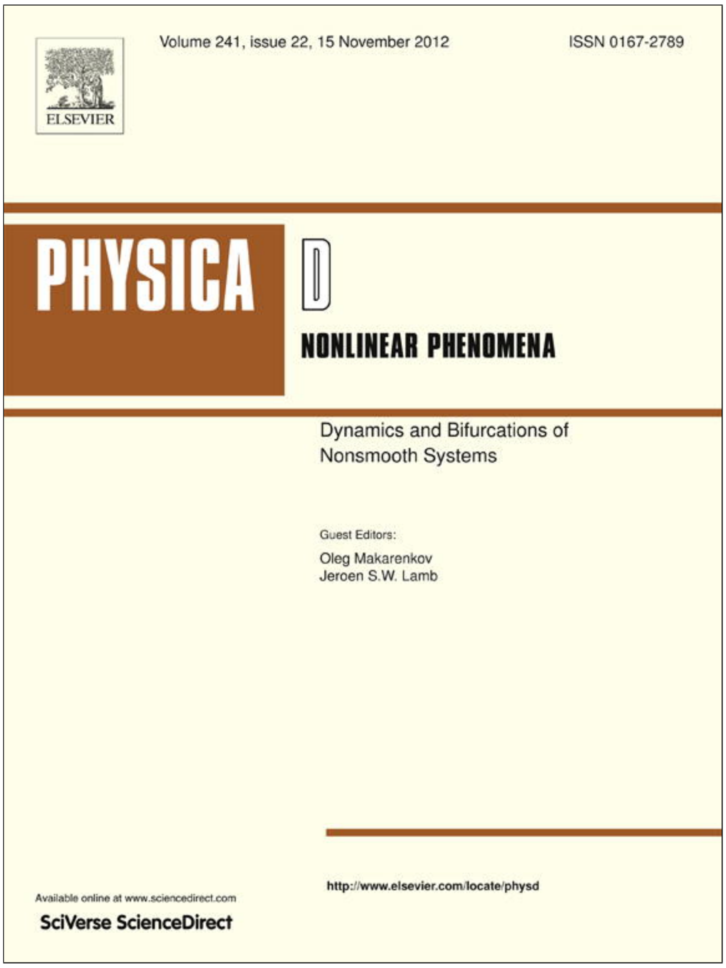

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NONLINEAR PHENOMENA

Dynamics and Bifurcations of Nonsmooth Systems

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Properties of solutions to a class of differential models incorporating Preisach hysteresis operator

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ABSTRACT

We consider a class of equations which have been recently proposed as a mathematical tool for modelling dynamics of hydrological, economic and biological systems exhibiting hysteresis and related memory effects. The detail of the modelling approach is illustrated by an example from the hydrological context where a balance equation is coupled with a hysteretic constitutive relationship between the water content θ in the soil and the matric potential ψ_m of the soil matrix and where the Preisach hysteresis operator is used as a model of this constitutive relationship. In particular, we present assumptions which eliminate spatial variation and lead to balance equations in the form of ODEs; two examples of such hydrological models are considered followed by a less detailed discussion of applications of similar modelling approach and equations in economics and biology. In the proposed formalism, the closed system is described by an operator-differential equation where the rate of change of the output of the Preisach operator is a function of its input and time. In the main part of the paper, for such operator-differential equations, we study the initial value problem: uniqueness, existence, extendability of solutions, their dependence on initial data, and the structure of the projection of the phase portrait onto the (t, ψ_m)-plane. Solutions are characterised by jumps of the derivative induced by either of the two reasons; one is the memory of past extremum values of the solution; the other is a singularity at the lines of zero flow. We analyse the singularity and calculate the value of the jumps thus providing an important input to numerical solution of the equation. Furthermore, we identify possible non-uniqueness points and points of sensitivity to small perturbations of initial data as well as conditions that ensure uniqueness and stability to such perturbations. Regularisation of the equation and natural monotonicity conditions ensuring global stability of a periodic solution for the equation with periodic input are discussed. Rigorous analysis of the well-posedness of the models is presented for the first time.

1. Introduction

1.1. Systems with hysteresis

The term hysteresis was first coined by Ewing in the late XIXth century referring to "a persistence of previous states" observed when ferric materials are magnetised [1]. The study of hysteresis produced a number of phenomenological and empirical models and techniques, of which the Preisach operator, introduced in [2], is of the main interest here. A mathematical framework and rigour was applied to models of hysteresis by a group of mathematicians led by Krasnosel'skii in their seminal work initiated in the 1970s [3].

Complex hysteresis operators can be conceived as being constructed from simpler, elementary hysteresis operators, or *hysterons*. The exact type of hysteron used and the nature of the connection between them determine the properties of the complex operator which they combine to form. Once such an operator is constructed, many of its properties can be analysed and deduced from those of the simpler hysterons. Examples of these hysterons include the *play* and *stop* operators, and the operator which underpins the Preisach model, the two-state *non-ideal relay* (also termed the thermostat nonlinearity, lazy switch, Schmitt trigger and hys-

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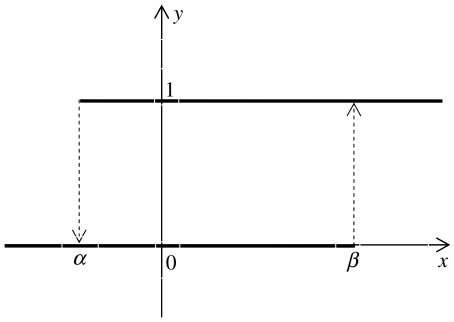

Fig. 1. Input–output diagram of the non-ideal relay.

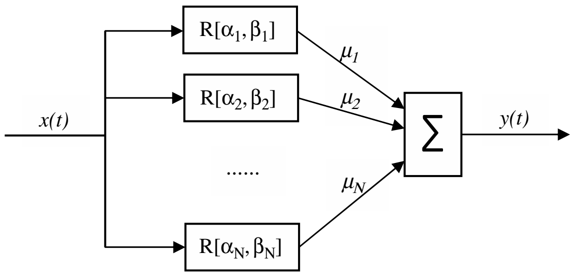

Fig. 2. Aggregations of non-ideal relays in the Preisach model. Relays $R[\alpha, \beta]$ with all possible threshold values $\alpha < \beta$ have a common input $x(t)$. These relays function independently of each other and contribute to the output $y(t)$ of the model, which is defined as the weighted sum (integral) of the outputs of the individual relays $R[\alpha, \beta]$ with weights $\mu(\alpha, \beta)$.

teretic relay in different applications).[2] For more details on the fundamentals of hysteresis models, including the Preisach operator, and their applications; see [6–11].

A non-ideal relay is characterised by two parameters α and β, the threshold values, with $\alpha < \beta$. The output $y(t)$ of the non-ideal relay equals either 0 or 1 at any moment for any continuous input $x(t)$. If the input value at some instant is below the lower threshold α then the output at this instant is 0 and it remains equal to 0 as long as the input is below the upper threshold value β. When the input reaches the value β the output at this moment jumps to the value 1 (the relay switches on). The output then remains equal to 1 as long as the input stays above the lower threshold value α — when the input reaches the value α the output jumps to back 0 (the relay switches off). This dynamics is captured by the input–output diagram in Fig. 1. If the input oscillates between two values X_-, X_+ such that $X_- < \alpha < \beta < X_+$, then the point $(x(t), y(t))$ moves along the rectangular counterclockwise hysteresis loop on this figure.

The Preisach model can be thought of as a collection of non-ideal relays (with all possible pairs of threshold values (α, β)) which respond to the same input $x = x(t)$ independently. The output of the model is the weighted sum of the outputs of all the relays as illustrated by the block diagram on Fig. 2. An interactive demonstration of the Preisach model can be found at [12].

A fundamental property of hysteresis nonlinearities is that they always define a rate-independent input–output relationship (operator), i.e. this operator is invariant with respect to the action of the group of affine transformations of the time scale. Omitting a few technicalities, hysteresis nonlinearities are defined in [7] as deterministic rate-independent operators with non-local memory (see also [13]). This general definition entails a set of non-trivial properties of hysteresis operators which are sufficient for developing formal concepts with various applications.

The main challenge of analysis of closed dynamical systems involving hysteresis operators is that these operators are intrinsically *non-smooth*. Moreover, the phase space of a system with a complex hysteresis operator, such as the Preisach operator, includes an infinite-dimensional component in the space of states (also known as memory configurations) of the hysteresis nonlinearity. This component is typically a metric space without a local linear structure. Therefore, the existing theory of such dynamical systems misses important tools of the smooth theory such as invariant manifolds, dimension reduction, and normal forms.[3] However, several methods of the theory of smooth dynamical systems have been adapted, and alternative methods have been developed, to analyse dynamics of systems with hysteresis operators including stability, oscillations, averaging, bifurcations, chaos and control (for a survey of some recent results; see, for example [15]).

1.2. Objectives

A new method of modelling rate-dependent hysteretic systems has been recently proposed and detailed in the context of hydrological, economic and biological applications in [16–20]. The method is based on a few assumptions about the response of the system to inputs varying on different time scales. In particular, it is assumed that in the adiabatic limit of slowly varying input the input–output characteristic is described by the Preisach model (this type of adiabatic response can be justified either by the phenomenological argument illustrated by Fig. 2 or by verifying that input–output data satisfy the conditions of Mayergoyz's identification theorem; see Section 2). In order to describe system response to arbitrary inputs, including dynamics near and far from the adiabatic limit, the Preisach input–output operator is coupled with an ordinary differential equation. The resulting operator-differential equations proposed as models of real-world systems (or their components) and tested on a number of particular examples, have the form

$$(Px)' = f(t, x) \tag{1}$$

where P is the Preisach operator, i.e. $y(t) = (Px)(t)$ is the output of the Preisach model with input x; here and henceforth prime demotes the time derivative.

While the theory of equations with hysteresis operators in the right-hand part (for example, those of the form $x' = g(t, x, Px)$) is fairly well developed, equations with the time derivative of the Preisach nonlinearity received less attention. The results in this direction refer to ordinary and partial differential equations where the time derivative applies to the combination $\varepsilon x + Px$ with $\varepsilon > 0$ (see, e.g. [7,21,22]). Eq. (1) features a number of quite different properties when compared to both these known types of equations with hysteresis. For example, the presence of the term $(Px)'$ is visibly revealed by the jumps of the derivative of solutions. Another feature is a possibility of non-uniqueness points. These features are manifestations of nonlinear effects of two types, namely, (a) non-local memory effects, and (b) a singularity, specific to equation (1), on particular curves of the (t, x)-plane.

The objective of this paper is to provide theoretical analysis of these nonlinear effects and their implications for dynamics of systems (1). In particular, we present a rigorous characterisation of, and explicit expression for, jumps of the derivative of

2 An input–output relation of a hysteron can be defined either explicitly or through a variational inequality, differential inclusion, or vibrostable differential equation as, for example, in dry friction models [4,5].

3 Systems with hysteresis operators exhibit dynamical and bifurcation scenarios different from those observed in smooth systems of ODEs; see, for example [14].

an individual solution, thus providing an important input to numerical algorithms such as modifications of the Runge–Kutta method developed in [23–25] for solving equation (1); these algorithms use special routines based on estimation of the jump of the derivative at points of non-smoothness. We then study the effect of negative feedback loops in the structure of the system on the well-posedness of the Cauchy problem (where initial data include an initial state of the Preisach operator). According to [26], monotonicity of the function f, which can be interpreted as dynamics driven by positive feedbacks in applications, ensures well-posedness of system (1) for arbitrary initial data, and, when coupled with further assumptions, leads to global stability. We show that the presence of negative feedback loops can result in more complex behaviour such as appearance of isolated non-uniqueness points defined by simple algebraic relationships. We develop further insight into the global properties of the semiflow by analysing its projection onto the (t, x)-plane, which allows us to identify domains of sensitivity to perturbation of initial data associated with non-uniqueness points. However, the structure of the semiflow within a cusp of solutions stemming from a non-uniqueness point remains unclear.

Surprisingly, some aspects of the behaviour of solutions, which we consider below, make analysis of, and modelling with, Eqs. (1) simpler, than that of the equations where the hysteresis term appears in the right-hand part (see, Theorem 3.5 and Corollary 3.6). The right-hand side of (1) can have quite different form depending on the components of the system: some examples are discussed in the next section. Motivated by this fact, as well as by potential applications of model (1) to power electronics, economics and biological sciences discussed below, we consider common properties of solutions of Eqs. (1) with right-hand parts f from general classes.

Equations $(\varepsilon x + Px)' = f(t, x)$ with $\varepsilon > 0$ are more regular than (1). Using the inverse operator $(\varepsilon I + P)^{-1}$, one can reduce them to equivalent equations of the form $x' = g(t, x, (\varepsilon I + P)^{-1}x)$ and thus get rid of the time derivative of the hysteresis term. When applied to (1), this approach leads however to a different situation, because the inverse P^{-1} of the Preisach operator is not Lipschitz continuous. Systems $(\varepsilon x + Px)' = f(t, x, u)$, $u' = g(t, x, u)$ and similar higher order systems have been used for modelling ferroresonance in power electronics systems [27] and as prototypal models of electronic oscillators [28,29] with the Preisach operator introducing the hysteretic constitutive relationship between the magnetisation and the magnetic field in transformers and other inductance elements with ferromagnetic core. The reason to consider equation (1) in the hydrological context is that identification by fitting experimental data performed in [17] tends to give rise to these equations rather than to those containing the time derivative of the combination $\varepsilon x + Px$. It should be noted that in applications to electrical circuits with magnetic hysteresis elements the parameter ε in $\varepsilon x + Px$ is small. Eq. (1) can be obtained as the limit of equations $(\varepsilon x + Px)' = f(t, x)$ as $\varepsilon \to 0$ [26].

This paper is organised as follows. In the next section we present a detailed discussion of hydrological models leading to equations of the form (1) and briefly survey further applications of this equation to modelling economic flows and biological systems. This section contains also concise description of the Preisach operator. The discussion of hydrology follows the presentation in [16]; complete mathematical proofs are presented for the first time in this paper. In Section 3 we consider general properties of solutions of the initial value problem for equation (1) with a smooth right-hand part, including uniqueness, existence and continuation of solutions and characterisation of the jumps of the derivative. In particular, we identify isolated points where non-uniqueness can occur; then, stratification of the projections of solutions onto the (t, x)-plane, which seems to be a peculiar property of equation (1), and sensitivity of solutions to perturbations of initial data in the domain of uniqueness are addressed. Section 4 contains a survey of further existing results on dynamics of equation (1) and the corresponding literature. We briefly discuss global stability of periodic solutions in systems with decreasing right-hand side driven by periodic inputs; regularisation of equation (1), which underpins numerical algorithms for solving this equation; and, equations driven by discontinuous inputs. Proofs of the results from Section 3 are presented in Section 5. The last section contains conclusions.

Equation (1) can be used to model a component of a larger system. In this case, the complete model can naturally consist of a system of ordinary differential equations coupled with operator-differential equation (1); see, for example [20]. Some results of this paper (for example, existence of solutions) can be extended to such higher order systems straightforwardly. However, uniqueness, stability and systematic rigorous analysis of non-smoothness points are an open problem. Stratification property (see, Theorem 3.5) seems to be specific to the scalar equation and does not carry over to higher-dimensional systems.

2. Examples of models

2.1. Hydrological models

2.1.1. Hysteresis in hydrology

The important role of hysteresis in hydrology and soil physics has been known for a long time. Hysteresis manifests itself through the fact that it is easier (i.e., less thermo-mechanical work is required) to put water into soil than to remove it afterwards [30]. The physical nature of this effect, often referred to as hysteresis in porous media, or moisture hysteresis, or capillary hysteresis, is rather complicated. Small pours of the solid soil matrix in unsaturated soil are filled partly with air and partly with water. The origin of the hysteretic is attributed to capillary effects and adhesion forces acting on the boundaries between the three phases. In [30] several factors causing hysteresis are considered, namely, (a) rapid flipping in the liquid–solid contact angle; (b) geometric nonuniformity of individual pores; (c) different spatial connectivity of pores during drying and wetting; and (d) air entrapment. However, it is now understood that the fundamental mechanism of hysteresis is (a), whereas (b)–(c) enhance the effect [31].

The soil moisture hysteresis is quantified in terms of different physical characteristics of the soil, most commonly as hysteresis relation between the moisture content θ and the matric potential ψ_m (or the capillary pressure). The variation of θ with ψ_m describing the ability of the soil to store or release water is called water retention characteristic. When considered on the time scales of water flow this characteristic is commonly assumed to be rate-independent. Rate-independence allows one to use two-dimensional images to illustrate water retention curves on the (ψ_m, θ) plane. The essence of the soil-water hysteresis is in the fact that there are infinitely many pairwise different water retention curves, hence the dependence between water content and matric potential is non-functional. The value of θ at a given instant depends not only on the simultaneous value of ψ_m but also on the previous history of wetting and drying processes in the soil. The point (ψ_m, θ) follows different water retention characteristic curves when $\psi_m = \psi_m(t)$ increases (wetting curves) and when ψ_m decreases (drying curve) switching to a new curve each time the function $\psi_m = \psi_m(t)$ has an extremum and creating hysteresis loops; see Fig. 3. This picture is similar to the familiar description of hysteresis in ferromagnetic materials in terms of the nested structure of magnetisation curves and

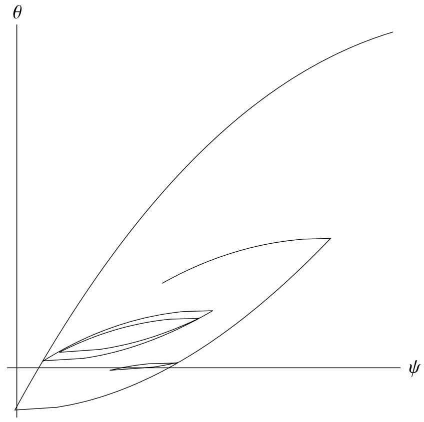

Fig. 3. Water retention curve of a soil with hysteretic soil-water characteristic (arbitrary units).

hysteresis loops on the diagram showing the magnetisation of the material vs the applied magnetic field. The rate-independence property distinguishes hysteretic memory from other memory types such as in systems with delayed argument or convolution operators. It implies that an instant value of θ (considered as an output) is defined by a sequence of certain extremum values (called the shock values) of the input ψ_m in the past. In other words, it is the shock values of the input which are memorised and have permanent effect on the output in the future. The magnetisation of ferromagnetic material by a fluctuation of external magnetic field which creates a permanent magnet is a classical example of such a memory.

Several phenomenological and mathematical models, including those based on the independent domain theory, have been proposed to describe the hysteretic constitutive relationship between ψ_m and θ [32–40], among them the Preisach model.[4] (see, e.g. [2,6,8]). These models were successfully used in prediction and detection problems where the input or the output is known (measured).

Models of closed hydrological systems predicting dynamics of the soil-water content in a slab of soil have two components: a balance equation relating the rate of change of θ to the gradient of potentials $\nabla\psi_m$, and the constitutive relationship between θ and ψ_m describing the water retention characteristic of the soil. These two components are independent and form together a closed model. In classical equations of soil physics the constitutive relationship is functional, i.e. it relates the simultaneous values of θ and ψ_m. This functional relationship can be replaced by a hysteretic constitutive relationship in order to account for hysteresis effects and refine the model.

In particular, this idea has been implemented in [18,16] leading to a class of models of the form (1) where the hysteretic constitutive relationship between the moisture content and the matric potential was modelled by the Preisach operator. The theory of rate-independent hysteresis operators and operator-differential equations with such operators provided the modelling framework and the necessary mathematical tools for the analysis of the resulting equations. Parameters of the Preisach operator used as part of the model were identified from a large experimental soil database [17]. A crucial step for identification was to design a class of Preisach measures broad enough to accommodate all the data from diverse soil samples, but at the same time depending on a small number of parameters to ensure a simple fitting procedure. The results showed that the Preisach nonlinearity with measures from the proposed class provide better accuracy than most standard models. The main clue at the stage of non-parametric identification preceding the parametric one was Mayergoyz's theorem (see, [6]) that identifies Preisach hysteresis in terms of a few simple criteria applied to the shape of hysteresis loops; these criteria are considered in Section 2.1.3. The closed model coupling the balance equation with the Preisach constitutive relationship was tested by rainfall and moisture content data collected from the lower Feale watershed in co. Kerry, Ireland [43].

[4] The Preisach model is also a classical tool for modelling magnetic hysteresis (see e.g. [10,41]). In recent years, its applications have been extended to plasticity, superconductivity, piezoelectricity, phase transitions, economics and several other areas [42].

2.1.2. *Examples of balance equation*

Accurate models of hydrological systems require partial derivatives for the local description of water potential and the conservation of soil water. However, under particular assumptions all spatial variation can be excluded from the full Philip–Richards equation (see, [18]). Taking the soil slab characteristics and outer conditions to be uniform in the horizontal directions, we eliminate spatial variation parallel to the Earth surface. We further assume that

1. A preferential flow network due to decayed roots, animal burrows, and wormholes in the soil presents the rain uniformly to the porous matrix allowing it to infiltrate; the preferential flow paths have cross sections that are much larger than the paths in the soil matrix where surface forces are active.
2. For vegetated soil, transpiration is treated as capillary flow through a space-filling 'wick' of roots extending from all points in the slab to a uniformly transpiring canopy above the surface of the slab, and
3. A uniform matrix of soil surrounds the slab-filling wick and the preferential flow network.

These assumptions eliminate spatial variation in vertical direction, hence the water balance in the soil can be described by the ordinary differential equation

$$\theta' = J(t) \tag{2}$$

that says the rate of change of water volume $\theta(t)$ per unit area of soil column (moisture content) is due to a vertical flow of water through horizontal unit area; here $J(t)$ is the specific flow rate (flux). We assume that the flux is a function of the total potential ψ of the water in the soil sample and outer conditions. Hence $\theta' = J(t, \psi)$ where ψ is the total potential defined as the sum of the two partial potentials: the matric and gravitational potentials. The matric potential ψ_m is defined to be the thermo-mechanical work that must be performed on the soil-water to bring it to a common reference state against all surface forces at the same reference level as the gravitational potential.

The first example of the balance equation we obtain by adjoining to (2) the simplest possible form of Darcy's law for flow through porous media

$$J(t) = k(\psi_{ref}(t) - \psi_m(t)), \tag{3}$$

which says the specific flow rate is proportional to a difference in matric potential while the corresponding difference in gravitational potential is negligible. Here ψ_{ref} is an arbitrary oscillating reference potential that drives the flow; it is defined by the outer conditions on the surface of the soil and below the soil slab, which are driven by the climate and weather factors such as the rainfall pattern. Coupling equations (2), (3) with the Preisach constitutive relationship $\theta(t) = (P\psi_m)(t)$ between ψ_m and θ results in an important example of equation (1), which has been examined numerically in [23]. An extension of this model called the 'hysteretic reservoir', which involves a modified Preisach type constitutive relationship, has been suggested and considered in the hydrological context in [44,45]. A stochastic counterpart of model (2), (3) with the rainfall modelled by a stochastic process has been studied in [46]. For an overview of applied modelling hysteretic open-loop systems with stochastic inputs and outputs by the Preisach operator we refer to [47]; see also [48,49].

As a next example, we consider a more involved model suggested in [18] to provide a testbed for examining and demonstrating as clearly as possibly the effect of hysteresis on selected processes in a one square metre column of vegetated soil reacting to the atmosphere. It is a model of a fully vegetated slab of soil of constant thickness L with transpiring plants, called the FEST model. We assume that the water flow J to and from the soil slab has three components, namely, infiltration of rainfall, drainage to and capillary rise from a water table below the slab and transpiration from plant leaves.

Hence, equation (2) reads

$$\theta' = J_I - J_D - J_T, \tag{4}$$

where J_I is the rate of infiltration of rain water, J_D the rate of drainage below the soil slab, and J_T the rate of transpiration from the soil slab. The water volume θ per unit area of soil column is measured in the units of length; it is nonnegative and satisfies $\theta \le \theta_{sat} L$ where $\theta_{sat} < 1$ is the dimensionless saturation concentration of water in the soil. The concentration of water is assumed to be uniform throughout the slab.

In the engineering system of units the soil water matric potential ψ_m has the dimension of length (the so-called hydraulic head) and the gravity potential is $-z$, where the distance z is measured downwards from the soil surface; the reference zero state is free water at the surface of the slab. When writing the flows J_I, J_D, and J_T as functions of the matric potential ψ_m and time, we again assume that each flow is driven by the appropriate difference in potential energy. Accordingly, the rate J_D of drainage is supposed to be driven by the difference in total potential between the centre of the soil slab and its base. We assume saturation immediately below the slab and a matric potential of zero. Consequently,

$$J_D = (\psi_m + L/2)/B \tag{5}$$

where the parameter B is the associated adjustment time. This can be interpreted as a negative feedback loop with the local equilibrium of $\psi_m = -L/2$ where matric forces in the soil hold a quantity of water against gravity.

The rate J_I of infiltration is the lesser of two quantities at any time

$$J_I = \min(-\psi_m/A, Q(t)), \tag{6}$$

where $Q = Q(t)$ is the rainfall rate per unit area and the parameter A is a second adjustment time. According to our assumption, the soil slab contains a network of macro-pores, which bring rainfall into uniform contact with the micro-pores of the soil.[5] When the first term in the expression for J_I is less than the rainfall rate $Q(t)$, ponding of water on the surface of the slab is said to occur, which means that the soil cannot absorb, or imbibe, the available rain from the macro-pores (the excess rain runs off immediately into a surface drain). In this case, absorption, or imbibition, of water by the soil is assumed to be driven by the difference between the potential of the water at saturation in the macro-pores, and its potential ψ_m in the soil. When the first term in the expression for J_I is greater than $Q(t)$, all the rain is absorbed immediately. This can be read as a negative feedback loop driving the moisture concentration to saturation and the associated matric potential to zero.

When considering the last component J_T of the flow, we assume vegetation to have a uniform system of roots, which allows transpiration from all parts of the soil slab, and no evaporation from the surface of the soil (because it is prevented by the vegetation).[6] The rate of transpiration is taken to be

$$J_T = E_p(h_0 - h_a)/(1 - h_a), \quad 1 \ge h_0 \ge h_a, \tag{7}$$

where $E_p = E_p(t)$ is the maximum rate of transpiration controlled by conditions in the atmosphere, $h_a = h_a(t)$ is the relative humidity of the atmosphere, and h_0 is the relative humidity of the vacuoles in the leaves through which the plants transpire. Consequently, when $h_0 = 1$, transpiration is at the maximum rate; when $h_0 < 1$, due to the suction of the plant water in iso-potential contact with soil water near the plant roots, transpiration up the plant xylem is less than the maximum rate; transpiration ceases when $h_0 = h_a$. The plant humidity h_0 can be related to the total potential $\psi_m + z$ at the canopy height $z = z_c$ by means of the thermodynamic equation of soil physics:

$$h_0 = \exp(M_w(\psi_m + z_c)/(RT)),$$

where R is the gas constant, T the absolute temperature and M_w the molecular weight of water. Substituting this expression in (7) gives the rate of transpiration as a function of the soil matric potential ψ_m. This relation, completing the balance equation, can be interpreted as a negative feedback loop, where the potential of soil-water drops, until the humidity of the plant vacuoles becomes equal to the humidity of the atmosphere, and transpiration ceases.

The above examples demonstrate that components of the water flow can represent diverse classes of functions even in simple models. If the coefficients in the expressions for the partial flows in FEST model are constant (which is a first instance approximation), then balance equation (4) contains linear, saturation and exponential terms in ψ_m. In more accurate models these coefficients however should become functions of ψ_m and t, reflecting more complicated feedback loops, which leads to further broadening of the class of different nonlinear terms appearing in the balance equation. For example, the hydraulic conductivity k in equation (3) is more correctly regarded as a function of ψ_m or θ, see [50], and can also exhibit hysteresis; A and B in (5), (6) are also not constant. Positive feedback loops can be present in the system too; for example, the surface runoff flow increases when the soil dries. This effect generates a positive feedback loop leading to important bifurcation scenarios [51]. As stated in the introduction, this motivates analysis of equations of the general form (1).

Equations presented in this section capture important features of dynamics of more complicated spatially distributed systems, thus helping understand them and providing an insight into the affect of hysteresis on the system. Inhomogeneous effects in hysteretic soils are modelled, for example, in [52].

2.1.3. Preisach constitutive relationship

The phenomenology of the Preisach model described in the introduction is used universally in physics applications and across several other disciplines.[7] In hydrology, the non-ideal relays shown in Fig. 2 model the behaviour of small pores of the soil matrix, which act like 'ink bottle' according to Haines [32]. A pore is filled at one threshold value of the pressure and emptied at

5 The macro-pores have a volume of zero in this simplified treatment.

6 Evaporation from the slab surface cannot be included in FEST because it requires a time-dependent supply of water upwards through the soil and a minimum of one space dimension is required to model this. Transpiration is a different process because the plant roots penetrate the soil matrix throughout the root zone.

7 The class of the so-called *operators of Preisach type* [8], also known as *return point memory operators*, is even more universal. As shown in [53], it includes the class of spin interaction models such as the Ising model, which are common in modelling avalanches and phase transition with applications including magnetic materials, earthquakes, percolation and random networks [54–60].

another threshold value of the pressure due to capillary effect. The variation of pore size and geometry leads to the variation of the threshold values. In the independent domain model, the filling/emptying processes in different pores are deemed to be independent. This interpretation of the wetting/drying processes in the soil matrix is used as justification of modelling the constitutive relationship between ψ_m and θ with the Preisach operator on the phenomenological level.

An empirical approach to validating the applicability of the Preisach operator for modelling the constitutive relationship between ψ_m and θ is based on the identification theorem. This approach also provides tools for fitting the model from real data. The experiments described by Haverkamp et al. in [39] show that the hysteresis relation between the matric potential ψ_m and the water content θ exhibits the so-called return point memory (or wiping-out property). That is, in case of a cyclic process, every minor hysteresis loop on the (ψ_m, θ)-plane returns back to its starting point and thus is closed. Furthermore, it is expected from experiments that periodic processes on the same potential level may take place with different amounts of water content. Assuming that the mass balance is given by the equation $\theta'(t) = f(t, \psi_m(t))$, this observation leads to the conclusion that all possible θ's corresponding to the same process $\psi_m(t)$ differ only by an additive constant independent of t. In the (ψ_m, θ) phase plane, this means that all simple closed loops with the same projection on the ψ_m axis have the same shape. This property is called the congruency of loops and a classical result by Mayergoyz (see, [6]) states that every hysteresis relation with return point memory and congruent loops can be represented by the Preisach model.

It is important to note again that modelling of the constitutive relationship with a hysteresis operator implies rate-independence of this relationship. We believe this to be a good approximation for the processes in soil.

2.2. Applications to economics and biology

Many processes in nature and society may be interpreted as flows modelling an aggregated result of a large number of elementary exchange operations. Often a characteristic feature of an elementary exchange is that there is an associated cost to be paid for it: it may be a direct commission charge, or a sunk cost, or it may be a psychological or risk-based cost (thus, the resulting flows are irreversible). In these situation, equations of the type (1) with a hysteresis operator can provide an adequate mathematical model of the flows. Below we discuss in brief some recent developments in this area.

2.2.1. Macro-economical flows

The current mainstream models of macro-economics originate in the "neoclassical revolution" period of the late XIXth century. The protagonists in this revolution, such as Walras, Edgeworth and Jevons, applied paradigms and analogies drawn from Newtonian mechanics to economic systems. A commonly used metaphor compared economic systems with a set of connected water reservoirs at different levels, Edgeworth himself is one of those who used this analogy. Later economists such as Fisher, [61], and Phillips, [62], constructed actual hydro-mechanical machines for the determination of market prices and of macro-economic flow variables such as output (respectively). Indeed, a number of "Phillips machines", or MONIACs, were built to order, both for study and policy making.

One further extension of the above metaphor, which could be fruitful (for a number of reasons) when applied to modelling economics systems, appeals to water flows through porous media as an analogue in physics. A model of an economic flow based on aggregation of elementary exchange operation and using this metaphor has been proposed in [19]. The model has the form of a hysteretic differential-operator equation similar to (1). In a further paper [63], the model has been applied to two fundamental problems in economic analysis concerning the determination of aggregate output, and the determination of market prices and quantities. The paper [64] dealt with the question whether the recessions following a financial crisis should have permanent effects on output, employment and unemployment. The model proposed in this paper is again a slight modification of equation (1). A survey of applications of hysteresis to economic systems can be found in [65–67].

2.2.2. Biological applications

Adapting the behaviour to the changes in the environment, as well as memory of the previous history, is typical for human communities and animal populations. An obvious example of such adaptation is switching to a "safe mode of behaviour" when a danger, either from a predator species, or from an infectious disease, becomes apparent. In the human society such switching to a safe behaviour is particularly evident in the instance of an epidemic.

Mathematically, the changes of behaviour in response to changes in the conditions can be described by models with switching. In most cases the switching assumed to depend on the system state disregards the history and the memory. Memory can be introduced into a mathematical model by non-ideal switches (relays) and aggregated sums of such switches such as the Preisach hysteresis operator. In [20], it is demonstrated how hysteresis, and, in particular, equation (1) can arise in this type of model, and how it may be applied to describe the memory effects. Another objective of this paper has been to introduce a new unified paradigm for mathematical modelling of certain memory effects in epidemiology and ecology.

2.3. Mathematical formulation of Preisach operator

In the following analysis, we use an alternative definition of the Preisach model due to [3,9], which is equivalent to the phenomenological approach based on the aggregation of relays. It uses the decomposition of the Preisach operator into the superposition of elementary hysterons of another type, namely plays, and provides a simpler geometrical interpretation of the evolution of states as well as a convenient mathematical framework for the analysis. Namely, the output of the model is a scalar function defined by the formula

$$y(t) = \text{mes}_{\mu}\,\omega(t) := \int\int_{\omega(t)} \mu(\alpha, \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \tag{8}$$

where $\mu : \Pi \to \mathbb{R}$ is a nonnegative continuously differentiable integrable function, $\Pi = \{(\alpha, \beta) : 0 \le \beta - \alpha \le d\}$ is a strip in the (α, β)-plane, and the domain $\omega(t) \subset \Pi$ of integration changes in time. Points (α, β) of the domain $\omega(t)$ represent the relays $R[\alpha, \beta]$ which have the state 1 at the moment t, the other relays are in the state 0 (cf. Fig. 2). For each t, the domain has the form

$$\omega(t) = \{(\alpha, \beta) \in \Pi : \alpha + \beta \le 2x(t) + \eta(t; \beta - \alpha)\}, \tag{9}$$

where $x(t)$ is a scalar continuous input of the model and the function $\eta(t; \cdot) : [0, d] \to \mathbb{R}$, called the state of the model, satisfies

$$\eta(t; 0) = 0; \qquad |\eta(t; \xi_1) - \eta(t; \xi_2)| \le |\xi_1 - \xi_2| \tag{10}$$

for all t and $0 \le \xi_1, \xi_2 \le d$. The evolution of the state $\eta = \eta(t; \cdot)$ (and hence, the evolution of ω) is determined by the input with the following simple rules. If the continuous input is monotone on a segment $[t_1, t_2]$, then on this segment

$$\eta(t; \xi) = \max\{-\xi, \eta(t_1; \xi) - 2x(t) + 2x(t_1)\} \tag{11}$$

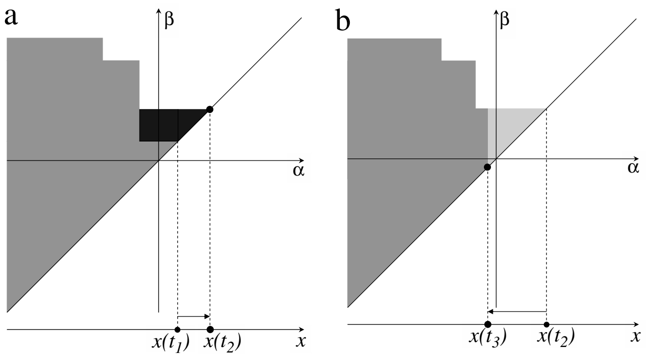

Fig. 4. Evolution of the domain of integration $\omega(t)$ in (8) defined by formulae (9)–(12). This domain is shown as the coloured area. The input $x(t)$ moves along the horizontal axis and controls the point on the diagonal $\alpha = \beta$ above itself. When moving towards the upper right corner (x increases), this point on the diagonal drags the horizontal line and colours the domain below this line and above the diagonal. When moving towards the bottom left corner (x decreases), the diagonal point drags the vertical line and subtracts the area to the right of this line from the coloured domain. (a) Domain $\omega(t_1)$ is grey; when x increases from the value $x(t_1)$ to the value $x(t_2)$, the coloured area is increased by the black trapezium; $\omega(t_2)$ is the join of the grey and black areas. (b) Input decreases from the value $x(t_2)$ to the value $x(t_3)$; the coloured area is decreased by the light grey triangle; the dark grey area is $\omega(t_3)$. An online applet explaining and implementing these evolution rules is available at [12].

for any x increasing input, and

$$\eta(t;\xi) = \min\{\xi, \eta(t_1;\xi) - 2x(t) + 2x(t_1)\} \tag{12}$$

for any decreasing input; see Fig. 4. Now, a standard two-step procedure is used to define the memory state and the output for any continuous input $x : \mathbb{R}_+ \to \mathbb{R}$ (see, e.g. [6,8,3]). First, formulae (11), (12) are combined with the semigroup identity to define the memory state $\eta(t;\cdot)$ at every moment $t \geq t_0$ for any continuous piecewise monotone input $x : [t_0, \infty) \to \mathbb{R}$ and any initial state $\eta(t_0;\cdot) = \eta_0(\cdot) \in W$, where t_0 is an initial moment and W is the class of functions satisfying (10). Then, formulae (8), (9) define the corresponding output $y : [t_0, \infty) \to \mathbb{R}$, which we denote by

$$y(t) = P[\eta_0]x(t), \tag{13}$$

reflecting the fact that it depends both on input and initial state. Because the function $\eta = \eta(t;\xi)$ is continuous in both arguments due to (11), (12), the output of the Preisach model is also continuous. This completes the definition of the model for piecewise monotone continuous inputs.

As the second step, one extends the input–output operator (13) to the whole class $C([t_0,\infty),\mathbb{R})$ of continuous inputs from the subset of piecewise monotone ones by continuity, based on the estimate $\|P[\eta_0^1]x_1 - P[\eta_0^2]x_2\|_{C[t_0,t]} \leq M(\rho(\eta_0^1,\eta_0^2) + \|x_1 - x_2\|_{C[t_0,t]})$ that holds for every $t \geq t_0$, where $\rho(\eta_0^1,\eta_0^2) = \max_{\xi\in[0,d]} |\eta_0^1(\xi) - \eta_0^2(\xi)|$ measures the distance between initial states in the state space W and the uniform norm $\|x_1 - x_2\|_C = \max_{s\in[t_0,t]} |x_1(s) - x_2(s)|$ in the space of continuous inputs and outputs is used. The operators $P[\eta_0]$ are globally Lipschitz continuous in $C([t_0,\infty),\mathbb{R})$ but not differentiable. A similar continuous extension argument defines the state $\eta(t) = \eta(t;\xi)$ at any moment $t > t_0$ for any continuous input $x : [t_0,\infty) \to \mathbb{R}$ and any initial state $\eta_0 \in W$. We note that because the state and output on any finite interval $J = [t_0, t_1)$ or $J = [t_0, t_1]$ depend on the input values on the same interval only (i.e., the Preisach model satisfies *causality* property) and do not depend on the 'future', the outputs and states are naturally defined for continuous inputs with a finite domain J.

Everywhere in this paper we assume that there is a strip $0 \leq \beta - \alpha < d_1$ with $d_1 \leq d$ where the function $\mu = \mu(\alpha,\beta)$, which is used to define the output of the Preisach model, is positive:

$$\mu(\alpha,\beta) > 0 \quad \text{for } 0 \leq \beta - \alpha < d_1. \tag{14}$$

3. Cauchy problem

3.1. Problem statement

Consider the equation

$$y' = f(t,x) \tag{15}$$

with $y = y(t)$ and $x = x(t)$ related by the Preisach operator (13):

$$y(t) = (P[\eta(t_0)]x)(t), \quad t \geq t_0, \tag{16}$$

where $\eta(t_0)$ is an initial state of the Preisach nonlinearity. The function f is assumed to be continuously differentiable. System (15), (16) is equivalent to (1).

We start with a standard range of questions related to the Cauchy problem, namely existence, uniqueness, continuability of solutions and continuous dependence on initial data. The initial conditions for system (15), (16) have the form

$$x(t_0) = x_0, \qquad \eta(t_0) = \eta_0. \tag{17}$$

Because inputs and outputs of the Preisach model are defined for $t \geq t_0$ only (the hysteresis nonlinearity is not time-reversible[8]), system (15), (16) is considered for the same t, i.e. its solutions are not defined for $t < t_0$.

An initial state η_0 can be considered either as initial data or as an infinite-dimensional parameter of system (15), (16). Given any $\eta_0 \in W$, we shall call a continuous function $x = x(t)$ defined on an interval with the left end at the point t_0 a solution of system (15), (16) if the output (16) of the Preisach nonlinearity is continuously differentiable and equation (15) is satisfied everywhere on this interval.

We shall see that the set

$$F_0 = \{(t,x) : f(t,x) = 0\}$$

of zeros of f plays a special role, because solutions $x = x(t)$ have extrema and jumps of the derivative at the points of this set. Solutions starting from the subset

$$\Omega = \{(t,x) : f(t,x) = 0,\ f_t(t,x) = 0\}$$

of the set F_0 are excluded from our consideration. Moreover, we first consider solutions in the domain $\mathbb{R}^2 \setminus \Omega$ of the (t,x)-plane $\mathbb{R}^2$ only. Solutions that intersect the set Ω require additional analysis; they are discussed in Section 3.4 to some extent. We shall call Ω the *dangerous set*.

Generically, the set F_0 is a collection of curves on the (t,x)-plane. For example, this is the case if the gradient of f is non-zero on the set F_0. The dangerous set Ω is defined by a system of two equations with two arguments. Hence, generically, the set Ω consists of isolated points, i.e. Ω is at most countable and has at most finite number of points in any bounded part of the (t,x)-plane (for example, this is true if f is twice continuously differentiable and its derivatives satisfy proper non-degeneracy conditions on the set F_0).

[8] An attempt to continue solutions backwards in time leads to multiplicity of solutions with infinitely many branching points.

3.2. Well-posedness results

Well-posedness includes existence and uniqueness results and continuous dependence of solutions of system (15), (16) on initial data (17).

Suppose that $(t_0, x_0) \notin \Omega$. Moreover, in order to be simple we impose a natural additional requirement on initial conditions. Let us say that a state $\eta_0 = \eta_0(\xi) \in W$ has a *vertical* initial segment if $\eta_0(\xi) = \xi$ on a nonempty interval $0 \le \xi < \xi_1$, and a *horizontal* initial segment if $\eta_0(\xi) = -\xi$ on a nonempty interval $0 \le \xi < \xi_1$.[9] We then call initial data (17) *admissible* (or, alternatively, we say that the state η_0 is *admissible* for initial data t_0, x_0) if η_0 has a vertical initial segment whenever either $f(t_0, x_0) < 0$ or $f(t_0, x_0) = 0, f_t(t_0, x_0) > 0$, and η_0 has a horizontal initial segment whenever either $f(t_0, x_0) > 0$ or $f(t_0, x_0) = 0$, $f_t(t_0, x_0) < 0$. As it is shown below, admissibility is a generic property in the sense that if initial data (17) are either admissible or $(t_0, x_0) \notin F_0$ then the state $\eta(t)$ is admissible for the pair $(t, x(t))$ at any moment $t > t_0$ where $x(t)$ is a solution of problem (15)–(17).

Theorem 3.1 (*Local Existence and Uniqueness*). *For any admissible initial data the Cauchy problem (15)–(17) has a unique solution on some interval $t_0 \le t < t_1$.*

Let us call a solution $x : [t_0, \tau) \to \mathbb{R}$ of the Cauchy problem (15)–(17) *admissible* if its graph lies in the domain $\mathbb{R}^2 \setminus \Omega$ of the (t, x)-plane for all $t \in [t_0, \tau)$ and the initial data are admissible. The theorem above implies that an admissible solution $x = x(t)$ can be extended to a maximal interval $t_0 \le t < T$ with a finite or infinite $T = T(t_0, x_0, \eta_0)$ within the domain $\mathbb{R}^2 \setminus \Omega$. In other words, any solution of problem (15)–(17) such that its graph does not intersect the dangerous set Ω is a restriction of the unique admissible solution defined on the maximal interval $t_0 \le t < T$ to a smaller interval. Further information is contained in the following theorem.

Theorem 3.2 (*Extension of Solutions*). *For any admissible initial data the solution $x = x(t)$ of the Cauchy problem (15)–(17) can be uniquely extended to the maximal interval $t_0 \le t < T$ (finite or infinite) such that the graph of x lies in the domain $\mathbb{R}^2 \setminus \Omega$ for all t from this interval. If this interval is finite ($T < \infty$), then either $x(t) \to -\infty$, or $x(t) \to +\infty$, or $x(t) \to x_*$ as $t \to T - 0$ and $(T, x_*) \in \Omega$ in the latter case.*

Given $x_0 \in \mathbb{R}$ and $\eta_0 = \eta_0(\xi) \in W$, let $\alpha_*(\beta)$ with $\beta \ge x_0$ be the maximal solution of the equation $\alpha + \beta = \eta_0(\beta - \alpha) + 2x_0$ if any exists and $\alpha_*(\beta) = \beta - d$ otherwise. Similarly, let $\beta_* = \beta_*(\alpha)$ with $\alpha \le x_0$ be the minimal solution of $\alpha + \beta = \eta_0(\beta - \alpha) + 2x_0$ if it exists and $\beta_*(\alpha) = \alpha + d$ otherwise. Consequently, $\alpha_*(\cdot)$ is a decreasing left-continuous function, $\beta_*(\alpha)$ is an increasing right-continuous function, and $\alpha_*(x_0) = \beta_*(x_0) = x_0$. Define a nonnegative function $a_*(\cdot) = a_*(\cdot; x_0, \eta_0)$ by

$$a_*(\beta) = \int_{\alpha_*(\beta)}^{\beta} \mu(\alpha, \beta)\mathrm{d}\alpha, \quad \beta \ge x_0; \tag{18}$$

$$a_*(\alpha) = \int_{\alpha}^{\beta_*(\alpha)} \mu(\alpha, \beta)\mathrm{d}\beta, \quad \alpha \le x_0. \tag{19}$$

This function has finite left and right limits at every point. It is left-continuous on the interval (x_0, ∞), right-continuous on $(-\infty, x_0)$ and $a_*(x_0) = 0$. The continuity points of $\alpha_*(\cdot)$ and $\beta_*(\cdot)$ are continuity points of $a_*(\cdot)$.

The next theorem, where we use the notation

$\sigma = \operatorname{sign} f(t_0, x_0) \quad \text{if } f(t_0, x_0) \ne 0;$
$\sigma = \operatorname{sign} f_t(t_0, x_0) \quad \text{if } f(t_0, x_0) = 0,$

summarises further properties of admissible solutions.

[9] The terms 'vertical' and 'horizontal' are natural in (α, β) coordinates.

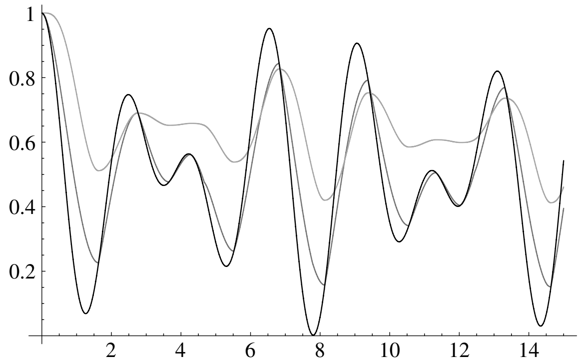

Fig. 5. Numerical solution of equation $(Px)' = b(t) - x$ with $b(t) = \frac{1}{4}(\cos t + \cos(\sqrt{2}t) + 2)$; the measure of the Preisach nonlinearity $P = P[\eta(t_0)]$ has a uniform distribution. Horizontal axis: t; vertical axis: x and y. The black line is the zero of the right-hand side: $b(t) - x = 0$. The dark grey line is the time series of a solution $x = x(t)$. Extrema of x lie on the black line and the derivative x' jumps at the extrema. The smooth light grey line is the time series of the output $y(t) = (Px)(t)$ of the Preisach nonlinearity.

Theorem 3.3. *Any admissible solution $x : [0, T) \to \mathbb{R}$ of the Cauchy problem (15)–(17) is absolutely continuous on each segment $[t_0, \tau] \subset [t_0, T)$ and has both left and right finite derivatives at each point $t \in (t_0, T)$ and a right finite derivative at the point t_0. Furthermore, solutions of the equation $f(t, x(t)) = 0$ on the interval (t_0, T) form an increasing finite (possibly empty) or infinite sequence $t_1 < t_2 < \cdots < t_k < \cdots$ and:*

- *The moments $t_1, t_2, \ldots$ when the graph of $x = x(t)$ intersects the set F_0 are all local extrema of the function x on the interval (t_0, T). The solution x is strictly monotone on each interval (t_{k-1}, t_k), $k = 1, 2, \ldots$.*
- *On each of the time intervals (t_{k-1}, t_k) the function $\sigma(-1)^k x(t)$ strictly decreases and the relation $\sigma(-1)^k f(t, x(t)) < 0$ holds; the same is true also on the interval (t_K, T) for $k = K + 1$ if the sequence $\{t_k\}$ consist of $K < \infty$ points; if the sequence $\{t_k\}$ is infinite, then $t_k \to T - 0$.*
- *The derivative of the solution x jumps at all points of the sequence $\{t_k\}$: the left derivative of x at each point t_k, $k = 1, 2, \ldots$, is zero, while its right non-zero derivative equals.*

$$\sigma(-1)^k\left(c_k + \sqrt{c_k^2 + 4b_k\mu_k}\right)/(2\mu_k), \tag{20}$$

where $c_k = f_x(t_k, x(t_k))$ and

$$b_k = \sigma(-1)^k f_t(t_k, x(t_k)), \qquad \mu_k = \mu(x(t_k), x(t_k))$$

satisfy $b_k > 0$, $\mu_k > 0$ for all $k = 1, 2, \ldots$. The same formula for the right derivative of x holds also for $k = 0$ at the point t_0 if $f(t_0, x_0) = 0$.

- *At any point $t \in (t_{k-1}, t_k)$, the left and right derivatives x'_- and x'_+ of the solution x equal*

$$x'_-(t) = \frac{f(t, x(t))}{a_*(x(t))}, \qquad x'_+(t) = \frac{f(t, x(t))}{a_*(x(t) + 0)} \tag{21}$$

if x increases on the interval (t_{k-1}, t_k) and

$$x'_-(t) = \frac{f(t, x(t))}{a_*(x(t))}, \qquad x'_+(t) = \frac{f(t, x(t))}{a_*(x(t) - 0)} \tag{22}$$

if x decreases on the interval (t_{k-1}, t_k), where $a_*(\cdot) = a_*(\cdot; x(t_{k-1}), \eta(t_{k-1}))$ and $\eta(t)$ is the state of the Preisach nonlinearity at the moment t.

Fig. 5 presents a numerical solution of equation $(Px)' = b(t) - x$ with $b(t) = \frac{1}{4}(\cos t + \cos(\sqrt{2}t) + 2)$; the measure of the Preisach nonlinearity has a uniform distribution.

In the formulation of the following statement we use the metric $\rho(\cdot,\cdot)$ in the state space W of the Preisach nonlinearity. Remark that W with this metric, introduced in the previous section, is a complete metric space.

Theorem 3.4 (*Continuity w.r.t. Initial Conditions*). *Let $x = x(t)$ be an admissible solution of problem (15)–(17) on some finite interval $t_0 \le t < T$. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x_0 - \tilde{x}_0| \le \delta$, $\rho(\eta_0, \tilde{\eta}_0) \le \delta$ and the initial state $\tilde{\eta}_0$ is admissible for the initial data $t_0, \tilde{x}_0$, then the initial value problem for system (15), (16) with the initial data $x(t_0) = \tilde{x}_0$, $\eta(t_0) = \tilde{\eta}_0$ has a solution $\tilde{x}$ defined on the interval $t_0 \le t < T - \varepsilon$, which satisfies $|x(t) - \tilde{x}(t)| < \varepsilon$ on this interval.*

3.3. Structure of solutions on (t, x)-plane

A natural phase space for system (15), (16), as for other systems with hysteresis nonlinearities, is the space of pairs (x_0, η_0) where shifts along the trajectories of the system form a semiflow (it is important to recall that initial conditions (17) define a solution for $t \ge t_0$; generically, we cannot send a solution back in time from a point (x_0, η_0) of the phase space). On the other hand, a simple way to visualise the x-component of solutions $(x(t), \eta(t))$, which is of main interest, is to consider the graphs of the x-component of solutions on the (t, x)-plane for a range of different initial values $x(t_0) = x_0$ and the same fixed initial state $\eta(t_0) = \eta_0$. However, it is a typical situation for systems with hysteresis that the graphs of the x-component on the (t, x)-plane intersect each other, while the solutions $(x(t), \eta(t))$ in the phase space do not. The situation is very much the same as for systems of ordinary differential equations with two components x and z, where the Cauchy problem has a unique solution for any set of initial data x_0, z_0, but graphs of the x-component of solutions with different x_0 and the same z_0 are in general intersecting, which means that the (t, x)-plane does not stratify into distinct graphs of the x-component.

Surprisingly, the intersection of solutions x corresponding to different initial values x_0 and a fixed initial state η_0 of the Preisach nonlinearity is not the case for (15).

Let us introduce a semiordering in the space W of states $\eta = \eta(\xi)$ of the Preisach nonlinearity writing $\eta_1 \ge \eta_2$ if this estimate holds pointwise for all $\xi \in [0, d]$. Also, we write $\eta_1 - \eta_2 \ge c$ with $c \in \mathbb{R}$ if this estimate holds on the whole segment $[0, d]$. In these terms, a well-known monotonicity property of the Preisach nonlinearity is that if $x_1(t) \ge x_2(t)$, then

$$\begin{aligned}&\eta_{01} - \eta_{02} \ge -2(x_1(t_0) - x_2(t_0))\\ &\Rightarrow \eta_1(t) - \eta_2(t) \ge -2(x_1(t) - x_2(t))\end{aligned} \tag{23}$$

for all $t \in [t_0, t_1]$, where $\eta_j = \eta_j(t)$ is the variable state of the Preisach nonlinearity, corresponding to the input $x_j : [t_0, t_1] \to \mathbb{R}$ and the initial state $\eta_j(t_0) = \eta_{0j}$.

Theorem 3.5. *Assume that Ω consists of isolated points and $f_x(t, x) \neq 0$ at every point $(t, x) \in \Omega$. Let $x_k : [t_0, T) \to \mathbb{R}$ with $T < \infty$ be an admissible solution of the Cauchy problem for system (15), (16) with initial data $x_k(t_0) = x_{0k}$, $\eta_k(t_0) = \eta_{0k}$, where $k = 1, 2$. Then the relations $\eta_{01} \ge \eta_{02}$ and $x_{01} > x_{02}$ imply $x_1(t) \ge x_2(t)$ for all $t \in [t_0, T)$ and moreover, $\inf_{t\in[t_0,T)}(x_1(t) - x_2(t)) > 0$ if the functions x_k are bounded.*

Corollary 3.6. *Let $x_1, x_2 : [t_0, T) \to \mathbb{R}$ be two admissible solutions of system (15), (16) for the same initial state $\eta(t_0) = \eta_0$ of the Preisach nonlinearity. Let $x_1(t_0) \neq x_2(t_0)$. Then the graphs of x_1 and x_2 do not intersect. Moreover, $\lim_{t\to T-0} x_1(t) \neq \lim_{t\to T-0} x_2(t)$ if T and both these limits are finite.*

Let us fix an initial moment t_0 and an arbitrary initial state η_0 of the Preisach nonlinearity and consider graphs of all admissible solutions of system (15), (16) starting from different initial values $x(t_0) = x_0$ on the line $t = t_0$ and lying in the half-plane $t \ge t_0$ of the (t, x)-plane. Denote by $x = x_{t_0,\eta_0}(\cdot; x_0)$ a unique solution defined by an initial value $x(t_0) = x_0$ on a maximal interval $t_0 \le t < T(x_0)$ such that the graph of x lies in the domain $\mathbb{R}^2 \setminus \Omega$. According to Theorem 3.2, for any x_0 either $T(x_0) = \infty$, or $T(x_0) < \infty$ and the solution $x = x_{t_0,\eta_0}(\cdot; x_0)$ goes to one of the infinite limits, or the graph of x ends at a point of the set Ω. Now, from Corollary 3.6 it follows that the solutions do not intersect and, moreover, there is at most one solution ending at any particular point of the set Ω. Thus, if the set Ω is finite, then the set of initial values x_0 such that the solution of $x_{t_0,\eta_0}(\cdot; x_0)$ ends in Ω is also finite; if the set Ω is countable, then the corresponding set of initial values x_0 is at most countable.

Theorem 3.4 implies that given an initial value x_0^* (here t_0 and η_0 are again fixed) and a moment $t_1 \in (t_0, T(x_0^*))$, the formula $X_{t_0,t_1,\eta_0}(x_0) = x_{t_0,\eta_0}(t_1; x_0)$ defines a continuous map $X_{t_0,t_1,\eta_0} : I_\delta \to \mathbb{R}$ on some segment $I_\delta = \{x_0 \in \mathbb{R} : |x_0^* - x_0| \le \delta\}$ with $\delta > 0$. From Corollary 3.6 it follows that this map (function) is injective and consequently continuously invertible.

A nice structure of the set of graphs of solutions $x_{t_0,\eta_0}(\cdot; x_0)$ on the (t, x)-plane, which follows from Theorems 3.1–3.5 and Corollary 3.6, reminds that of an ordinary differential equation. However, an essential difference is the lack of a semiflow property. In this regard, we note that the state $\eta(t_1)$ is generically different for different solutions $x_{t_0,\eta_0}(\cdot; x_0)$, i.e. for different x_0, and depends on the values of $x_{t_0,\eta_0}(\cdot; x_0)$ on the segment $t_0 \le t \le t_1$ (while the initial state η_0 is the same for all these solutions by definition). Hence, the restriction of solutions $x_{t_0,\eta_0}(\cdot; x_0)$ with varying x_0 to the interval $t \ge t_1$ is not a set of the form $x_{t_1,\eta_1}(\cdot; x_1)$ with some fixed η_1 and varying x_1. To further illustrate this situation, consider a problem of constructing (say, numerically) a solution $x_{t_0,\eta_0}(\cdot; x_0)$ passing through a given point (t_1, x_1) with $t_1 > t_0$, where t_0, η_0 are also given and x_0 is unknown. The initial data for the Cauchy problem at the moment t_1 include the state $\eta(t_1)$, which is not known *a priori*. Hence the data is not sufficient for starting at the point (t_1, x_1) and continuing the solution from this point, even if we are interested in finding the solution $x_{t_0,\eta_0}(\cdot; x_0)$ for $t > t_1$ only.[10] One can resort to the shooting method instead, starting at the moment t_0 and varying x_0.

3.4. Dangerous set and non-uniqueness of solutions

Let us consider the Cauchy problem (15)–(17) with initial data (t_0, x_0) in the dangerous set. Consequently, $f(t_0, x_0) = 0$ and $f_t(t_0, x_0) = 0$. We assume that

$$f(t, x) = b(t - t_0)^2 + c(x - x_0) + o((t - t_0)^2 + |x - x_0|) \tag{24}$$

near the point $(t_0, x_0) \in \Omega$ with non-zero partial derivatives $b = f_{tt}(t_0, x_0)$ and $c = f_x(t_0, x_0)$. In order to deal with this situation, we extend the definition of admissible initial data in a natural way: a state η_0 is admissible for initial data $(t_0, x_0) \in \Omega$ if it has a vertical initial segment in case $b < 0$ and a horizontal initial segment in case $b > 0$. One can show that generically, if representation (24) holds, then uniqueness of solutions is determined by the sign of the partial derivative c of f. Namely, if $c < 0$, then the Cauchy problem (15)–(17) with any admissible initial state η_0 has a unique solution on some interval $t_0 \le t < t_1$ and the right derivative of

[10] Continuation to the interval $t_0 \le t \le t_1$ from the point t_1 would be prevented by the irreversibility of the Preisach nonlinearity even from a known initial point $x(t_1) = x_1$, $\eta(t_1) = \eta_1$, as was discussed earlier.

this solution at t_0 equals zero. If $c > 0$, then problem (15)–(17) has a continuum of solutions on an interval $t_0 \le t < t_1$ and this continuum contains an upper solution x^+ and a lower solution x^- with the positive angle $c/\mu(x_0, x_0)$ between them. More precisely, if $b > 0$, then the right derivative of x^+ is zero and the right derivative of x^- is $-c/\mu(x_0, x_0)$ at the initial moment t_0, while if $b < 0$, then the right derivative of x^- is zero and the right derivative of x^+ is $c/\mu(x_0, x_0)$.

We do not prove these statements: their proof can be obtained by modification of the proofs of Theorems 3.1–3.5 presented in the next section.

As a consequence, if a solution $x_{t_0,\eta_0}(\cdot; x_0)$ starting outside the dangerous set Ω comes to a point $(t_1, x_1) \in \Omega$ of the dangerous set where $f_x(t_1, x_1) \ne 0$, $f_{tt}(t_1, x_1) \ne 0$ at some moment $t_1 > t_0$, i.e. $x_{t_0,\eta_0}(t; x_0) \to x_1$ as $t \to t_1 - 0$ (as we know, there is at most one such solution for any point $(t_1, x_1) \in \Omega$), then this solution can be continued to a larger interval $[t_0, T)$ with $T > t_1$ and the continuation is unique if $f_x(t_1, x_1) < 0$ and non-unique if $f_x(t_1, x_1) > 0$. Moreover, in case of non-uniqueness solutions $x_{t_0,\eta_0}(\cdot; x_0 + \delta)$ with the initial values nearby x_0 converge to the upper solution of problem (15)–(17) as δ goes to zero from above, and to the lower solution of this problem as δ goes to zero from below. Since the upper and lower solutions bifurcate after the moment t_1 (before this moment they coincide according to Theorem 3.2), we see that the Cauchy problem (15)–(17) is unstable with respect to small perturbations $x(t_0) = x_0 + \delta$ of initial data if its solution $x_{t_0,\eta_0}(\cdot; x_0)$ arrives at a point $(t_1, x_1) \in \Omega$ with $f_x(t_1, x_1) > 0$, $f_t(t_1, x_1) \ne 0$ at some moment $t_1 > t_0$. Otherwise, it is stable. We do not know what is the structure of the continuum of solutions that fill in the cusp between the upper and lower solutions after the bifurcation moment t_1: for example, whether these solutions have intersections for $t > t_1$.

In the previous sections, we considered the solutions lying in the complement $\mathbb{R}^2 \setminus \Omega$ of the dangerous set Ω in order to exclude points of the (t, x)-plane where non-uniqueness of solutions may arise. Now we see that if the relations $f_x(t_0, x_0) \ne 0$, $f_{tt}(t_0, x_0) \ne 0$ and (24) hold in Ω, then Theorems 3.1–3.5 can be made more accurate, because non-uniqueness arises on the part $\Omega_+ = \{(t_0, x_0) \in \Omega : f_x(t_0, x_0) > 0\}$ of the set Ω only. Hence, one can redefine the dangerous set by replacing Ω with its subset Ω_+. Then all the conclusions of Theorems 3.1–3.5 remain valid if everywhere in their formulations the set Ω is replaced by the set Ω_+ and one understands admissible solutions to be those with graphs in the complement in $\mathbb{R} \setminus \Omega_+$ of the set Ω_+ and with admissible initial data. The only minor remark regarding the formulations is that in the new version of Theorem 3.3 the local extrema t_k of the solution x are the moments when $f(t, x(t)) = 0$, $f_t(t, x(t)) \ne 0$. The moments of possible intersections of the solution with the lines $f = 0$ at the points of the set $\Omega \setminus \Omega_+$ are not local extrema of x. The derivative $x'(\tau)$ of the solution at the points where $(\tau, x(\tau)) \in \Omega \setminus \Omega_+$ is zero and does not jump unlike at the extremum points of x.

4. Survey of further results

4.1. Regularising equations

Equation (15) can be regularised by adding a small term $\varepsilon x'$ in the left-hand side, which leads to the equation

$$y' + \varepsilon x' = f(t, x) \tag{25}$$

with $\varepsilon > 0$, where y and x are related by (16) and f is continuously differentiable. Solutions x of (25) do not have jumps of the derivative on the lines $f(t, x) = 0$. This fact can be derived from the inequality

$$\varepsilon \max_{a\le s\le\tau\le b} |x(s) - x(\tau)| \le \max_{a\le s\le\tau\le b} |(y + \varepsilon x)(s) - (y + \varepsilon x)(\tau)|$$

which follows from the definition of the Preisach model and holds for an arbitrary time interval $[a, b]$. For a point (t_0, x_0) of intersection of a solution $x = x(t)$ with the line $f = 0$, this inequality implies

$$|x(t) - x(t_0)| \le \varepsilon^{-1}|t - t_0| \max_{\min\{t_0,t\}\le s\le\max\{t_0,t\}} |(y' + \varepsilon x')(s)|$$

where the value of $\max |(y' + \varepsilon x')(s)|$ tends to zero as $t \to t_0$, because $(y' + \varepsilon x')(t_0) = f(t_0, x_0) = 0$ and the continuity of f in equation (25) ensures continuity of the derivative $(y+\varepsilon x)'$.[11] Hence $x' = 0$ at every point where $f(t, x) = 0$.

Furthermore, the Cauchy problem (16), (17), (25) has a unique solution $x : [t_0, T) \to \mathbb{R}$ for any initial data t_0, x_0 and η_0: in this sense, equation (25) is a regularisation of equation (15). In order to prove uniqueness and existence of solutions, one can use the inverse $(\varepsilon I + P[\eta_0])^{-1}$ of the operator $\varepsilon I + P[\eta_0]$ (where I is the identity) and rewrite equation (25) equivalently as a system

$$z' = f(t, x), \qquad x(t) = ((\varepsilon I + P[\eta(t_0)])^{-1} z)(t).$$

Since the operator $(\varepsilon I + P[\eta_0])^{-1}$ with $\varepsilon > 0$ is globally Lipschitz continuous in the space $C = C([t_0, t_1]; \mathbb{R})$ (see, [8]), the Cauchy problem for this system can be approached in a standard way by passing to the integral equation in C and applying the contracting mapping principle.

The same method, when applied to the Cauchy problem for equation (15), leads to the existence result based on the Schauder principle, but does not work in the uniqueness problem, because the inverse $(P[\eta_0])^{-1}$ of the Preisach operator is not Lipschitz continuous. Indeed, as we have seen, the uniqueness is not always the case for the Cauchy problem (15)–(17). In the proofs presented below we use another more constructive approach both for analysing uniqueness and proving existence of solutions.

Omitting some technicalities, for any given initial data (17), the solution x_ε of the Cauchy problem for equation (25) converges to that of (15) as $\varepsilon \to 0$ and this convergence is uniform on any segment $[t_0, t_1]$ where all the solutions x_ε are defined (see [26] for a rigorous statement). If the limit problem (15)–(17) has multiple solutions, then the functions x_ε converge either to the upper or to the lower solution of this limit problem.

4.2. Equations with discontinuities in time

Natural classes of hydrological models lead to a balance equation which includes the terms discontinuous in time, for example, describing precipitation. Systems with such terms are the subject of [25] where equation (15) with the right-hand side

$$f(t, x) = g(t, x) + b(t) \tag{26}$$

is considered. Here g is continuously differentiable, while the term b has discontinuities. If the function $b = b(t)$ has a finite number of discontinuity points on any finite interval $-\tau < t < \tau$, is continuously differentiable on each interval of its continuity, and both left and right limits of b and b' at each discontinuity point are defined and are finite (for example, $b(t) = \text{sign}\ \sin t$), then the results of the previous section can be easily extended to equations with a function f of the form (26). Essentially, it suffices to extend the set Ω by including in this set some points (t, x) where f jumps

[11] Note that the derivative x' of a solution $x = x(t)$ to equation (25) can have jumps, however not on the lines $f = 0$ (cf. the last bullet point of Theorem 3.3). Furthermore, the continuity of the right-hand side of equation (25) ensures that the linear combination $(y + \varepsilon x)(t)$ in the left-hand side is continuously differentiable everywhere, hence the jumps of x' and y', when they happen, are simultaneous.

and at least one of the limits $f(t-0,x), f(t+0,x)$ is zero. If we define the dangerous set by

$$\Omega = \{(t,x) : 0 = f(t,x) = f_t(t,x)\} \cup \{(t,x) : t \in \mathbb{T}, 0 = f(t+0,x)f(t-0,x)\}$$

with $\mathbb{T} = \{t \in \mathbb{R} : b(t-0) \neq b(t+0)\}$, then all the theorems of the previous section (and Corollary 3.6) with the exception of Theorem 3.3 are valid without any change in formulation for equation (15) with the right-hand side (26). Theorem 3.3 requires a minor adaptation, because solutions have additional extrema at the points where f jumps and changes sign, i.e. on the set $D = \{(t,x) : f(t-0,x)f(t+0,x) < 0\}$, and the right derivative of a solution at these points is infinite. More precisely, if a solution x passes through a point $(t, x(t)) \in D$, then $\tau^{-1/2}(x(t+\tau) - x(t))$ goes to a finite limit as $\tau \to +0$.

Equations with the right-hand side (26) involving a Poisson type stochastic process $b(t)$ and white noise have been studied in [46,68], respectively.

4.3. Systems with negative feedbacks

In hydrological models presented in Section 2 all the relationships describing the flows can be interpreted as negative feedback loops. This implies that the function $f(t,x)$ in equation (15) is decreasing in x. The class of decreasing functions f is thus natural in the hydrological context. If f is non-increasing in x then the Cauchy problem for equation (15) has a unique solution; see [26] for details. Hydrological systems including positive feedback loops have been considered in [51,69].

4.4. Stable periodic solutions

If time-dependent terms in the balance equation are periodic or almost periodic (which is a typical situation), then the system can naturally exhibit periodic or almost periodic behaviour. For example, assume that the right-hand side f of equation (15) or (25) is decreasing in x and periodic in t. In this case, the equation has a unique periodic solution, which is globally asymptotically stable, whenever there is a segment $[x_-, x_+]$ such that $f(t, x_+) \le 0 \le f(t, x_-)$ for all t and some non-degeneracy conditions are satisfied; see [26]. Global stability refers to convergence from arbitrary admissible initial data, including an arbitrary initial state η_0 of the Preisach operator, to the periodic solution. Locally stable periodic solutions of equation (25) with non-monotone right-hand side and their simple bifurcations have been studied in [69].

5. Proofs

5.1. Auxiliary lemmas

We start with three auxiliary lemmas on monotone solutions. We say that x increases if $t_1 < t_2$ implies $x(t_1) \le x(t_2)$ and strictly increases if $t_1 < t_2$ implies $x(t_1) < x(t_2)$. By definition, if the input x of the Preisach nonlinearity increases, then its output y also increases. If x is absolutely continuous, then y is absolutely continuous.

Lemma 5.1. *If an output $y = y(t)$ increases, then the input $x = x(t)$ increases. If y strictly increases, then x strictly increases.*

Proof. If x is not increasing, then there are moments such that $t_1 < t_2$ and $x(t_1) > x(t_2)$. Consequently, there is a moment t_3 such that $t_3 < t_2$, $x(t_3) > x(t_2)$, and $x(t_2) \le x(t) \le x(t_3)$ for $t_3 \le t \le t_2$. By definition of the Preisach nonlinearity with μ satisfying equation (14) (see, the end of Section 2.3), this implies $y(t_2) < y(t_3)$, hence y is not increasing. Thus, x increases if y does. Now, if x is constant on some segment, then y is constant on the same segment, hence x strictly increases if y strictly increases. □

Lemma 5.2. *If x and y are increasing and y is absolutely continuous, then x is absolutely continuous.*

Proof. Let $t_1 \le \tau_1 \le t_2 \le \tau_2 \le \cdots \le t_n \le \tau_n$. Define

$$x(t_k) = x_k, \qquad x(\tau_k) = \tilde{x}_k, \qquad y(t_k) = y_k, \qquad y(\tau_k) = \tilde{y}_k,$$

$\Delta x_k = \tilde{x}_k - x_k$, $\Delta y_k = \tilde{y}_k - y_k$, and

$$A_k = \{(\alpha,\beta) : x_1 \le \alpha \le \tilde{x}_k, x_k \le \beta \le \tilde{x}_k, \alpha < \beta\},$$

Since the input increases, $\Delta y_k \ge \operatorname{mes}_\mu A_k$, consequently

$$\Delta y_1 + \cdots + \Delta y_n \ge \operatorname{mes}_\mu A_1 + \cdots + \operatorname{mes}_\mu A_n. \tag{27}$$

The Lebesgue measure of the set A_k is $\operatorname{mes} A_k = \Delta x_k(x_k + \tilde{x}_k - 2x_1)/2$, therefore $\operatorname{mes} A_k \ge \Delta x_k(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_{k-1} + \Delta x_k/2)$ and

$$\operatorname{mes} A_1 + \cdots + \operatorname{mes} A_n \ge (\Delta x_1 + \cdots + \Delta x_n)^2/2.$$

The conclusion of the lemma follows from this estimate, estimate (27) and assumption (14). □

Lemma 5.3. *If x and y are increasing and absolutely continuous, then almost everywhere $y'(t) = a(x(t))x'(t)$ with*

$$a(\beta) = \int_{\alpha(\beta)}^{\beta} \mu(\alpha,\beta)\mathrm{d}\alpha \ge 0, \tag{28}$$

where $\alpha = \alpha(\beta)$ is a solution of the equation $\alpha + \beta = \eta_0(\beta - \alpha) + 2x_0$ if it exists on the interval $0 \le \beta - \alpha \le d$ and $\alpha = \beta - d$ otherwise. Here x_0 and η_0 are an initial value of x and an initial state of the Preisach nonlinearity.

Proof. Recall that by $\alpha_*(\beta)$ we denote the maximal solution of the equation $\alpha + \beta = \eta_0(\beta - \alpha) + 2x_0$ if any exists and $\alpha_*(\beta) = \beta - d$ otherwise, consequently $\alpha_*(\cdot)$ is a decreasing left-continuous function satisfying $\alpha_*(x_0) = x_0$. The fact that $y = y(t)$ is constant on any segment where $x = x(t)$ is constant implies that the function $y = y(x)$ is defined. For $x_1 < x_2$, the formula

$$y(x_2) - y(x_1) = \operatorname{mes}_\mu\{(\alpha,\beta) : \alpha \in (\alpha_*(\beta), \beta), \beta \in (x_1, x_2)\},$$

implies that the left and right derivatives of the function $y(x)$ are defined at each point x and equal $a_*(x)$ and $a_*(x+0)$, where the nonnegative function a_* is defined by (18). Therefore the derivative of $y(x)$ is defined by $\mathrm{d}y/\mathrm{d}x = a_*(x)$ at the continuity points of $\alpha_*(x)$ and is not defined on the countable set E of the points x where $\alpha_*(\cdot)$ jumps.

Consider a sequence of open sets $E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots \supset E$ with $\operatorname{mes} E_n \le 1/n$. Denote by E^t the preimage of E and by E_n^t the preimage of E_n under the mapping $t \to x(t)$. By construction, the sets E_n^t are open, $E_1^t \supset E_2^t \supset \cdots \supset E_n^t \supset \cdots \supset E^t$, and $\int_{E_n^t} x'(t)\mathrm{d}t = \operatorname{mes} E_n$. Passing to the limit in the last relation, we arrive at $\int_{\tilde{E}^t} x'(t)\mathrm{d}t = 0$ with $\tilde{E}^t = \cap_{n=1}^\infty E_n^t \supset E^t$, therefore $x' = 0$ almost everywhere on E^t. Thus, for almost every t either $x(t) \notin E$ and $x'(t) > 0$ or $x'(t) = 0$.

As we have seen, if $x(t) \notin E$, then the derivative $\mathrm{d}y/\mathrm{d}x = a(x)$ is defined at the point $x = x(t)$. Therefore, if $x(t) \notin E$ and $x'(t) > 0$, then passing to the limit in the relation

$$\frac{y(x(t+\tau)) - y(x(t))}{\tau} = \frac{y(x(t+\tau)) - y(x(t))}{\Delta x(t)} \cdot \frac{\Delta x(t)}{\tau}$$

with $\Delta x(t) = x(t+\tau) - x(t)$ as $\tau \to 0$ and taking into account that $\Delta x(t) = \tau x'(t) + o(\tau) \neq 0$ for small τ, we obtain $y'(t) = a(x(t))x'(t)$. If $x'(t) = 0$, then $y'(t) = 0$, because both the left and right derivatives of $y(x)$ are defined and are finite at the point $x = x(t)$, which implies $y'(t) = a(x(t))x'(t)$ in this case too. □

Analogs of the above lemmas are true for decreasing inputs and outputs. Formula (28) for this case takes the form

$$a(\alpha) = \int_{\alpha}^{\beta(\alpha)} \mu(\alpha, \beta)\mathrm{d}\beta \geq 0, \tag{29}$$

where $\beta = \beta(\alpha)$ is a solution of the equation $\alpha + \beta = \eta_0(\beta - \alpha) + 2x_0$.

5.2. Proof of Theorem 3.1 in case $f(t_0, x_0) \neq 0$

5.2.1. Existence of solution

To be definite, assume that $f(t_0, x_0) > 0$. If an input $x = x(t)$ with the initial value x_0 increases, then the output (16) of the Preisach nonlinearity with an initial state η_0 equals $y = \varphi(x)$, where the continuous function $\varphi(\cdot) = \varphi_{x_0,\eta_0}(\cdot)$ is defined for $x \geq x_0$ by

$$\begin{aligned} &\varphi_{x_0,\eta_0}(x) = \mathrm{mes}\,_{\mu} B, \\ &B = \{(\alpha, \beta) : \alpha + \beta < \eta_0(\beta - \alpha) + 2x_0\} \cup \{(\alpha, \beta) : \beta < x\}. \end{aligned} \tag{30}$$

By assumption (14) on the measure density μ, this function is strictly increasing and therefore the inverse function $\psi = \varphi^{-1}$ is defined on some interval $[y_0, y_1)$ with $y_0 = \varphi(x_0)$. Let us extend the function ψ continuously to an interval $(y^1, y_1) \supset [y_0, y_1)$ by setting $\psi(y) = \psi(y_0) = x_0$ for $y < y_0$ and consider the ordinary differential equation $y' = f(t, \psi(y))$. Since f and ψ are continuous, this equation has a solution y with the initial condition $y(t_0) = y_0$. From the relations $f(t_0, x_0) = f(t_0, \psi(y_0)) > 0$ it follows that this solution increases on a sufficiently small interval $[t_0, t_0 + \delta)$. Therefore $y(t) \geq y_0$ for $t_0 \leq t < t_0 + \delta$, which implies that the solution $y = y(t)$ and the function $x = x(t) = \psi(y(t))$ are related by $x = \varphi^{-1}(y)$, $y = \varphi(x)$ and x increases on the same interval. Consequently, $y'(t) = f(t, x(t))$, $y = P[\eta_0]x$, and both x and y increase on $[t_0, t_0 + \delta)$, i.e. x is a solution of the Cauchy problem (15)–(17). This proves local existence of a solution.

5.2.2. Uniqueness of solution

Assume that the Cauchy problem (15)–(17) has two solutions $x_1, x_2 : [t_0, t_0 + \delta) \to \mathbb{R}$. To be definite, assume again that $f(t_0, x_0) > 0$. Then $f(t, x_i(t)) > 0$ and the output $y_i = P[\eta_0]x_i$ strictly increases on a sufficiently small interval $[t_0, t_0 + \delta)$. Consequently, Lemmas 5.1–5.3 imply that x_i strictly increases on the same interval and $y_i'(t) = a_*(x_i(t))x_i'(t)$ almost everywhere. Moreover, since x_i strictly increases, $a_*(x_i(t)) > 0$ for $t > t_0$ and therefore $y_i' = f(t, x_i)$ is equivalent to the relation

$$x_i' = f(t, x_i)/a_*(x_i), \quad i = 1, 2,$$

which is valid for almost every $t \in (t_0, t_0 + \delta)$. Hence,

$$\begin{aligned} x_1' - x_2' &= f(t, x_1)(1/a_*(x_1) - 1/a_*(x_2)) \\ &\quad + (f(t, x_1) - f(t, x_2))/a_*(x_2). \end{aligned} \tag{31}$$

For a given t assume without loss of generality that $x_2(t) \geq x_1(t)$. Hence $\alpha_*(x_2) \leq \alpha_*(x_1)$ at this moment and

$$\begin{aligned} a_*(x_2) - a_*(x_1) &= \int_{\alpha_*(x_2)}^{x_2} \mu(\alpha, x_2)\mathrm{d}\alpha - \int_{\alpha_*(x_1)}^{x_1} \mu(\alpha, x_1)\mathrm{d}\alpha \\ &\geq \int_{\alpha_*(x_1)}^{x_1} (\mu(\alpha, x_2)\mathrm{d}\alpha - \mu(\alpha, x_1))\mathrm{d}\alpha \\ &\quad + \int_{x_1}^{x_2} \mu(\alpha, x_2)\mathrm{d}\alpha. \end{aligned}$$

Using (14) and taking into account that μ is continuously differentiable, we obtain

$$a_*(x_2) - a_*(x_1) \geq (x_2 - x_1)(c_1 - c_2(x_1 - \alpha_*(x_1)))$$

with $c_1, c_2 > 0$. Since $f(t, x_i) > 0$, $a_*(x_i) > 0$, and f is continuously differentiable, this relation and (31) imply the estimate

$$x_1' - x_2' \geq (x_2 - x_1) \cdot \frac{c_1^* - c_2^*(x_1 - \alpha_*(x_1)) - c_3^* a_*(x_1)}{a_*(x_1)a_*(x_2)}$$

with all $c_i^* > 0$. Multiplying by $\Delta = x_2 - x_1 \geq 0$ we arrive at

$$(\Delta^2)' \leq -2\Delta^2 \cdot \frac{c_1^* - c_2^*(x_1 - \alpha_*(x_1)) - c_3^* a_*(x_1)}{a_*(x_1)a_*(x_2)}. \tag{32}$$

Recall that α_* is a decreasing left-continuous function and $\alpha_*(x_0) = x_0$. Consider separately the cases $\alpha_*(x_0 + 0) < x_0$ and $\alpha_*(x_0 + 0) = x_0$. Because x_i strictly increases and $x_i(t_0) = x_0$, in the former case $x_i - \alpha_*(x_i) \geq x_0 - \alpha_*(x_0 + 0) > 0$ for $t > t_0$, which implies $a_*(x_i) \geq c > 0$. Consequently, from (32) the estimate $((x_1 - x_2)^2)' \leq c_*(x_1 - x_2)^2$ with some $c_* > 0$ follows. Therefore, the relations $x_1(t_0) = x_2(t_0) = x_0$ imply $x_1 \equiv x_2$ for $t > t_0$.

In the other case, $\alpha_*(x_0 + 0) = x_0$, the value $x_i - \alpha_*(x_i)$ tends to zero as $t \to t_0 + 0$ and consequently $a_*(x_i) \to 0$ as well. Hence, for sufficiently small $t - t_0 > 0$, relation (32) implies $((x_1 - x_2)^2)' \leq 0$. Thus, $x_1 \equiv x_2$ in this case too, which completes the proof of uniqueness of a solution for $f(t_0, x_0) > 0$. For $f(t_0, x_0) < 0$, the proof follows the same line.

5.3. Proof of Theorem 3.1 in case $f(t_0, x_0) = 0$

5.3.1. Existence of a solution

Assume that $f(t_0, x_0) = 0$ at a point $(t_0, x_0) \notin \Omega$. Consequently, $f_t(t_0, x_0) \neq 0$. To be definite, let us assume that $f_t(t_0, x_0) > 0$ (this assumption is supposed to hold until the end of Section 5.3) and show that in this case problem (15)–(17) has an increasing local solution $x : [t_0, t_0 + \delta) \to \mathbb{R}$.

Define the continuous function

$$\tilde{f}(t, x) = \begin{cases} f(t, x) & \text{if } t \geq t^*(x), \\ 0 & \text{if } t < t^*(x), \end{cases}$$

where $t^*(x)$ is a continuous branch of the implicit function $f(t, x) = 0$ passing through the point (t_0, x_0), which is well defined and smooth due to $f_t(t_0, x_0) \neq 0$.

Define the function $\varphi(x) = \varphi_{t_0,x_0}(x)$, its inverse $\psi = \varphi^{-1}$, and the extension of ψ from the interval $[y_0, y_1)$ to an interval $(y^1, y_1) \supset [y_0, y_1)$ as in Section 5.2.1 and consider the equation

$$y' = \tilde{f}(t, \psi(y))$$

for $t \geq t_0$ with the initial condition $y(t_0) = y_0$ where $y_0 = \varphi(x_0)$. Since $\tilde{f}$ is continuous, this initial value problem has a local solution $y : [t_0, t_0 + \delta) \to \mathbb{R}$. Relations $f(t_0, x_0) = 0$, $f_t(t_0, x_0) > 0$ imply $f(t, x) > 0$ for $t > t^*(x)$ in some vicinity of the point (t_0, x_0). Therefore $\tilde{f}(t, x) \geq 0$ in this vicinity and hence the solution $y = y(t)$ increases. Consequently, $y(t) \geq y_0$ for all $t \in [t_0, t_0 + \delta)$, which implies that the functions y and $x(t) = (\psi(y))(t)$ are related by $y = \varphi(x)$ or, equivalently, $y = P[\eta_0]x$ and that x increases.

It remains to show that the graph of $x : [t_0, t_0 + \delta) \to \mathbb{R}$ lies in the domain $t \geq t^*(x)$ of the (t, x)-plane where $\tilde{f} = f$ and hence x is a solution of problem (15)–(17). In order to prove the estimate $t \geq t^*(x)$ on the solution $x = x(t)$, assume that the opposite relation $\tau < t^*(x(\tau))$ holds for some $\tau > t_0$. Then $\tilde{f} = 0$ in some vicinity of the point $(\tau, x(\tau)) = (\tau, \psi(y(\tau)))$ and therefore $y' = 0$ and $y(t) \equiv y(\tau)$ in some neighbourhood of the point τ. Consequently, there is a minimal $\tau_* \in [t_0, \tau)$ such that $y(t) \equiv y(\tau)$ on the segment $[\tau_*, \tau]$. But the relations $\tau_* < \tau < t^*(x(\tau)) = t^*(\psi(y(\tau)))$ and $y(\tau) = y(\tau_*)$ imply that τ_*, like τ, satisfies $\tau_* < t^*(\psi(y(\tau_*))) = t^*(x(\tau_*))$, hence $y(t) \equiv y(\tau_*)$ in some neighbourhood of the point τ_*, and from the definition of this point it follows that $\tau_* = t_0$. Thus, $t_0 < t^*(x(t_0)) = t^*(x_0)$, which contradicts the assumption that $f(t_0, x_0) = 0$. This contradiction shows that $t \geq t^*(x)$ on the solution $x = x(t)$ and therefore x is an increasing solution of the Cauchy problem (15)–(17).

5.3.2. Auxiliary lemma

To prove uniqueness, we need a simple auxiliary lemma.

First remark that $y(t) > y_0$ for $t > t_0$ for any increasing solution of problem (15)–(17). Indeed, if $y \equiv y_0$ on some initial segment $t_0 \le t \le t_1$, then $x \equiv x_0$ and hence $y' \equiv f(t, x_0) \equiv 0$ on the same segment. This cannot be the case for a segment $[t_0, t_1]$ of non-zero length, because $f_t(t_0, x_0) > 0$ by assumption. Thus, $x(t) > x_0$ and $y(t) > y_0$ for $t > t_0$.

Similarly, $x(t) < x_0$ and $y(t) < y_0$ for $t > t_0$ for any decreasing solution.

Lemma 5.4. *Any increasing solution of problem (15)–(17) satisfies $x(t) - x_0 \le k(t - t_0)$ with a $k > 0$ on some interval $t_0 < t < t_1$.*

Proof. From $f(t_0, x_0) = 0$ it follows $|f(t, x)| \le c_1|t-t_0|+c_2|x-x_0|$ in some vicinity of the point (t_0, x_0). Furthermore, $a(x) \ge \mu_0(x - \alpha_*(x))$, where $\mu_0 > 0$ due to (14), hence $a(x) \ge \mu_0(x - x_0)$ for $x > x_0$, because $\alpha_*(\cdot)$ decreases and $\alpha_*(x_0) = x_0$. Consequently, for $t > t_0$, $x > x_0$,

$$\frac{f(t,x)}{a(x)} \le \frac{c_1(t-t_0)+c_2(x-x_0)}{\mu_0(x-x_0)} \le \tilde{c}_1\frac{t-t_0}{x-x_0} + \tilde{c}_2.$$

Let $x(t) - x_0 \ge k(t - t_0)$ for some point $t > t_0$ such that both the derivatives $x'(t)$, $y'(t)$ are defined and $y'(t) = a(x(t))x'(t)$. Consequently, $a(x)x' = f(t, x)$ at this point t and therefore

$$x'(t) \le \tilde{c}_1(t - t_0)/(x(t) - x_0) + \tilde{c}_2 \le \tilde{c}_1/k + \tilde{c}_2.$$

Let us choose k so that $\tilde{c}_1/k + \tilde{c}_2 \le k$. We see that for such a k the implication $x(t)-x_0 \ge k(t-t_0) \Rightarrow x'(t) \le k$ holds for almost every $t > t_0$. This implication proves the estimate $x(t) - x_0 \le k(t - t_0)$ for all $t \ge t_0$. □

5.3.3. Uniqueness of an increasing solution

We prove uniqueness of a solution in two steps. In this subsection, we show that there is a unique increasing solution. In the next subsection, it is shown that there are no other solutions.

Assume that problem (15)–(17) has two increasing solutions $x_1, x_2 : [t_0, t_0+\delta) \to \mathbb{R}$. The assumption that x_0, η_0 are admissible initial data implies $\alpha_*(x_j(t)) \equiv x_0$ on a sufficiently small interval $t_0 \le t < t_0 + \delta$. Consequently,

$$a_*(x_j) = \mu_0(x_j - x_0) + o(x_j - x_0), \quad x_j \to x_0,$$
$$a_*(x_2) - a_*(x_1) = \mu_0(x_2 - x_1) + o(x_2 - x_1)$$

with $\mu_0 = \mu(x_0, x_0)$. Taking into account that $0 \le x_j(t) - x_0 \le k(t - t_0)$ by Lemma 5.4, we obtain

$$a_*(x_j) = \mu_0(x_j - x_0) + o(x_j - x_0),$$
$$a_*(x_2) - a_*(x_1) = \mu_0(x_2 - x_1) + o(x_2 - x_1)$$

as $t \to t_0+0$. Furthermore, the relations $0 \le x_j(t)-x_0 \le k(t-t_0)$ imply

$$f(t, x_j) = b(t - t_0) + c(x_j - x_0) + o(t - t_0),$$
$$f(t, x_1) - f(t, x_2) = c(x_1 - x_2) + o(x_1 - x_2),$$

as $t \to t_0 + 0$ with $b = f_t(t_0, x_0)$, $c = f_x(t_0, x_0)$. Substituting the above four asymptotic equalities in (31) gives

$$x_1' - x_2' = \frac{x_2 - x_1}{a_*(x_1)a_*(x_2)}(b\mu_0(t - t_0) + o(t - t_0)),$$

where we use $x_1 - x_0 = O(t - t_0)$. Since $b > 0$, this implies $((x_1 - x_2)^2)' \le 0$ almost everywhere on some interval $t_0 < t < t_0 + \delta$ and therefore $x_1 \equiv x_2$ on this interval, which proves uniqueness of an increasing solution.

5.3.4. Absence of non-monotone solutions

Consider separately the cases where $f_x(t_0, x_0)$ has different signs.

Case $f_x(t_0, x_0) > 0$. Let $x : [t_0, t_0 + \delta) \to \mathbb{R}$ be a solution of problem (15)–(17). If we assume that $f(t, x(t)) \le 0$ on some initial interval $[t_0, t_0 + \delta_1)$, then y and consequently x decrease on this interval, which implies $x(t) < x_0$ for $t > t_0$. Because the initial data are admissible, the initial state has a vertical initial segment. This implies that the increasing function $\beta_*(\cdot)$ jumps at the left of the point x_0, i.e. $\beta_*(x) \ge \beta_*(x_0 - 0) > x_0$ for $x < x_0$. Consequently, $a_*(x(t)) \ge a_0 > 0$ for $t > t_0$. Taking into account the relations $a(x(t))x'(t) = f(t, x(t))$ and $f(t_0, x_0) = 0$, we conclude that $|x'|$ is small for small $t - t_0 > 0$: more precisely, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|x'(t)| < \varepsilon$ for $0 < t - t_0 < \delta$.

On the other hand, from the relations $f(t, x(t)) \le 0$ and $f_x(t_0, x_0) > 0$, $f(t_0, x_0) = 0$ it follows that $x(t) \le \xi(t)$, where $\xi(t)$ denotes the implicit function defined by the equality $f(t, x) = 0$ in a vicinity of the point (t_0, x_0). Since $\xi'(t_0) = -f_t(t_0, x_0)/f_x(t_0, x_0) < 0$ and $\xi(t_0) = x(t_0) = x_0$, the estimate $x(t) \le \xi(t)$ contradicts our conclusion that $|x'|$ is arbitrarily small on appropriately small interval $(t_0, t_0 + \delta)$, which proves that the assumption $f(t, x(t)) \le 0$ on $(t_0, t_0 + \delta)$ is wrong for any $\delta > 0$.

Now assume that $f(t, x(t)) > 0$ on some time interval $(t_1, t_2) \subset (t_0, t_0 + \delta)$ and $f(t_1, x(t_1)) = f(t_2, x(t_2)) = 0$. The estimate $f(t, x(t)) > 0$ implies that y and x strictly increase on (t_1, t_2), hence $x(t_1) < x(t_2)$. At the same time, the relation $f(t_j, x(t_j)) = 0$ is equivalent to $x(t_j) = \xi(t_j)$ and, because $\xi'(t_0) < 0$, we arrive at the opposite estimate $x(t_1) > x(t_2)$ if $\delta > 0$ is sufficiently small. This contradiction shows that the relation $f(t, x(t)) > 0$, valid on some initial interval $(t_0, t_0 + \delta)$, which implies that the solution x increases, is the only possibility in the case considered.

Case $f_x(t_0, x_0) < 0$. In this case $\xi'(t_0) > 0$, hence ξ strictly increases in some neighbourhood of the point t_0 and the relations $f(t_0, x_0) = 0$, $f_t(t_0, x_0) > 0$ imply that $f(t, x) > 0$ for $x < \xi(t)$ and $f(t, x) < 0$ for $x > \xi(t)$. If we assume that $f(t, x(t)) \le 0$ on some interval $(t_0, t_0 + \delta)$, then x decreases on this interval, hence $x(t) \le x_0 = \xi(t_0) < \xi(t)$ for $t > t_0$ and consequently $f(t, x(t)) > 0$, which is a contradiction. That is, this assumption is wrong. If we assume that $f(t, x(t)) < 0$ on some interval $(t_1, t_2) \subset (t_0, t_0 + \delta)$ and $f(t_1, x(t_1)) = f(t_2, x(t_2)) = 0$, then we arrive at the contradiction, because $f(t, x(t)) < 0$ implies that x decreases on (t_1, t_2), while the relation $f(t_j, x(t_j)) = 0$ is equivalent to $x(t_j) = \xi(t_j)$ for $j = 1, 2$ and ξ strictly increases. Thus, in this case the only possibility is again that $f(t, x(t)) \ge 0$ in some right neighbourhood of t_0 and hence x is an increasing solution.

Case $f_x(t_0, x_0) = 0$. The same argument as in the case $f_x(t_0, x_0) > 0$ shows that if $f(t, x(t)) \le 0$ on some interval $(t_0, t_0+\delta)$, then $|x'|$ is small on this interval. At the same time, the relations $f(t, x(t)) \le 0$ and $f_t(t_0, x_0) > 0$, $f(t_0, x_0) = 0$ imply $t \le \tau(x(t))$, where $\tau(x)$ denotes the implicit function defined by the equality $f(t, x) = 0$ in a vicinity of the point (t_0, x_0). However, from the relations $\tau'(x_0) = -f_x(t_0, x_0)/f_t(t_0, x_0) = 0$ and $\tau(x(t_0)) = \tau(x_0) = t_0$ it follows $\tau(x(t)) = t_0 + o(x(t) - x_0)$, $t \to t_0$, therefore the estimate $t \le \tau(x(t))$ contradicts the fact that $|x'|$ is arbitrarily small on appropriately small interval $(t_0, t_0 + \delta)$, which proves that the assumption $f(t, x(t)) \le 0$ on $(t_0, t_0 + \delta)$ is wrong for any $\delta > 0$ as in the other cases above.

Now assume that $f(t, x(t)) > 0$ on an interval $(t_1, t_2) \subset (t_0, t_0 + \delta)$ and $f(t_1, x_1) = f(t_2, x_2) = 0$ for $x_j = x(t_j)$, $j = 1, 2$. Consequently, $x = x(t)$ strictly increases on $[t_1, t_2]$ and $a_*(x(t))x'(t) = f(t, x(t))$ almost everywhere on (t_1, t_2). Considering t_1, $x(t_1)$ and $\eta(t_1)$ as initial data, we obtain for $t_1 < t < t_2$

$$a_*(x(t)) \ge \mu_0(x(t) - \alpha_*(x(t))), \tag{33}$$

hence $a_*(x(t)) \ge \mu_0(x(t) - x_1)$ for $t > t_1$ with $\mu_0 > 0$, because $\alpha_*(\cdot)$ decreases and $\alpha_*(x_1) = x_1$. Consequently,

$$\mu_0(x - x_1)x' \le a_*(x)x' = f(t, x) = f(t, x) - f(t_1, x_1)$$
$$\le A(t - t_1) + B(x - x_1)$$

and hence

$$x'(t) \le \tilde{A}(t-t_1)/(x(t)-x_1)+\tilde{B}, \quad t_1<t<t_2.$$

If $x(t)-x_1 \ge k(t-t_1)$, then $x'(t) \le \tilde{A}/k+\tilde{B}$. Choosing k so that $\tilde{A}/k+\tilde{B} \le k$, we arrive at the implication

$$x(t)-x_1 \ge k(t-t_1) \quad \Rightarrow \quad x'(t) \le k,$$

which ensures that $x(t)-x_1 \le k(t-t_1)$ on $[t_1,t_2]$. Hence

$$(t_2-t_1)/(x_2-x_1) \ge 1/k. \tag{34}$$

On the other hand, because $\tau'(x_0)=0$, for every $\varepsilon>0$ there is a $\delta>0$ such that the estimate $(t_2-t_1)/(x_2-x_1)<\varepsilon$ holds for every pair of points (t_j,x_j) lying in a δ-vicinity of the point $(t_0,x_0)=(\tau(x_0),x_0)$ on the curve $f(t,x)=0$, which is the graph of the function $t=\tau(x)$. Since this estimate contradicts (34), we conclude that $f(t,x(t))>0$ on some interval $(t_0,t_0+\delta)$ and hence x increases on this interval. This completes the proof.

5.4. Lemma on the sign of f_t

Lemma 5.5. *Let a solution x of the Cauchy problem (15)–(17) satisfy $f(t,x(t))>0$ for $t\in(t_0,t_1)$ and $f(t_1,x(t_1))=0$, $f_t(t_1,x(t_1))\neq 0$. Then $f_t(t_1,x(t_1))<0$ and the left derivative $x'_-(t_1)$ of x at the point t_1 is zero. Similarly, if $f(t,x(t))<0$ for $t\in(t_0,t_1)$ and $f(t_1,x(t_1))=0$, $f_t(t_1,x(t_1))\neq 0$, then $x'_-(t_1)=0$ and $f_t(t_1,x(t_1))>0$.*

Proof. We prove the lemma in case $f(t,x(t))>0$ on (t_0,t_1), in the other case the argument is the same. The estimate $f(t,x(t))>0$ implies that x strictly increases on the segment $[t_0,t_1]$ and $a_*(x(t))x'(t)=f(t,x(t))$ almost everywhere on this segment. Like in the last case of Section 5.3.4, relations (33) and $\alpha_*(x(t))\le\alpha_*(x_0)=x_0$ (which are now valid on the interval (t_0,t_1) in place of (t_1,t_2)) imply $a_*(x(t))\ge\mu_0(x(t)-x_0)$ with $\mu_0>0$, hence

$$0\le\mu_0(x(t)-x_0)x'(t)\le f(t,x(t)), \quad t_0<t<t_1.$$

Hence, from the relations $f(t_1,x(t_1))=0$ and $x(t_1)>x_0$ it follows that the left derivative $x'_-(t_1)$ of x at the point t_1 is zero.

Now, consider the function $y'(t)=f(t,x(t))$. The relation $x'_-(t_1)=0$ implies that the left derivative $y''_-(t_1)$ of this function at the point t_1 is defined and equals $y''_-(t_1)=f_t(t_1,x(t_1))+x'_-(t_1)f_x(t_1,x(t_1))=f_t(t_1,x(t_1))$. Finally, because $y'(t)=f(t,x(t))>0$ on (t_0,t_1) and $y'(t_1)=f(t_1,x(t_1))=0$, the left derivative $y''_-(t_1)=f_t(t_1,x(t_1))$ of y' at the point t_1 is non-positive, hence $f_t(t_1,x(t_1))\neq 0$ implies $f_t(t_1,x(t_1))<0$. □

Evidently, $f(t,x(t))>0$ on some interval $(t_0,t_0+\delta)$ if $f(t_0,x_0)>0$. The same is true for any admissible initial data such that $f(t_0,x_0)=0$, $f_t(t_0,x_0)>0$. Indeed, as we have proved in Section 5.3, x and y increase on a sufficiently small interval $(t_0,t_0+\delta)$ and hence $y'(t)=f(t,x(t))\ge 0$ on such an interval. If we assume that $f\equiv 0$ on some segment $[t_0,t_0+\delta_1]$, then $y'\equiv 0$, i.e. $x\equiv x_0$, $y\equiv y_0$ and hence $f(t,x_0)\equiv 0$ on this segment, which contradicts the relation $f_t(t_0,x_0)>0$. Consequently, in any right neighbourhood of the point t_0 there is a point τ such that $f(\tau,x(\tau))>0$. Now note that according to Lemma 5.5 if $f(t,x(t))>0$ for $t\in[\tau,t_1)$ and $f(t_1,x(t_1))=0$, then $f_t(t_1,x(t_1))\le 0$. Because in some vicinity of the point (t_0,x_0) the opposite estimate $f_t(t,x)>0$ holds, we conclude that $f(t,x(t))>0$ on some initial interval $(t_0,t_0+\delta)$ with $\delta>0$.

Similarly, if $f(t_0,x_0)=0$ and $f_t(t_0,x_0)<0$, then $f(t,x(t))<0$ on some initial interval $(t_0,t_0+\delta)$.

5.5. Proof of Theorem 3.2

By Theorem 3.1, any admissible initial data define a unique solution x of the Cauchy problem (15)–(17) on some initial interval $[t_0,t_0+\delta)$. Moreover, according to the remark after the proof of Lemma 5.5, the relation $f(t,x(t))\neq 0$ holds on $(t_0,t_0+\delta)$ if $\delta>0$ is sufficiently small. To be definite, suppose that $f(t,x(t))>0$ for $t_0<t<t_0+\delta$ and hence x and y strictly increase on this interval. Consequently, for $t\in(t_0,t_0+\delta)$

$$y'(t)=f(t,\varphi^{-1}(y(t))) \tag{35}$$

and $x=\varphi^{-1}(y)$, where φ^{-1} is the inverse of function (30).

Recall that μ is a nonnegative integrable function on the strip Π of the half-plane $\alpha<\beta$, hence $\bar{\mu}:=\text{mes}_\mu\Pi<\infty$, and that (14) implies the strict estimate $y(t)<\bar{\mu}$ for all outputs of the Preisach operator. Consider the extension of the solution y of (35) from $[t_0,t_0+\delta)$ to the maximal interval $[t_0,t_1)$ such that $f(t,\varphi^{-1}(y(t)))>0$ and $y(t)<\bar{\mu}$ on this interval. By definition, y increases, $x=\varphi^{-1}(y)$ is a solution of problem (15)–(17) on $[t_0,t_1)$, and moreover, either $t_1=\infty$; or $t_1<\infty$ and $y\to\bar{\mu}$ as $t\to t_1-0$, which implies $x\to+\infty$ as $t\to t_1-0$, because $\varphi(\infty)=\bar{\mu}$; or $t_1<\infty$ and $y\to\tilde{y}<\bar{\mu}$, $x\to\tilde{x}=\varphi^{-1}(\tilde{y})<\infty$ as $t\to t_1-0$ with $f(t_1,\tilde{x})=0$. Thus, either the solution x of (15)–(17) is defined on the infinite interval $t\ge t_0$, or goes to $+\infty$ in a finite time, or it is extendable to a zero point $(t_1,\tilde{x})$ of f.

Assume that in the latter case $(t_1,\tilde{x})\notin\Omega$. Then by Lemma 5.5, $f_t(t_1,\tilde{x})<0$. Furthermore, because x strictly increases on the interval (t_0,t_1), the state $\tilde{\eta}_1=\eta(t_1)$ of the Preisach nonlinearity at the moment t_1 has a horizontal initial segment and therefore $t_1,x(t_1)=\tilde{x}$, $\eta(t_1)=\tilde{\eta}$ are admissible initial data for the Cauchy problem for system (15)–(16). Consequently, by Theorem 3.1, the solution x can be extended from the interval $[t_0,t_1)$ to some interval $[t_0,t_1+\delta)$. Moreover, according to the results of the previous subsection (see the last paragraph), the relations $f(t_1,\tilde{x})=0$, $f_t(t_1,\tilde{x})<0$ imply that $f(t,x(t))<0$ on some interval $(t_1,t_1+\delta_1)$.

Using the same type of argument as in the above case of $f(t,x(t))>0$, one shows that the solution x can be extended from the interval $(t_1,t_1+\delta_1)$ to the maximal interval (t_1,t_2) where x and y strictly decrease, $f(t,x(t))<0$ holds, and either $t_2=\infty$, or $t_2<\infty$ and $x\to-\infty$ as $t\to t_2-0$, or $t_2<\infty$ and $x\to\tilde{\tilde{x}}>-\infty$ as $t\to t_2-0$ with $f(t_2,\tilde{\tilde{x}})=0$, $f_t(t_2,\tilde{\tilde{x}})\ge 0$.

We can now continue to extend x successively to new intervals (t_{i-1},t_i) such that $f(t,x(t))$ is non-zero on each interval and has different signs on each two successive intervals. As a result, we arrive at the alternative of the following four cases.

Case 1. There is a sequence of moments $t_0<t_1<\cdots<t_N<T<\infty$ such that the solution x of problem (15)–(17) is defined on the finite interval $[t_0,T)$; the relations

$$f(t_i,x(t_i))=0, \qquad (-1)^i f_t(t_i,x(t_i))>0 \tag{36}$$

with $i=1,\dots,N$, are valid (recall that we consider an initial problem such that $f(t,x(t))>0$ for sufficiently small $t-t_0>0$); the solution x is strictly monotone on each of the intervals (t_{i-1},t_i) and on (t_N,T), all the moments t_i are extrema points of x, and either $x\to+\infty$ or $x\to-\infty$ as $t\to T-0$.

Case 2. There is a sequence of moments $t_0<t_1<\cdots<t_N<T<\infty$ such that the solution x of problem (15)–(17) is defined on the finite interval $[t_0,T)$; the relations (36) are valid; x is strictly monotone on each of the intervals (t_{i-1},t_i) and on (t_N,T), all the moments t_i are extrema points of x, and x goes to a finite limit x_* such that $(T,x_*)\in\Omega$ as $t\to T-0$.

Case 3. The solution x is defined on the infinite interval $[t_0,\infty)$; there is a sequence of moments $t_0<t_1<\cdots<t_N<\infty$ such that relations (36) hold, x is strictly monotone on each of the intervals (t_{i-1},t_i) and on (t_N,∞), and all the moments t_i are extrema points of x.

Case 4. There is an infinite sequence $t_0<\cdots t_1<\cdots<t_n<\cdots$ with a finite or infinite limit $t_n\to T$ such that the solution x is defined on the interval $[t_0, T)$, relations (36) are valid for all $i \geq 1$, the solution x is strictly monotone on each interval (t_{i-1}, t_i), and t_i are extrema points of x.

Consider the latter case in more detail. If $T = \infty$, then the solution x is defined on the infinite interval $[t_0, \infty)$.

Assume that $T < \infty$. Let us show that in this case $y \to \tilde{y}$ as $t \to T - 0$. For this purpose, consider the upper and lower limits

$$y_+ = \limsup_{t\to T-0} y(t), \qquad y_- = \liminf_{t\to T-0} y(t). \tag{37}$$

This limits satisfy $0 \leq y_- \leq y_+ \leq \bar{\mu}$, because the range of output values of the Preisach nonlinearity is $(0, \bar{\mu})$. Suppose $y_+ > y_-$. Consequently, $y_+ - y_- > 2\varepsilon$ for a sufficiently small $\varepsilon > 0$. Since an output of the Preisach nonlinearity tends to 0 and $\bar{\mu}$ as an input goes to $-\infty$ and $+\infty$, respectively, it follows that if $y_- + \varepsilon < y(t) < y_+ - \varepsilon$ at any moment t, then $|x(t)| \leq K$ and consequently $|y'(t)| = |f(t, x(t))| \leq K'$ at this moment with some positive finite $K = K(\varepsilon), K' = K'(\varepsilon)$. Hence, $K'|t_1 - t_2| \geq y_+ - y_- - 2\varepsilon > 0$ for any two moments t_1, t_2 such that $y(t_1) \geq y_+ - \varepsilon$ and $y(t_2) \leq y_- + \varepsilon$, which contradicts (37). It means that the assumption $y_- < y_+$ is wrong, i.e. y converges to a limit $\tilde{y}$ as $t \to T - 0$.

If $\tilde{y} = 0$, then $x \to -\infty$; if $\tilde{y} = \bar{\mu}$, then $x \to +\infty$ as $t \to T - 0$.

Let $0 < \tilde{y} < \bar{\mu}$. This implies $|x(t)| \leq K$ in some left neighbourhood of the point $t = T$. If we assume that x does not have a limit as $t \to T - 0$, then there is a $\varepsilon > 0$ and a sequence of segments $[t_n, \tau_n]$ such that $t_n, \tau_n \to T - 0$ and

$$|x(t_n) - x(\tau_n)| = \max_{s',s''\in[t_n,\tau_n]} |x(s') - x(s'')| > \varepsilon.$$

From these relations and assumption (14) on the measure density μ of the Preisach operator, it follows that $|y(t_n) - y(\tau_n)| \geq \delta > 0$ with

$$\delta = \min_{|\alpha'|\leq K,\ |\beta'|\leq K,\ \beta'-\alpha'\geq\varepsilon} \operatorname{mes}_\mu\{(\alpha, \beta) : \alpha' \leq \alpha < \beta \geq \beta'\}$$

for all n, thus we arrive at the contradiction with the fact that y converges. Consequently, x has a finite limit $\tilde{x}$ as $t \to T - 0$. Recall that in Case 4, we consider now, there is an infinite sequence $t_i \to T$ satisfying (36). Passing to the limit in (36), one obtains $f(T, \tilde{x}) = 0, f_t(T, \tilde{x}) = 0$, i.e. $(T, \tilde{x}) \in \Omega$.

We conclude that in any case x is uniquely extendable to a maximum interval $[t_0, T)$, finite or infinite, such that $(t, x(t)) \notin \Omega$ for all $t \in [t_0, T)$, and that if $T < \infty$, then x goes either to $+\infty$, or to $-\infty$, or to a finite value $\tilde{x}$ such that $(T, \tilde{x}) \in \Omega$ as $t \to T - 0$. This completes the proof.

5.6. *Proof of Theorem 3.3*

A number of statements of Theorem 3.3 are proved in the previous section. To complete the proof, it remains to show that a solution x of the Cauchy problem (15)–(17) has both the left and right derivatives at each point t and that for $t \neq t_k$ these derivatives are defined by formulae (21), (22) while for $t = t_k$ the right derivative is defined by (20), where t_k are all the moments such that $f(t_k, x(t_k)) = 0$ (by Lemma 5.5, the left derivative of x at any point t_k is zero).

First consider a point t_k. Because the state $\eta(t)$ is admissible for the current input value $x(t)$ at any moment t, the argument is the same for all k. To be definite, let $k = 0$ and $f_t(t_0, x_0) > 0$. As we know, this implies that x, y strictly increase and $f(t, x(t)) > 0$ on some interval $[t_0, t_0 + \delta)$.

From $f(t_0, x_0) = 0$, it follows that

$$f(t, x) = b(t - t_0) + c(x - x_0) + o(|t - t_0| + |x - x_0|) \tag{38}$$

as $t \to t_0$, $x \to x_0$ with $b = f_t(t_0, x_0) > 0$, $c = f_x(t_0, x_0)$. Furthermore, since η_0 is an admissible state for t_0, x_0, it has a vertical initial segment, i.e. $\alpha_*(x) \equiv x_0$ for all sufficiently small $x - x_0 > 0$. Hence

$$a_*(x) = \mu_0(x - x_0) + o(x - x_0), \quad x \to x_0 + 0, \tag{39}$$

with $\mu_0 = \mu(x_0, x_0) > 0$. Consequently,

$$f(t, x)/a_*(x) \geq \tilde{b}(t - t_0)/(x - x_0) - \tilde{c}, \quad t > t_0,\ x > x_0,$$

in a small vicinity of the point (t_0, x_0) with some $\tilde{b}, \tilde{c} > 0$. If $x(t) - x_0 \leq k'(t - t_0)$ for a sufficiently small $k' > 0$, then

$$x'(t) = f(t, x(t))/a_*(x(t)) \geq \tilde{b}/k' - \tilde{c} > k'.$$

Thus, the implication $x(t) - x_0 \leq k'(t - t_0) \Rightarrow x'(t) \geq k'$ holds almost everywhere on an interval $[t_0, t_0 + \delta)$ and therefore $x(t) - x_0 \geq k'(t - t_0)$ on this interval. Recalling Lemma 5.4, we conclude that for $t_0 < t < t_0 + \delta$

$$0 < k'(t - t_0) \leq x(t) - x_0 \leq k(t - t_0). \tag{40}$$

Since $\alpha_*(x) \equiv x_0$, the function $a_*(x)$ is continuous for small $x - x_0 \geq 0$. Hence, the solution x is continuously differentiable in a right vicinity of the point t_0 and $x'(t) = f(t, x(t))/a_*(x(t))$ everywhere on $(t_0, t_0 + \delta)$. Introducing the new variable $u = (x - x_0)/(t - t_0)$, we obtain

$$(t - t_0)u' + u = \frac{f(t, (t - t_0)u + x_0)}{a_*((t - t_0)u + x_0)}.$$

Relations (38)–(40) imply that the right-hand part here equals $(b + cu(t))/(\mu_0 u(t)) + \varphi(t)$ with $\varphi(t) \to 0$ as $t \to t_0 + 0$ and

$$0 < k' \leq u(t) \leq k, \qquad t_0 < t < t_0 + \delta. \tag{41}$$

Hence, $(t - t_0)u' = b/(\mu_0 u) + c/\mu_0 - u + \varphi(t)$.

Let us fix an arbitrary $\varepsilon > 0$. Then $|\varphi| < \varepsilon$ on any sufficiently small interval $t_0 < t \leq t_0 + \delta(\varepsilon)$ and therefore

$$\frac{b}{\mu_0 u} + \frac{c}{\mu_0} - u - \varepsilon \leq (t - t_0)u' \leq \frac{b}{\mu_0 u} + \frac{c}{\mu_0} - u + \varepsilon$$

on this interval. From the Theorem on Differential Inequalities it follows that u lies between the solutions $v_\pm$ of the differential equations

$$(t - t_0)v' = b/(\mu_0 v) + c/\mu_0 - v \pm \varepsilon,$$

that start from the point $v_\pm(t_0 + \delta) = u(t_0 + \delta) > 0$ at the moment $t_0 + \delta$ and are continued to the left. Both these equations have the form

$$(t - t_0)v' = Av^{-1} + B - v, \quad A > 0, \tag{42}$$

admitting an explicit positive solution for each initial value $v(t_0 + \delta) = v_0 > 0$ on some interval $t_1(v_0) < t \leq t_0 + \delta$. Namely, if $v(t_0 + \delta) = \xi$, where

$$\xi = (B + \sqrt{B^2 + 4A})/2$$

is the positive root of the equation $A + Bv - v^2 = 0$, then $v \equiv \xi$. Otherwise, $v = v(t)$ is defined by the equality

$$-2\ln(\delta^{-1}(t - t_0)) = \left(1 + \frac{B}{D}\right)\ln\frac{2v - B - D}{2v_0 - B - D} + \left(1 - \frac{B}{D}\right)\ln\frac{2v - B + D}{2v_0 - B + D}, \tag{43}$$

where $D = \sqrt{B^2 + 4A}$. Here the derivative of the right-hand part equals $2v/(v^2 - Bv - A)$, which is positive for $v > \xi$ and negative for $0 < v < \xi$. Hence, the positive solution v of (42) strictly decreases if $v(t_0 + \delta) = v_0 > \xi$, strictly increases if $0 < v_0 < \xi$, and is constant if $v_0 = \xi$. Moreover, if $\xi < v_0 \in [k', k]$, then there is a moment $\tilde{t} \in (t_0, t_0 + \delta)$ such that $v(\tilde{t}) = 2k$; if $\xi > v_0 \in [k', k]$, then there is a moment $\tilde{t} \in (t_0, t_0 + \delta)$ such that $v(\tilde{t}) = k'/2$.

Set $A = b/\mu_0$, $B_+ = c/\mu_0 + \varepsilon$ and consider the solution v_+ of (42) with this A and $B = B_+$, satisfying $v_+(t_0 + \delta) = u(t_0 + \delta)$. By the Theorem on Differential Inequalities, $v_+(t) \le u(t)$ on any segment $[\tilde{t}, t_0 + \delta] \subset [t_0, t_0 + \delta]$ where v_+ is defined. As we have seen, if we assume that

$$u(t_0 + \delta) > \xi_+ := (B_+ + \sqrt{B_+^2 + 4A})/2,$$

then there is a moment $\tilde{t} \in (t_0, t_0 + \delta)$ such that $v_+(\tilde{t}) = 2k$ (we use the estimate $k' \le u(t_0 + \delta) \le k$) and consequently $u(\tilde{t}) \ge 2k$. The latter estimate contradicts (41), hence our assumption is wrong, i.e. $u(t_0 + \delta) \le \xi_+$. Similarly, the Theorem on Differential Inequalities implies that the solution v_- of (42) with $B = B_- := c/\mu_0 - \varepsilon$, defined by the same initial value $v_-(t_0 + \delta) = u(t_0 + \delta)$, satisfies $v_-(t) \ge u(t)$. Here, the assumption

$$u(t_0 + \delta) < \xi_- := (B_- + \sqrt{B_-^2 + 4A})/2$$

leads to the conclusion that $v_-(\tilde{t}) = k'/2$ and hence $u(\tilde{t}) \le k'/2$ for some $\tilde{t} \in (t_0, t_0 + \delta)$, which contradicts (41) again. Thus, $\xi_- \le u(t_0 + \delta) \le \xi_+$. In other words, we have proved that for any $\varepsilon > 0$ and any δ from a sufficiently small interval $(0, \delta(\varepsilon))$,

$$\xi_- \le u(t_0 + \delta) = (x(t_0 + \delta) - x_0)/\delta \le \xi_+$$

with $\xi_\pm = (\frac{c}{2\mu_0} \pm \frac{\varepsilon}{2}) + \sqrt{(\frac{c}{2\mu_0} \pm \frac{\varepsilon}{2})^2 + \frac{b}{\mu_0}}$. This implies formula (20) for the right derivative of x at the points t_k.

Finally, consider any point $\tau \ne t_k$, i.e. $\tau \in (t_{k-1}, t_k)$. As we know, x is strictly monotone in some neighbourhood of τ: to be definite, assume that x strictly increases and $t_{k-1} = t_0$. Therefore, $x(t) - \alpha_*(x(t)) \ge \alpha_0 > 0$ in a sufficiently small vicinity I of the point τ, hence $a_*(x(t)) \ge a_0 > 0$ in I, and Lemma 5.3 implies

$$x(t) - x(\tau) = \int_\tau^t \frac{f(s, x(s))}{a_*(x(s))} ds, \quad t \in I. \tag{44}$$

Since $\alpha_*(x)$ is a decreasing function, it has both the left and right limit at each point x. Consequently, the same is true for the functions $a_*(x)$ and hence the function $f(t, x(t))/a_*(x(t))$ has the left and right limits at the point $t = \tau$ defined by

$$\lim_{s \to \tau \pm 0} f(s, x(x))/a_*(x(s)) = f(\tau, x(\tau))/a_*(x(\tau) \pm 0),$$

which are finite, because $a_*(x(t)) \ge a_0 > 0$. These relations and (44) imply that x has left and right derivatives at the point τ and formulae (21) hold. A similar argument applies if x decreases in I. This completes the proof.

5.7. Proof of Theorem 3.4

Let us assume that the conclusion of the theorem is wrong and show that this assumption leads to a contradiction. More precisely, we assume that there is a number $c > 0$, sequences x_0^n and η_0^n with $x_0^n \to x_0$, $\rho(\eta_0^n, \eta_0) \to 0$ and a sequence of moments $T^n \in (t_0, T - c]$ such that the solution $x = x(t)$ of initial problem (15)–(17) and the solution $x^n = x^n(t)$ of system (15)–(16) with the initial values $x^n(t_0) = x_0^n$, $\eta^n(t_0) = \eta_0^n$ satisfy

$$|x^n(T^n) - x(T^n)| = \varepsilon_0; \qquad |x^n(t) - x(t)| < \varepsilon_0 \quad \text{for } t_0 \le t < T^n$$

with the same $\varepsilon_0 > 0$ for all n. Without loss of generality, we can also assume that the sequence T^n converges.

We consider the function x^n on the segment $[t_0, T^n]$. By assumption, the norms $\|x^n\|_C = \max_{[t_0,T^n]} |x^n(t)|$ are uniformly bounded for all n, i.e. $\sup_n \|x^n\|_C \le \bar{C} < \infty$. Let us show that the sequence of functions x^n is equicontinuous. For this purpose, note that if $|x^n(t_1) - x^n(t_2)| \ge \delta > 0$, then there is a segment $[t_1', t_2'] \subset [t_1, t_2]$ such that $|x^n(t_1') - x^n(t_2')| \ge \delta$ and x^n reaches its maximum and minimum values on the segment $[t_1', t_2']$ at its ends. This implies $|y^n(t_1') - y^n(t_2')| \ge \varepsilon(\delta)$ with

$$\varepsilon(\delta) := \min_{-\bar{C} \le r \le \bar{C}} \text{mes}_\mu \{(\alpha, \beta) : r - \delta/2 \le \alpha < \beta \le r + \delta/2\},$$

where $\varepsilon(\delta) > 0$ due to (14). Combining the estimate $|y^n(t_1') - y^n(t_2')| \ge \varepsilon(\delta)$ with the relations

$$|(y^n)'| = |f(t, x^n(t))| \le \sup_{t_0 \le t \le T, |x| \le \bar{C}} |f(t, x)| =: \bar{\bar{C}} < \infty,$$

we obtain $\bar{\bar{C}}|t_1' - t_2'| \ge |y^n(t_1') - y^n(t_2')| \ge \varepsilon(\delta)$ and therefore $|t_1 - t_2| \ge \varepsilon(\delta)/\bar{\bar{C}}$. Thus, the implication $|x^n(t_1) - x^n(t_2)| \ge \delta \Rightarrow |t_1 - t_2| \ge \varepsilon(\delta)/\bar{\bar{C}}$ holds uniformly with respect to n, which proves equicontinuity of the sequence x^n.

Equicontinuity of x^n implies that $T^n - t_0 \ge \nu > 0$ (indeed, if we assume the opposite, then $T_n \to t_0 + 0$, hence $x(T^n) \to x_0$ and $x^n(T^n) - x_0^n \to 0$ due to the equicontinuity, which implies $x(T^n) - x^n(T^n) \to 0$ and therefore contradicts $|x(T^n) - x^n(T^n)| = \varepsilon_0$). Consequently, the limit $\tilde{T}$ of the sequence T^n satisfies $t_0 < \tilde{T} \le T$. Moreover, because $|x(T^n) - x^n(T^n)| = \varepsilon_0$, $T^n \to \tilde{T}$ and x^n is equicontinuous, there is a $\hat{T} \in (t_0, \tilde{T})$ such that all the functions x^n with sufficiently large n are defined on the segment $[t_0, \hat{T}]$ and

$$|x^n(\hat{T}) - x(\hat{T})| \ge \varepsilon_0/2. \tag{45}$$

Now, let us consider the restrictions of the functions x_n to the segment $[t_0, \hat{T}]$. Since the sequence $x^n : [t_0, \hat{T}] \to \mathbb{R}$ is uniformly bounded and equicontinuous, it is compact in $C = C[t_0, \hat{T}]$ and we can assume without loss of generality that $\|x^n - \tilde{x}\|_{C[t_0,\hat{T}]} \to 0$ for some $\tilde{x} \in C$. This relation and $\rho(\eta_0^n, \eta_0) \to 0$ imply the relation $\|y^n - \tilde{y}\|_{C[t_0,\hat{T}]} \to 0$ for the outputs $y_n = P[\eta_0^n]x^n$ and $\tilde{y} = P[\eta_0]\tilde{x}$ of the Preisach nonlinearity. The fact that x^n is a solution of system (15)–(16) implies $y^n(t) = y^n(t_0) + \int_{t_0}^t f(s, x^n(s))ds$ for $t_0 \le t \le \hat{T}$. Passing here to the limit as $n \to \infty$, we see that $\tilde{x} : [t_0, \hat{T}] \to \mathbb{R}$ is also a solution of (15)–(16). Because $\tilde{y} = P[\eta_0]\tilde{x}$ and $\tilde{x}(t_0) = \lim_{n\to\infty} x^n(t_0) = \lim_{n\to\infty} x_0^n = x_0$, the solutions $\tilde{x}$ and x have the same initial data (17) and therefore should coincide, according to Theorem 3.1. However, (45) implies $|\tilde{x}(\hat{T}) - x(\hat{T})| \ge \varepsilon_0/2 > 0$. Thus, we arrive at a contradiction, which proves the theorem.

5.8. Proof of Theorem 3.5

Lemma 5.6. *If $\eta_{01} \ge \eta_{02}$, $x_1(t) \ge x_2(t) + c$ for $t_0 \le t \le t_1$, and $x_1(t_1) = x_2(t_1) + c$ with any $c \in \mathbb{R}$, then $\eta_1(t_1) \ge \eta_2(t_1)$, where $\eta_j = \eta_j(t)$ is the output of the Preisach nonlinearity, corresponding to the input $x_j = x_j(t)$ and the initial state $\eta_j(t_0) = \eta_{0j}$.*

Proof. Consider the variable state $\tilde{\eta} = \tilde{\eta}(t)$, corresponding to the input $\tilde{x} \equiv x_1 - c$ and the initial state $\tilde{\eta}(t_0) = \eta_{01}$. It is a straightforward consequence of the definition of the Preisach nonlinearity that if a difference between two inputs is constant and an initial state is the same, then the state is the same at any moment $t \ge t_0$. Hence, $\tilde{\eta} \equiv \eta_1$. Now, because $\tilde{x} \ge x_2$ on the segment $[t_0, t_1]$ and $\eta_{01} - \eta_{02} \ge 0 \ge -2(\tilde{x}(t_0) - x_2(t_0))$, the monotonicity property (23) applied to the inputs $\tilde{x}$, x_2 and the initial states η_{01}, η_{02}, implies the estimate $\eta_1(t) - \eta_2(t) \ge -2(\tilde{x}(t) - x_2(t))$ for all $t \in [t_0, t_1]$. In particular, for $t = t_1$ from this estimate and the assumption that $x_2(t_1) = x_1(t_1) - c = \tilde{x}(t_1)$, we obtain $\eta_1(t_1) \ge \eta_2(t_1)$. □

Let us proceed by assuming that under the conditions of the theorem two solutions $x_1, x_2 : [t_0, T) \to \mathbb{R}$ with initial data $x_{01} > x_{02}$, $\eta_{01} \ge \eta_{02}$ and graphs lying in the domain $\mathbb{R}^2 \setminus \Omega$ can intersect and bring this assumption to a contradiction, which will prove the theorem. If the solutions x_1, x_2 are bounded, then they have limits as $t \to T - 0$, according to Theorem 3.2, and consequently they can be extended by continuity to the segment $[t_0, T]$. Thus, our assumption reads as

$$x_1(t) > x_2(t), \quad [t_0, t_*); \qquad x_1(t_*) = x_2(t_*);\ \eta_{01} \geq \eta_{02} \tag{46}$$

for some $t_* \in (t_0, T]$. We are going to show that it is wrong. For this purpose, consider the following alternative.

Case 1. There is a sequence of moments $t_1 < t_2 < \cdots < t_n < \cdots$ with $t_n \to t_*$ such that $(-1)^n f(t_n, x_k(t_n)) > 0$ for either $k = 1$ or $k = 2$ (or both).

Case 2. There is a $\tilde{t}_0 \in [t_0, t_*)$ such that the relation $f(t, x_1(t))\, f(t, x_2(t)) < 0$ holds on the interval $\tilde{t}_0 \leq t < t_*$.

Case 3. There is a $\tilde{t}_0 \in [t_0, t_*)$ such that the relation $f(t, x_1(t))\, f(t, x_2(t)) > 0$ holds on the interval $\tilde{t}_0 \leq t < t_*$.

By Theorem 3.3, one of the above three cases takes place. However, the assumptions of Theorem 3.5 that each point of the set Ω is isolated and that f_x is not zero at every such point exclude Case 1. Indeed, in this case on each interval (t_n, t_{n+1}) there is a $\tilde{t}_n$ such that $f(\tilde{t}_n, x_k(\tilde{t}_n)) = 0$, $(-1)^n f_t(\tilde{t}_n, x_k(\tilde{t}_n)) < 0$ and from $t_n \to t_*$ it follows that $\tilde{t}_n \to t_* - 0$, $x_k(\tilde{t}_n) \to x(t_*)$ and $f(t_*, x_k(t_*)) = 0, f_t(t_*, x_k(t_*)) = 0$, i.e. $(t_*, x_k(t_*)) \in \Omega$ (which implies $t_* = T$ too). By our assumption, this inclusion implies $f_x(t_*, x_k(t_*)) \neq 0$, hence the equality $f(t, x) = 0$ determines a smooth curve $\Gamma = (t, X(t))$ in some neighbourhood of the point $M_* := (t_*, x_k(t_*)) \in \Gamma$. Because $f(\tilde{t}_n, x_k(\tilde{t}_n)) = 0$ and the points $M_n = (\tilde{t}_n, x_k(\tilde{t}_n))$ converge to M_*, they lie on Γ for all sufficiently large n. Because $(-1)^n f_t(\tilde{t}_n, x_k(\tilde{t}_n)) < 0$, for every pair of points M_n and M_{n+1} of the curve Γ there is a point $N_n = (\hat{t}_n, X(\hat{t}_n))$ of Γ with $f_t(\hat{t}_n, X(\hat{t}_n)) = 0, \hat{t}_n \in (\tilde{t}_n, \tilde{t}_{n+1})$. Thus, $N_n \in \Omega$, the points N_n converge to M_*, and $\hat{t}_n < t_*$, hence $M_* \in \Omega$ is a limit point of the set Ω. This contradicts the other assumption of Theorem 3.5, namely that all points of Ω are isolated. Hence, Case 1 is not possible.

Let us consider Case 2 and show, how to reduce it to Case 3. Recall that $x_k(t)$ increases if $f(t, x_k(t)) > 0$ and decreases if $f(t, x_k(t)) < 0$ on an interval. Hence, relations (46) and $f(t, x_1(t)) f(t, x_2(t)) < 0, t \in [\tilde{t}_0, t_*)$, imply

$$f(t, x_1(t)) < 0, \qquad f(t, x_2(t)) > 0 \quad \text{for } \tilde{t}_0 \leq t < t_*.$$

Since $x_1(t_*) = x_2(t_*)$, these relations imply $f(t_*, x_*) = 0$ where $x_* = x_k(t_*)$. Moreover, from Lemma 5.6 it follows $f_t(t_*, x_*) = 0$, hence $(t_*, x_*) \in \Omega$ (and therefore $t_* = T$). Using the assumption that all points of Ω are isolated, let us fix a neighbourhood U of the point (t_*, x_*) where there are no other points of Ω. Then fix a $\tilde{t} \in [\tilde{t}_0, t_*)$, which is sufficiently close to t_*, such that

$$\{(t, x) :\ \tilde{t} \leq t \leq t_*,\ x_2(t) \leq x \leq x_1(t)\} \subset U. \tag{47}$$

Finally, take a point $\tilde{x}_{02} \in (x_{02}, x_{01})$ sufficiently close to x_{01} such that the solution $\tilde{x}_2 = \tilde{x}_2(t)$ of system (15)–(16) with the initial values $\tilde{x}_2(t_0) = \tilde{x}_{02}, \tilde{\eta}_2(t_0) = \eta_{02}$ is defined on the segment $[t_0, \tilde{t}]$ and satisfies

$$\max_{t_0 \leq t \leq \tilde{t}} |\tilde{x}_2(t) - x_1(t)| < \delta := \min_{t_0 \leq t \leq \tilde{t}} (x_1(t) - x_2(t))$$

(the existence of x_{02} follows from Theorem 3.4 on continuous dependence of solutions of (15)–(16) on initial data). This estimate implies $\tilde{x}_2(t) > x_2(t)$ on the segment $[t_0, \tilde{t}]$.

Assume that there is a moment $\tilde{t}_* \in [t_0, \tilde{t}]$ such that $\tilde{x}_2(\tilde{t}_*) = x_1(\tilde{t}_*)$; because $\tilde{x}_{02} < x_{01}$, we can additionally suppose without loss of generality that $\tilde{x}_2(t) < x_1(t)$ for $t \in [t_0, \tilde{t}_*)$. Then, considering the pair of solutions $x_1, \tilde{x}_2$ in place of x_1, x_2 and the segment $[t_0, \tilde{t}_*]$ in place of $[t_0, t_*]$, we arrive at Case 3 (which means that $f(t, x_1(t)) f(t, \tilde{x}_2(t)) > 0$ on some interval $[t', \tilde{t}_*)$). Indeed, as we have seen, Case 1 is not possible for any solution and in Case 2 the pair of solutions considered meets first time at a point of the set Ω. However, the point $(\tilde{t}_*, x_1(\tilde{t}_*))$ where x_1 meets $\tilde{x}_2$ does not belong to Ω, because the graph of x_1 lies in $\mathbb{R}^2 \setminus \Omega$ for $t < T$ and $\tilde{t}_* \leq \tilde{t} < t_* \leq T$, by construction. Thus, under the assumption that $x_1(\tilde{t}_*) = \tilde{x}_2(\tilde{t}_*)$ for some $\tilde{t}_* \in [t_0, \tilde{t}]$ we have Case 3 for the new pair of solution $x_1, \tilde{x}_2$.

Now, assume the opposite, namely that the solutions $x_1, \tilde{x}_2$ do not intersect on the segment $[t_0, \tilde{t}]$, i.e. $x_2(t) < \tilde{x}_2(t) < x_1(t)$ on this segment. Let us extend the solution $\tilde{x}_2$ to the right to a maximal interval $[t_0, \tilde{T})$ such that its graph lies in $\mathbb{R}^2 \setminus \Omega$. Since $(t_*, x_1(t_*)) = (t_*, x_2(t_*))$ is the only point of the set Ω in the set (47), from Theorem 3.2 it follows that either there is a $\tilde{t}_* \in (\tilde{t}, t_*)$ such that

$$x_2 < \tilde{x}_2 < x_1 \quad \text{for } t \in [t_0, \tilde{t}_*), \qquad \tilde{x}_2(\tilde{t}_*) = x_k(\tilde{t}_*) \tag{48}$$

for one of the indices $k = 1, 2$; or $\tilde{T} = t_*$ and

$$x_2 < \tilde{x}_2 < x_1 \quad \text{for } t \in [t_0, t_*), \tag{49}$$

$$\lim_{t \to t_* - 0} \tilde{x}_2(t) = x_1(t_*) = x_2(t_*). \tag{50}$$

If (48) with $\tilde{t}_* < t_*$ holds, then we again have Case 3 for the pair of solutions x_k and $\tilde{x}_2$ on the segment $[t_0, \tilde{t}_*]$, by the same reasoning as in the previous paragraph (Case 2 would imply $(\tilde{t}_*, x_k(\tilde{t}_*)) \in \Omega$, which is not true for $\tilde{t}_* < t_* \leq T$). If relations (49), (50) hold, then we extend the solution $\tilde{x}_2$ by continuity to the segment $[t_0, t_*]$ from the interval $[t_0, t_*)$ and arrive at the situation with the three solutions $x_1, x_2, \tilde{x}_2$ meeting at the point $(t_*, x_1(t_*))$. Because Case 1 is excluded for any solution, $f_k(t, \tilde{x}_2(t)) \neq 0, f_k(t, x_k(t)) \neq 0$, $k = 1, 2$, on some interval $[\tilde{t}_0, t_*)$ and therefore we have Case 3 either for the pair $x_1, \tilde{x}_2$ or for the pair $x_2, \tilde{x}_2$.

Thus, we conclude that if Case 2 takes place for the solutions x_1, x_2, then we always have Case 3 for a pair $x_k, \tilde{x}_2$ with either $k = 1$ or $k = 2$ on some interval $[\tilde{t}_0, \tilde{t}_*] \subset [\tilde{t}_0, t_*]$. Moreover, by our construction,

$$x_1(t) > \tilde{x}_2(t) > x_2(t) \quad \text{for } t_0 \leq t < \tilde{t}_*, \quad x_k(\tilde{t}_*) = \tilde{x}_2(\tilde{t}_*)$$

and $\eta_{01} \geq \tilde{\eta}_2(t_0) = \eta_{02}$ for the initial states of these three solutions, therefore the relations analogous to (46) hold for the pair $x_k, \tilde{x}_2$.

Thus, it remains to consider Case 3. To be definite, assume that it holds for the pair x_1, x_2 itself, i.e. (46) is valid and $f(t, x_1(t)) f(t, x_2(t)) > 0$ on an interval $[\tilde{t}_0, t_*) \subset [t_0, t_*)$. Define $\delta = \min_{[t_0, \tilde{t}_0]}(x_1 - x_2) > 0$ and consider the smallest $\tau_0 \in [t_0, t_*)$ such that $x_1(\tau_0) - x_2(\tau_0) = \delta/2$. By Lemma 5.6, from the relations $\eta_{01} \geq \eta_{02}, x_1(t) > x_2(t) + \delta/2$ for $t \in [t_0, \tau_0]$, and $x_1(\tau_0) = x_2(\tau_0) + \delta/2$ it follows that $\eta_1(\tau_0) \geq \eta_2(\tau_0)$, hence the relations

$$\begin{aligned} &x_1(t) > x_2(t) \quad \text{for } \tau_0 \leq t < t_*, \\ &x_1(t_*) = x_2(t_*); \quad \eta_1(\tau_0) \geq \eta_2(\tau_0) \end{aligned} \tag{51}$$

similar to (46) with t_0 replaced by τ_0 are valid. In addition, $\tau_0 > \tilde{t}_0$ implies

$$f(t, x_1(t)) f(t, x_2(t)) > 0 \quad \text{for all } \tau_0 \leq t < t_*. \tag{52}$$

Moreover, because $f(t, x_k(t)) \neq 0$ on the segment $[\tilde{t}_0, \tau_0]$, the solutions x_k are strictly monotone on this segment and hence each of the states $\eta_k(\tau_0)$ has a vertical initial segment if $f(t, x_k(t)) < 0$ and a horizontal initial segment if $f(t, x_k(t)) > 0$. From now on we consider τ_0 as the initial moment and $\eta_k(\tau_0)$ as the initial states. The fact that they have either a vertical or a horizontal initial segment is equivalent to

$$\alpha_*(x_k(t)) \geq c > 0, \qquad a_*(x_k(t)) \geq c' > 0, \quad t > \tau_0. \tag{53}$$

Set $\Delta x(t) = x_1(t) - x_2(t)$, $v(t) = \min_{\tau \in [\tau_0, t]} \Delta x(\tau)$ for $\tau_0 \leq t \leq t_*$. By definition, the function v decreases, $v(t) > 0$ for $\tau_0 \leq t < t_*$ and $v(t_*) = 0$. Since x_1, x_2 are absolutely continuous on $[\tau_0, t_*]$, the function v is absolutely continuous on this segment too (indeed, for any segment $[t_1, t_2]$ there is a segment $[s_1, s_2] \subset [t_1, t_2]$ such that $|v(t_1) - v(t_2)| \le |\Delta x(s_1) - \Delta x(s_2)|$, hence for any N disjoint segments $[t_1^n, t_2^n]$ there are segments $[s_1^n, s_2^n] \subset [t_1^n, t_2^n]$ such that $\sum_{n=1}^N |v(t_1^n) - v(t_2^n)| \le \sum_{n=1}^N |\Delta x(s_1^n) - \Delta x(s_2^n)|$ and therefore the fact that Δx is absolutely continuous implies that v is absolutely continuous). We shall consider points t where all the three derivatives $x_1'(t), x_2'(t)$, and $v'(t)$ are defined, i.e. almost every $t \in (\tau_0, t_*)$.

From the definition of v it follows that $v(t) \le \Delta x(t)$ for all t and that if $v(t) < \Delta x(t)$ at some point t then $v \equiv const$ in a sufficiently small neighbourhood of t. Therefore, if v strictly decreases on some interval (τ_1, τ_2), then $v = \Delta x$ on (τ_1, τ_2). Consequently, $v = \Delta x$ in a small neighbourhood of a point t whenever $v'(t) < 0$. Thus, for almost every $t \in (\tau_0, t_*)$ either $v'(t) = 0$, or $v'(t) < 0$ and

$$v'(t) = \Delta x'(t) = \frac{f(t, x_1(t)) - f(t, x_2(t))}{a_*(x_2(t))} + \frac{a_*(x_2(t)) - a_*(x_1(t))}{a_*(x_2(t))a_*(x_1(t))} f(t, x_1(t)).$$

Here $f(t, x_1(t)) - f(t, x_2(t)) \ge -c_0 \Delta x(t) = -c_0 v(t)$ for some $c_0 > 0$. Hence if $v'(t) < 0$, then

$$v'(t) \ge \frac{a_*(x_2(t)) - a_*(x_1(t))}{a_*(x_2(t))a_*(x_1(t))} f(t, x_1(t)) - c_1 v(t), \tag{54}$$

where we take into account (53). Furthermore, because $\Delta x(s) \ge v(s)$ for $\tau_0 \le s \le t \le t_*$ and $\Delta x(t) = v(t)$ whenever $v'(t) < 0$, from Lemma 5.6 it follows that

$$\eta_1(t) \ge \eta_2(t) \quad \text{if } v'(t) < 0. \tag{55}$$

Relation (52) implies that the functions $f(t, x_k(t))$ are nonzero and have the same sign on the interval $[\tau_0, t_*)$. Assume that $f(t, x_k(t)) > 0$ and consequently both the solutions x_1, x_2 strictly increase on $[\tau_0, t_*)$. Therefore, if $\eta_1(t) \ge \eta_2(t)$ at a moment t, then

$$x_1(t) - \alpha_*(x_1(t)) \le x_2(t) - \alpha_*(x_2(t)). \tag{56}$$

Combining this with $x_1(t) \ge x_2(t)$ and (55), we obtain

$$\alpha_*(x_2(t)) \le \alpha_*(x_1(t)) + x_2(t) - x_1(t) \le \alpha_*(x_1(t)) \tag{57}$$

if $v'(t) < 0$. Consider the difference

$$a_*(x_2) - a_*(x_1) = -\int_{\alpha_*(x_1)}^{x_2} \mu(\alpha, x_1) d\alpha - \int_{x_2}^{x_1} \mu(\alpha, x_1) d\alpha + \int_{\alpha_*(x_1)}^{x_2} \mu(\alpha, x_2) d\alpha + \int_{\alpha_*(x_2)}^{\alpha_*(x_1)} \mu(\alpha, x_2) d\alpha$$

with $x_k = x_k(t)$. If $v'(t) < 0$, then the last integral here is nonnegative due to (57), and hence

$$a_*(x_2) - a_*(x_1) \ge \int_{\alpha_*(x_1)}^{x_2} (\mu(\alpha, x_2) - \mu(\alpha, x_1)) d\alpha - \int_{x_2}^{x_1} \mu(\alpha, x_1) d\alpha \ge -c_2(x_1 - x_2) = -c_2 v$$

with some $c_1 > 0$, where we use the fact that μ is bounded and locally Lipschitz continuous. From these relations and estimates (53), (54), and $0 < f(t, x_1(t)) \le \bar{c} < \infty$, it follows that for *a.e.* $t \in (\tau_0, t_*)$

$$v'(t) \ge -\kappa v(t) \quad \text{whenever } v'(t) < 0 \tag{58}$$

with a $\kappa > 0$ independent of t.

Now, suppose that $f(t, x_k(t)) < 0$, hence the solutions x_1, x_2 strictly decrease on $[\tau_0, t_*)$. Here, the same relations (56), (57) hold whenever $\eta_1(t) \ge \eta_2(t)$. Therefore, if $v'(t) < 0$, then the last integral in

$$a_*(x_2) - a_*(x_1) = \int_{x_2}^{x_1} \mu(x_2, \beta) d\beta + \int_{x_1}^{\alpha_*(x_2)} \mu(x_2, \beta) d\beta - \int_{x_1}^{\alpha_*(x_2)} \mu(x_1, \beta) d\beta - \int_{\alpha_*(x_2)}^{\alpha_*(x_1)} \mu(x_1, \beta) d\beta$$

is nonnegative (note that this relation has different forms in cases of decreasing and increasing inputs x_k), hence

$$a_*(x_2) - a_*(x_1) \le \int_{x_1}^{\alpha_*(x_2)} (\mu(x_2, \beta) - \mu(x_1, \beta)) d\beta + \int_{x_2}^{x_1} \mu(x_2, \beta) d\beta \le \bar{c}_2(x_1 - x_2) = \bar{c}_2 v.$$

Combining this with (53), (54), and $0 > f(t, x_1(t)) \ge -\bar{c} > -\infty$, we arrive at (58) again.

Thus, (58) is valid both in cases of positive and negative functions $f(t, x_k(t))$. Because $v'(t) \ge -\kappa v(t)$ trivially holds at every point t with $v'(t) = 0$ too, we conclude that $v'(t) \ge -\kappa v(t)$ almost everywhere in (τ_0, t_*). By the Gronwall inequality, this contradicts the relation $v(\tau_0) > 0$, $v(t_*) = 0$ that follow from (46). This contradiction proves that relation (46) and consequently the assumption (46) are wrong, which proves the theorem.

6. Conclusions

We have considered operator-differential equations which have been recently proposed for modelling dynamics of hydrological, economic and biological systems or their components. In these models the rate of change of the output of the Preisach operator is a function of the input variable x and time t; their dynamics is characterised by nonlinear effects of two types, namely, memory effects introduced by the Preisach operator and a singularity on certain lines of the (t, x)-plane originating from nonsmoothness of the Preisach operator. In particular, these effects account for discontinuities of the derivative of solutions. We have analysed the singularity and computed the jumps of the derivative at non-smoothness points, thus providing an input to numerical schemes for solving the equation. Previous studies proved well-posedness of the Cauchy problem for the equation with decreasing right-hand side; in particular, systems driven by positive feedbacks possess this monotonicity. Here, we have shown that the presence of negative feedback loops in the structure of the system can result in isolated non-uniqueness points. In the domain of uniqueness, the initial value problem is well posed. We have developed further insight into the global properties of the semiflow by establishing a stratification property for the projection of the semiflow on the (t, x)-plane. This property, specific to the equations considered and generally not valid for equations with hysteresis operators, ensures that projections of solutions on the (t, x)-plane do not intersect. Using this property, we have characterised domains of sensitivity to small perturbations of initial data originating from non-uniqueness points. We have presented rigorous analysis and proofs of the above results as well as a survey of further existing results. In particular, global stability of periodically driven systems with decreasing right-hand side and regularisation of the equation have been discussed. Overall, we have shown that equations of the type (1) have 'nice' mathematical characteristics and the nonlinear effects of memory and singularity manifested in their dynamics are accessible for analysis both by analytic and numerical methods.

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