n a_n^2 \leqslant 1$. Then for every $\varepsilon > 0$ there exists $x \in H, \|x\| = 1$, such that*

$$|\langle x, f_n\rangle| \geqslant a_n, \qquad n \in \mathbb{N}.$$

Unfortunately, the corresponding result for vector-valued functions is not known and it is not even clear how to formulate it in a right way. Thus the following problem seems to be quite interesting.

Problem. Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on a Hilbert space H, and let $f:[0,\infty)\to(0,\infty)$ be a function such that $f\in L^1(\mathbb{R}_+)$. Are there x and y in H such that

$$|\langle T(t)x,y\rangle|\geqslant f(t)\|T(t)\| \tag{8.1}$$

for all $t\in\mathbb{R}$?

The following statement illustrates the problem in a particular situation.

Theorem 8.3. *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on a Hilbert space H, with bounded generator A, and let $\varepsilon>0$. Then there exist $x,y\in H$ such that*

$$|\langle T(t)x,y\rangle|\geqslant \frac{\|T(t)\|}{(t+1)^{2+\varepsilon}}$$

for all $t\geqslant 0$.

Proof. We can assume that $\varepsilon\leqslant\frac{1}{6}$.

Let A be the generator of the semigroup $(T(t))_{t\geqslant 0}$. Let

$$s:=\max\{2,8\|A\|\sqrt{2+\varepsilon^{-1}}\}.$$

So for $0\leqslant t\leqslant s^{-1}$ we have

$$\|T(t)-I\|=\|e^{tA}-I\|\leqslant e^{t\|A\|}-1\leqslant 2t\|A\|\leqslant\frac{1}{4\sqrt{2+\varepsilon^{-1}}}, \tag{8.2}$$

where we use the estimate $e^a\leqslant 1+2a$ for $0\leqslant a\leqslant 1$. Thus $\|T(t)\|\leqslant 1+\|T(t)-I\|\leqslant\frac{3}{2}$.

For $n\in\mathbb{N}$ set $\beta_n=n^{-\frac{1+\varepsilon}{2}}$. Then

$$\sum_{n=1}^{\infty}\beta_n^2=1+\sum_{n=2}^{\infty}n^{-(1+\varepsilon)}<1+\int_1^{\infty}u^{-(1+\varepsilon)}\mathrm{d}u=1+\varepsilon^{-1}.$$

For $n\in\mathbb{N}$ set $t_n=s^{-1}\Big(1+\sum_{j=1}^{n-1}\beta_j\Big)$. In particular, $t_1=s^{-1}$. By Theorem 8.1, (i), there exists $x\in H$ such that $\|T(t_n)x\|\geqslant\beta_n\|T(t_n)\|$ for all $n\in\mathbb{N}$. Moreover, $\|x\|^2\leqslant 1+\varepsilon^{-1}$. We can clearly assume that $\|x\|\geqslant 2$.

Furthermore, by Theorem 8.2, there exists $y\in H$ such that for all $n\in\mathbb{N}$

$$|\langle T(t_n)x,y\rangle|\geqslant\beta_n\|T(t_n)x\|,\quad\text{and}\quad|\langle x,y\rangle|\geqslant\|x\|.$$

Moreover, $\|y\|^2\leqslant 2+\varepsilon^{-1}$.

We show that x,y satisfy

$$|\langle T(t)x,y\rangle|\geqslant\frac{\|T(t)\|}{s^3(t+1)^{2+6\varepsilon}} \tag{8.3}$$

for all $t\geqslant 0$.

For $0\leqslant t\leqslant s^{-1}$, by (8.2), we have

$$\begin{aligned}|\langle T(t)x,y\rangle| &\geqslant |\langle x,y\rangle|-|\langle (I-T(t))x,y\rangle|\\ &\geqslant \|x\|-\|I-T(t)\|\cdot\|x\|\cdot\|y\|\\ &\geqslant \|x\|\Big(1-2t\|A\|\sqrt{2+\varepsilon^{-1}}\Big)\geqslant\frac{3}{4}\|x\|\end{aligned}$$

$$\geqslant \quad \|T(t)\| \geqslant \frac{\|T(t)\|}{s^3(t+1)^{2+6\varepsilon}}.$$

Let $n \in \mathbb{N}$ and $t_n \leqslant t < t_{n+1}$. Set $h = t - t_n$. Then $h \leqslant t_{n+1} - t_n = \beta_n s^{-1}$. We have

$$\begin{aligned} |\langle T(t)x, y\rangle| &\geqslant |\langle T(t_n)x, y\rangle| - |\langle (T(t_n) - T(t))x, y\rangle| \\ &\geqslant \beta_n \|T(t_n)x\| - \|I - T(h)\| \cdot \|T(t_n)x\| \cdot \|y\| \\ &\geqslant \|T(t_n)x\|(\beta_n - 2h\|A\| \cdot \|y\|) \\ &\geqslant \|T(t_n)x\|\Big(\beta_n - \frac{\beta_n\|y\|}{4\sqrt{2+\varepsilon^{-1}}}\Big) \\ &\geqslant \frac{3\beta_n}{4}\|T(t_n)x\| \geqslant \frac{3\beta_n^2}{4}\|T(t_n)\|. \end{aligned}$$

Furthermore,

$$\begin{aligned} t &\geqslant t_n = s^{-1}\Big(1 + \sum_{j=1}^{n-1} j^{-\frac{1+\varepsilon}{2}}\Big) \geqslant s^{-1}\Big(1 + \int_1^n u^{-\frac{1+\varepsilon}{2}}\,\mathrm{d}u\Big) \\ &\geqslant s^{-1}(1 + 2(n^{\frac{1-\varepsilon}{2}} - 1)) \\ &= s^{-1}(2n^{\frac{1-\varepsilon}{2}} - 1). \end{aligned}$$

So

$$n \leqslant \Big(\frac{st+1}{2}\Big)^{\frac{2}{1-\varepsilon}}$$

and

$$\beta_n \geqslant \Big(\frac{st+1}{2}\Big)^{\frac{2}{1-\varepsilon}\cdot\frac{-(1+\varepsilon)}{2}} \geqslant \Big(\frac{2}{st+1}\Big)^{\frac{1+\varepsilon}{1-\varepsilon}} \geqslant \frac{2}{(st+1)^{1+3\varepsilon}}.$$

Hence

$$\begin{aligned} |\langle T(t)x, y\rangle| &\geqslant \frac{3\beta_n^2}{4}\|T(t_n)\| \geqslant \frac{3}{(st+1)^{2+6\varepsilon}}\|T(t_n)\| \\ &\geqslant \frac{3\|T(t)\|}{s^3(t+1)^{2+6\varepsilon}\|T(h)\|} \geqslant \frac{\|T(t)\|}{s^3(t+1)^{2+6\varepsilon}}. \end{aligned}$$

Thus, (8.3) holds.

Replacing y by $s^3 y$ and ε by $\varepsilon/6$ we get the required statement. □

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