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OPTIMAL CONTROL OF ODE SYSTEMS INVOLVING A RATE INDEPENDENT VARIATIONAL INEQUALITY

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Abstract. This paper is concerned with an optimal control problem for a system of ordinary differential equations with rate independent hysteresis modelled as a rate independent evolution variational inequality with a closed convex constraint $Z \subset \mathbb{R}^m$. We prove existence of optimal solutions as well as necessary optimality conditions of first order. In particular, under certain regularity assumptions we completely characterize the jump behaviour of the adjoint.

1. Introduction.

A main ingredient of the control problem to be considered is the evolution variational inequality (EVI)

$$\begin{aligned} \langle \dot{z} - \dot{v}\,,\, z - \zeta \rangle \le 0 \quad &\text{for all } \zeta \in Z, \text{ a.e. in } [0,T], \\ z(t) \in Z \quad \text{for all } t \in [0,T], &\quad z(0) = z_0 \in Z\,, \end{aligned} \tag{1}$$

on a fixed time interval $[0,T]$. It involves an input function $v : [0,T] \to \mathbb{R}^m$, an output function $z : [0,T] \to \mathbb{R}^m$ and a closed convex constraint $Z \subset \mathbb{R}^m$. It was introduced, in an equivalent formulation as a differential inclusion termed *sweeping process* (processus du rafle), by Moreau in [1, 2]. Its solution operator $z = \mathcal{W}[v; z_0]$ is rate independent and has the Volterra property; such operators are called hysteresis operators. The properties of 1 have been studied to a large extent, see e.g. [3, 4] and, for the rather general class of regulated input functions, [5, 6].

We consider the following control problem (P).

$$\text{Minimize} \qquad J(y,z,u) = \int_0^T \Big(L(t, y(t), z(t)) + \frac{1}{2} u(t)^T E u(t) \Big)\, \mathrm{d}t \tag{2}$$

subject to the dynamics defined by 1 coupled to

$$\begin{aligned} \dot{y} &= f(t,y,z) + Bu\,, \quad y(0) = y_0\,, \\ v &= Sy\,, \end{aligned} \tag{3}$$

and subject to the control constraint

$$u(t) \in \Omega\,, \tag{4}$$

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where $\Omega \subset \mathbb{R}^d$ is a given set. The state functions y and v, z take values in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The matrices B, E and S are constant and of appropriate dimension, f and L are given functions. Throughout the text, we work with functions of time $t \in [0,T]$ with values in vector spaces of different dimensions. For simplicity, we denote the Lebesgue and Sobolev spaces of such functions $L^p(0,T)$ and $W^{k,p}(0,T)$ without specifying the dimension, or, if no confusion can occur, simply L^p and $W^{k,p}$.

For the vectorial case $m > 1$ of the EVI, we are not aware of any publication where necessary optimality conditions are derived for the control problem above. For the scalar case $m = 1$, necessary optimality conditions have been obtained in [7, 8, 9] for hysteresis operators more general than the one represented by 1, including the Preisach operator. Those proofs used time discretization and smoothing of the hysteresis within the general setting presented in the monograph [10, 11] of Mark Krasnosel'skiĭ and Alexei Pokrovskiĭ, they were not based on variational inequalities. Reference [7] has been translated in [12, 13]. Recently, the case $m = 1$ of the EVI coupled to the harmonic oscillator was treated in [14] with variational techniques.

The main difficulty in the derivation of optimality conditions for such problems stems from the apparent lack of differentiability of the solution operator $(v, z_0) \mapsto z$ of the EVI. One way to overcome this, and this is what we do here, is to regularize the variational inequality by replacing it with an equation which includes an additional term in order to drive the state towards the constraint Z if it is outside of Z. In contrast to [14] where a non-smooth penalization has been used, we use a more standard smooth penalization in order to avoid complications which otherwise appear when trying to linearize the regularized problem. The optimality conditions for the original problem are then obtained by a limit process from those of the regularized problem. This is the difficult part, because a priori estimates are required which reflect the loss of regularity of the adjoint (as compared to the unconstrained case) and which are not an immediate consequence of the underlying monotonicity properties of the EVI.

In the present paper, we deal with the case of a smooth strictly convex constraint Z. The polyhedral case appears to require a different proof. But this phenomenon again and again occurs in the analysis of rate-independent situations, compare [3].

Let us also mention the recent paper [15], where the optimal control of the EVI itself (not coupled to another evolution) is treated. Optimality conditions are obtained for the case of a half-space Z, the control being position and direction of the hyperplane ∂Z. There, a time discretization is employed, on the other hand techniques of variational analysis and generalized derivatives are used on the inequality directly, without regularization.

Instead of characterizing optimal controls and states explicitly, one might also use the dynamic programming approach in order to derive an HJB equation resp. inequality for the optimal value function $V(y_0, z_0) = \inf_u J(y(u), z(u), u)$ of problem (P) parametrized by the initial conditions (y_0, z_0), and prove that V is its unique viscosity solution; properties of the optimal control and state should follow from this. This line of research started with [16], where a problem with a rate independent delayed relay was treated, and so far seems to be concerned with infinite horizon problems ($T = +\infty$) for various types of rate independent nonlinearities, see [17],[18] and the references therein.

Finally, let us remark that we concentrate on the interaction of the EVI and the ODE system and restrict ourselves to the specific form of the control problem given above. Standard techniques from optimal control in order to treat different cost functionals or more general right hand sides, or to obtain the "maximum" form of the optimality condition could be used as well. Additional terminal constraints or pointwise state constraints would raise the issue of controllability and of regularity of the corresponding multipliers which we do not discuss here.

2. **Existence of solutions of (P).**

2.1. **Wellposedness of the dynamics.**

It is well known that the EVI 1 has a unique solution $z \in W^{1,1}(0,T)$ for any given $v \in W^{1,1}(0,T)$ and $z_0 \in Z$. A proof can be found in [3]. Nevertheless let us recall the basic stability estimate; it shows in a nutshell the relevance of the total variation and of the maximum norm (both are "rate independent" in the sense that they are invariant under time transformation).

Lemma 2.1. *Let $z_1 = \mathcal{W}[v_1; z_{0,1}]$, $z_2 = \mathcal{W}[v_2; z_{0,2}]$. Then*

$$|z_1(t) - z_2(t)| \le |z_{0,1} - z_{0,2}| + \int_0^t |\dot v_1(s) - \dot v_2(s)|\,\mathrm{d}s\,, \quad \textit{for all } t \in [0,T]\,. \tag{5}$$

Proof. Testing the variational inequality for z_1 with z_2 and vice versa, we obtain pointwise a.e. in t

$$\begin{aligned} |z_1 - z_2|\frac{\mathrm{d}}{\mathrm{d}t}|z_1 - z_2| = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|z_1 - z_2|^2 = \langle \dot z_1 - \dot z_2\,,\, z_1 - z_2\rangle \le \langle \dot v_1 - \dot v_2\,,\, z_1 - z_2\rangle \\ \le |\dot v_1 - \dot v_2|\,|z_1 - z_2| \end{aligned} \tag{6}$$

Dividing by $|z_1 - z_2|$ where nonzero and integrating yields the assertion. □

Thus, $\mathcal{W}[\cdot, z_0]$ viewed as an operator from $W^{1,1}(0,T)$ to $C[0,T]$ is Lipschitz continuous with Lipschitz constant 1.

Hypothesis 2.2. *The right-hand side f in 3 has the following properties:*

(i) *f is measurable with respect to t and locally Lipschitz with respect to (y,z) in the sense that there exist $\lambda \in L^1(0,T)$ and $G : (0,\infty) \to (0,\infty)$ such that for all $|y_i| \le R, |z_i| \le R$, $i = 1,2$, and a.e. $t \in (0,T)$ we have*

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \le \lambda(t)G(R)(|y_1 - y_2| + |z_1 - z_2|); \tag{7}$$

(ii) *f satisfies a linear growth condition*

$$|f(t,y,z)| \le \alpha_0(t) + \alpha_1(|y| + |z|) \tag{8}$$

for some $\alpha_0 \in L^1(0,T)$ and some $\alpha_1 > 0$.

Under Hypothesis 2.2, the standard contraction argument applied to

$$y(t) = y_0 + \int_0^t \Big(f(s, y(s), (\mathcal{W}[Sy])(s)) + Bu(s)\Big)\,\mathrm{d}s$$

in the space $W^{1,1}(0,\delta)$ for small enough $\delta > 0$ shows that the coupled dynamics has a unique local solution for any given control $u \in L^1$. Since $|\dot z| \le |\dot v|$ holds pointwise a.e. for the EVI $z = \mathcal{W}[v; z_0]$, any solution (y,z) of the coupled dynamics satisfies, for any t within its interval of existence,

$$|\dot y(t)| + |\dot z(t)| \le c(\alpha_0(t) + |y(t)| + |z(t)| + |u(t)|)\,, \quad \text{for some } c > 0\,. \tag{9}$$

From Gronwall's lemma we now conclude that 1, 3 has a unique global solution $(y,z) \in W^{1,p}(0,T)$ if $\alpha_0, u \in L^p(0,T)$ and $p \in [1,\infty]$, which we denote as $(y(u), z(u))$. Moreover,

$$\|y(u)\|_\infty + \|z(u)\|_\infty \le C(1 + \|u\|_1)\,, \tag{10}$$

where C does not depend on u. In addition, if $\alpha_0, u \in L^p(0,T)$, we get by virtue of 9 and 10 that

$$\begin{aligned} \|\dot y(u)\|_p + \|\dot z(u)\|_p &\le c(\|\alpha_0\|_p + \|y(u)\|_p + \|z(u)\|_p + \|u\|_p) \\ &\le C(1 + \|u\|_p)\,, \end{aligned} \tag{11}$$

where again C does not depend on u.

Lemma 2.3. *If $u_k \to u$ weakly in L^2 and $\alpha_0 \in L^2$, then $y(u_k) \to y(u)$ and $z(u_k) \to z(u)$ uniformly.*

Proof. Let $y_k = y(u_k)$, $z_k = z(u_k)$. By 11, $(\dot y_k, \dot z_k) \to (p,q)$ weakly in L^2 for some subsequence. Since the embedding $H^1(0,T) \to C[0,T]$ is compact, $(y_k, z_k) \to (\tilde y, \tilde z)$ uniformly with $(\dot{\tilde y}, \dot{\tilde z}) = (p,q)$. We therefore may pass to the limit in 3 as well as in the integral form

$$\int_0^T \langle \dot z_k(s) - \dot v_k(s)\,,\, z_k(s) - \zeta(s)\rangle \,\mathrm{d}s \le 0\,, \quad \zeta \in L^2(0,T)\,,$$

of the EVI, and thus also in its pointwise form. Since moreover $\tilde z(t) \in Z$ for all t, because Z is closed, we conclude that $\tilde y = y(u)$ and $\tilde z = z(u)$ and that the whole sequence converges. □

2.2. **Existence of an optimal control.** Let the integrand L in the cost functional 2 satisfy a Carathéodory condition as well as the growth condition

$$|L(t,y,z)| \le \beta_0(t) \cdot \beta_1(y,z) \tag{12}$$

for some $\beta_0 \in L^1$ and some continuous function β_1, let E be symmetric and positive semidefinite, let $\Omega \subset \mathbb{R}^d$ be closed and convex. Assume further that L, E and Ω are such that that

every minimizing sequence of controls $u_k \in L^2(0,T)$ satisfies
$\|u_k\|_2 \le C$ for some constant C not depending on k. (13)

This is true if e.g. Ω is bounded, or if L is bounded from below by some constant and E is positive definite, that is, $u^T E u \ge \beta_2 |u|^2$ for some $\beta_2 > 0$.

Theorem 2.4. *Let 8 and 13 hold, let $L^2(0,T)$ be the admissible set of controls. Then there exists an optimal control $u_* \in L^2(0,T)$ for problem (P).*

Proof. Under the assumptions stated above, any weak limit u_* in L^2 of a subsequence of any minimizing sequence $\{u_k\}$ provides an optimal control for (P), due to Lemma 2.3. □

When $u_* \in L^2$, the corresponding state $y_* = y(u_*)$ satisfies $y_* \in H^1$, moreover $v_* = Sy_* \in H^1$. The function $\xi = v_* - z_*$ is called the play of v_*. We have (see [3, Proposition 4.1]) $|\dot z_*| \le |\dot v_*|$ and $|\dot \xi| \le |\dot v_*|$ pointwise a.e., thus $z_* \in H^1$ and $\xi \in H^1$. The optimal trajectory in general will consist of pieces on $\mathrm{int}(Z)$ and on ∂Z. Accordingly, we introduce the decomposition $[0,T] = I_0 \cup I_\partial$ into the disjoint sets

$$I_0 = \{t : z_*(t) \in \mathrm{int}(Z)\}\,, \quad I_\partial = \{t : z_*(t) \in \partial Z\}\,. \tag{14}$$

Note that I_0 is open and I_∂ is closed in $[0,T]$. We have $\dot{\xi} = 0$ a.e. on I_0 and

$$\dot{\xi}(t) = |\dot{\xi}(t)| n(t)\,, \quad \text{a.e. on } I_\partial\,, \tag{15}$$

where $n(t)$ denotes the outer unit normal to ∂Z in $z_*(t)$.

3. The regularized problem.

3.1. **Regularized dynamics.** We approximate the variational inequality 1 by the differential equation, for any $\varepsilon > 0$,

$$\dot{z} - \dot{v} = -\frac{1}{\varepsilon} \nabla \Psi(z)\,, \tag{16}$$

where $\Psi : \mathbb{R}^m \to \mathbb{R}$ is convex and twice continuously differentiable, $\Psi = 0$ on Z and $\Psi > 0$ outside of Z. In Subsection 3.4 below, Ψ will be constructed explicitly. Together with 3 this gives the initial value problem

$$\dot{y} = f(t,y,z) + Bu\,, \qquad\qquad y(0) = y_0\,, \tag{17}$$

$$\dot{z} = S(f(t,y,z) + Bu) - \frac{1}{\varepsilon}\nabla\Psi(z)\,, \quad z(0) = z_0\,. \tag{18}$$

Let $u \in L^1(0,T)$ be given, let $(y,z) \in W^{1,1}$ be the corresponding solution with maximal existence interval $I \subset [0,T]$. We have

$$|z(t)| - |z_0| = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}|z(s)|\,\mathrm{d}s = \int_0^t \left\langle \frac{z(s)}{|z(s)|}\,,\, \dot{z}(s) \right\rangle \mathrm{d}s\,,$$

note that the first integrand as well as $\dot{z}$ are zero a.e. on $\{z = 0\}$ because z is absolutely continuous. We insert $\dot{z}$ from 18, use 8 as well as the monotonicity of $\nabla\Psi$ (note that $\nabla\Psi(0) = 0$ since $0 \in Z$, and therefore $\langle \nabla\Psi(\zeta)), \zeta\rangle \geq 0$ for all $\zeta \in \mathbb{R}^n$) and obtain

$$|z(t)| \leq c_0 + c_1 \int_0^t |y(s)| + |z(s)| + |u(s)|\,\mathrm{d}s\,, \tag{19}$$

where the constants c_0 and c_1 do not depend upon ε and u. A corresponding estimate, with $|y(t)|$ instead of $|z(t)|$ in 19, follows from 17 and 8. Therefore, using Gronwall's lemma we conclude as above that $I = [0,T]$ and $(y^\varepsilon(u), z^\varepsilon(u)) := (y,z)$ satisfies

$$\|y^\varepsilon(u)\|_\infty \leq C(1 + \|u\|_1)\,, \quad \|z^\varepsilon(u)\|_\infty \leq C(1 + \|u\|_1)\,, \tag{20}$$

with a constant C independent from ε and u.

Lemma 3.1. *If $u_k \to u$ weakly in L^2 and $\alpha_0 \in L^2$, then $y^\varepsilon(u_k) \to y^\varepsilon(u)$ and $z^\varepsilon(u_k) \to z^\varepsilon(u)$ uniformly, for any given $\varepsilon > 0$.*

Proof. Using 20 instead of 10, an estimate analogous to 11 is obtained for $p = 2$, and the proof proceeds along the same lines as that of Lemma 2.3. The z-component of the system is treated in the same manner as the y-component. □

Let $\alpha_0 \in L^p$. From 17 we get as above in 11, due to 8 and 20, that

$$\|\dot{y}^\varepsilon(u)\|_p \leq C(1 + \|u\|_p)\,, \tag{21}$$

where C is independent from ε and u.

In the following we consider the case $p = 2$ and assume $\alpha_0 \in L^2$. Testing 18 with z we get, for any $t \in [0,T]$

$$\frac{1}{2}|z(t)|^2 - \frac{1}{2}|z_0|^2 = \int_0^t \langle \dot{z}, z\rangle\ \mathrm{d}s = \int_0^t \langle S\dot{y}, z\rangle\ \mathrm{d}s - \frac{1}{\varepsilon}\int_0^t \langle \nabla\Psi(z), z\rangle\ \mathrm{d}s\,.$$

Using 20, 21 and the monotonicity of $\nabla\Psi$ yields

$$0 \le \sup_{t\in[0,T]} \frac{1}{\varepsilon}\int_0^t \langle \nabla\Psi(z), z\rangle \,\mathrm{d}s \le C(1+\|u\|_2)\,, \tag{22}$$

where C is independent from ε and u. Testing 18 with $\dot z$ gives, for any $t\in[0,T]$,

$$\int_0^t \langle \dot z, \dot z\rangle \,\mathrm{d}s = \int_0^t \langle S\dot y, \dot z\rangle \,\mathrm{d}s - \frac{1}{\varepsilon}\int_0^t \langle \nabla\Psi(z), \dot z\rangle \,\mathrm{d}s\,,$$

so

$$\int_0^t |\dot z|^2 \,\mathrm{d}s \le \frac{1}{2}\int_0^t |\dot z|^2 \,\mathrm{d}s + C(1+\|u\|_2)^2 - \frac{1}{\varepsilon}(\Psi(z(t)) - \Psi(z_0))\,,$$

and therefore, since $\Psi(z_0)=0$,

$$\int_0^T |\dot z^\varepsilon(u)|^2 \,\mathrm{d}s + \sup_{t\in[0,T]} \frac{1}{\varepsilon}\Psi(z^\varepsilon(u)(t)) \le C(1+\|u\|_2)^2\,, \tag{23}$$

where C is independent from ε and u.

Lemma 3.2. *For any sequence $u_\varepsilon \in L^2(0,T)$ with $u_\varepsilon \to u$ weakly in $L^2(0,T)$, we have $y^\varepsilon(u_\varepsilon)\to y(u)$ and $z^\varepsilon(u_\varepsilon)\to z(u)$ weakly in $H^1(0,T)$ as well as uniformly in $C[0,T]$.*

Proof. Let $y_\varepsilon = y^\varepsilon(u_\varepsilon)$ and $z_\varepsilon = z^\varepsilon(u_\varepsilon)$. Since $\dot y_\varepsilon$ and $\dot z_\varepsilon$ are bounded in L^2 by 21 and 23, for some $(\tilde y, \tilde z)$ we have $(y_\varepsilon, z_\varepsilon)\to(\tilde y,\tilde z)$ weakly in $H^1(0,T)$ and thus uniformly in $C[0,T]$, for some subsequence. By 23, $\Psi(z_\varepsilon(t))\to 0$ pointwise in t, thus $\Psi(\tilde z(t))=0$ and $\tilde z(t)\in Z$ for all t. Moreover, for any $\zeta\in Z$ we have

$$\langle \dot z_\varepsilon(t) - S\dot y_\varepsilon(t), z_\varepsilon(t)-\zeta\rangle = -\frac{1}{\varepsilon}\langle \nabla\Psi(z_\varepsilon(t)), z_\varepsilon(t)-\zeta\rangle \le 0$$

due to the monotonicity of $\nabla\Psi$, since $\nabla\Psi(\zeta)=0$. Letting $\varepsilon\to 0$ we see that $(\tilde y,\tilde z)$ solves 1 and 3. Therefore $\tilde y = y(u)$, $\tilde z = z(u)$ and the whole sequence converges. □

3.2. **The regularized control problem.** The regularized control problem (P_ε) is defined by equations 16 – 17, with the cost functional

$$J_*(y,z,u;u_*) = J(y,z,u) + \frac{1}{2}\int_0^T |u(t)-u_*(t)|^2 \,\mathrm{d}t\,. \tag{24}$$

Here, u_* is an optimal control for problem (P) with corresponding state (y_*, z_*) and admissible control space L^2, according to Theorem 2.4. In this manner, we will enforce convergence of the optimal controls for the regularized problem towards any specifically chosen optimal control for (P); note that the solution of (P) might be nonunique.

Theorem 3.3. *For any $\varepsilon>0$, there exists a solution u_ε of problem (P_ε) with corresponding states $y_\varepsilon = y^\varepsilon(u_\varepsilon)$, $z_\varepsilon = z^\varepsilon(u_\varepsilon)$. Moreover, $u_\varepsilon\to u_*$ strongly in L^2 and $(y_\varepsilon, z_\varepsilon)\to(y_*,z_*)$ uniformly for $\varepsilon\to 0$.*

Proof. Due to Lemma 3.1 and because E is positive semidefinite and and Ω is convex, any weak limit u_ε of a subsequence of a minimizing sequence of controls furnishes a solution of (P_ε). We have

$$\begin{aligned} J(y^\varepsilon(u_*), z^\varepsilon(u_*), u_*) &= J_*(y^\varepsilon(u_*), z^\varepsilon(u_*), u_*; u_*) \\ &\ge J_*(y_\varepsilon, z_\varepsilon, u_\varepsilon; u_*) = J(y_\varepsilon, z_\varepsilon, u_\varepsilon) + \frac{1}{2}\|u_\varepsilon - u_*\|_2^2 \ge J(y_*, z_*, u_*)\,. \end{aligned}$$

Since $y^\varepsilon(u_*) \to y_*$ and $z^\varepsilon(u_*) \to z_*$ by Lemma 3.2 when $\varepsilon \to 0$, the assertions follow, once more applying Lemma 3.2. □

Problem (P_ε) is a standard optimal control problem for an ODE system.

3.3. **Necessary optimality conditions for the regularized problem.** Let $(y_\varepsilon, z_\varepsilon, u_\varepsilon)$ be a solution of (P_ε). For any admissible control $\tilde u$, that is $\tilde u \in L^2(0,T)$ with $\tilde u(t) \in \Omega$ a.e. in $(0,T)$, the derivative of J_* in the direction $w = \tilde u - u_\varepsilon$ must be nonnegative,

$$\lim_{h \downarrow 0} \frac{1}{h}\left(J_*(y(u_\varepsilon + hw), z(u_\varepsilon + hw), u_\varepsilon + hw; u_*) - J_*(y_\varepsilon, z_\varepsilon, u_\varepsilon; u_*)\right) \geq 0\,. \tag{25}$$

The standard way to evaluate 25 involves the linearization of 17, 18 – which we do not write down since we will not need it later – and its adjoint system given by

$$\dot p = -A_\varepsilon(t)^T p - A_\varepsilon(t)^T S^T q - \ell_\varepsilon^y(t)\,, \qquad p(T) = 0\,, \tag{26}$$

$$\dot q = -D_\varepsilon(t)^T p - D_\varepsilon(t)^T S^T q + \frac{1}{\varepsilon} D^2\Psi(z_\varepsilon(t))q - \ell_\varepsilon^z(t)\,, \quad q(T) = 0\,, \tag{27}$$

where

$$A_\varepsilon(t) = \partial_y f(t, y_\varepsilon(t), z_\varepsilon(t))\,, \quad D_\varepsilon(t) = \partial_z f(t, y_\varepsilon(t), z_\varepsilon(t))\,,$$
$$\ell_\varepsilon^y(t) = \partial_y L(t, y_\varepsilon(t), z_\varepsilon(t))\,, \quad \ell_\varepsilon^z(t) = \partial_z L(t, y_\varepsilon(t), z_\varepsilon(t))\,.$$

Hypothesis 3.4. *In addition to Hypothesis 2.2 and 12, we assume that the partial derivatives of f and L satisfy a Carathéodory condition and*

$$\|\partial_y f(t,y,z)\| + \|\partial_z f(t,y,z)\| \leq \alpha_2(t) \cdot \alpha_3(|y|,|z|) \tag{28}$$

for some $\alpha_2 \in L^\infty$ and some continuous function α_3, and

$$|\partial_y L(t,y,z)| + |\partial_z L(t,y,z)| \leq \beta_3(t) \cdot \beta_4(|y|,|z|) \tag{29}$$

for some $\beta_3 \in L^2$ and some continuous function β_4. Note that 28 and 29 are satisfied if f and L are C^1 with respect to all arguments.

Under Hypothesis 3.4, A_ε, D_ε are bounded in L^∞ and ℓ_ε^y, ℓ_ε^z are bounded in L^2 independently of ε. Moreover, the operator defined by $u \mapsto (y^\varepsilon(u), z^\varepsilon(u))$ is Fréchet-differentiable from L^1 to $C \times C$, and the functional defined by $(y,z) \mapsto \int_0^T L(t, y(t), z(t))\,dt$ is Fréchet-differentiable from $C \times C$ to $\mathbb{R}$. (Here, "C" stands for the space of continuous functions on $[0,T]$ with range $\mathbb{R}^n$ resp. $\mathbb{R}^m$.) As a consequence, the usual computations from optimal control theory are formally justified and yield the necessary optimality conditions for the regularized problem.

Theorem 3.5. *Let $(p_\varepsilon, q_\varepsilon)$ solve the adjoint system 26, 27. Then a.e. in t we have*

$$\left\langle B^T p_\varepsilon(t) + B^T S^T q_\varepsilon(t) + E u_\varepsilon(t) + (u_\varepsilon(t) - u_*(t))\,,\, w - u_\varepsilon(t)\right\rangle \geq 0\,, \quad \forall w \in \Omega\,. \tag{30}$$

3.4. **The penalty function.** Let $P : \mathbb{R}^m \to Z$ denote the projection onto the closed convex set Z, let $d(x) = |x - Px|$ denote the distance from x to Z. It is well known that

$$\nabla d(x) = \frac{x - Px}{|x - Px|}\,, \quad x \notin Z\,. \tag{31}$$

Lemma 3.6. *Assume that the boundary ∂Z is a manifold of dimension $m-1$ and regularity C^2. Then P is C^1 in $\mathbb{R}^m \setminus Z$, and consequently d is C^2 in $\mathbb{R}^m \setminus Z$.*

Proof. See [19], Theorem 2, and [20], Theorem 3.9. □

Let $d_S(x)$ denote the signed distance (or oriented distance) which is defined as the negative distance from x to the complement of Z if $x \in Z$, and as $d(x)$ otherwise.

Lemma 3.7. *Assume that the boundary ∂Z is a manifold of dimension $m-1$ and regularity C^2. Then d_S is C^2 in some neighbourhood of ∂Z.*

Proof. See [21], Theorem V.4.3 (ii). □

In the situation of Lemmas 3.6 and 3.7, ∇d and $D^2 d$ can be extended continuously from the complement of Z to ∂Z, setting $\nabla d(x) = \nabla d_S(x)$ and $D^2 d(x) = D^2 d_S(x)$ for $x \in \partial Z$, and $\nabla d(x)$ gives the unit outer normal for $x \in \partial Z$. We now define

$$\psi(x) = \frac{1}{2}(d(x)+1)^2 + \frac{1}{2}\,, \quad x \notin \operatorname{int}(Z), \tag{32}$$

and extend ψ to $\operatorname{int}(Z)$ such that ψ is C^2 in some neighbourhood V of Z and $\psi < 1$ on $\operatorname{int}(Z)$. (This is possible due to Lemma 3.7.) Furthermore, we define

$$\Psi(x) = \rho(\psi(x))\,, \tag{33}$$

where $\rho \in C^\infty(\mathbb{R})$ is a function which vanishes on $(-\infty, 1]$ such that ρ'' is nondecreasing and satisfies $\rho'' > 0$ on $(1,\infty)$. Note that this implies $\rho'(1) = 0 = \rho''(1)$ as well as $\rho' > 0$ and $\rho > 0$ on $(1,\infty)$. As a consequence of these definitions and of Lemma 3.7, $\Psi = 0$ on Z, $\Psi > 0$ outside Z, Ψ is C^2 in $\mathbb{R}^m$, and ψ as well as Ψ are convex on $\mathbb{R}^m$ since d is convex on $\mathbb{R}^m$.

Below we will need more information about the derivatives of ψ and Ψ.

Lemma 3.8. *We have $D^2 d(x)\nabla d(x) = 0$ for all $x \in \mathbb{R}^m \setminus Z$, and thus also for $x \in \partial Z$.*

Proof. One easily checks that d grows linearly in the normal direction, that is,

$$d(x + t\nabla d(x)) = d(x) + t \tag{34}$$

holds for any $x \in \mathbb{R}^m \setminus Z$. Differentiating 34 twice with respect to t and setting $t = 0$ gives $\nabla d(x)^T D^2 d(x) \nabla d(x) = 0$. Since $D^2 d(x)$ is symmetric and positive semidefinite, the assertion follows. □

For $x \notin \operatorname{int}(Z)$, the first two derivatives of ψ are given by

$$\nabla\psi(x) = (d(x)+1)\nabla d(x) \tag{35}$$

$$D^2\psi(x)h = \langle \nabla d(x), h\rangle\, \nabla d(x) + (d(x)+1) D^2 d(x) h\,. \tag{36}$$

Note that 35 implies that $|\nabla\psi(x)| \ge 1$ for $x \notin Z$ and that $\nabla\psi(x) = \nabla d(x)$ is the unit outer normal for $x \in \partial Z$. Moreover, for $x \in \partial Z$ let us denote by

$$T(x) = \{h : \langle \nabla d(x), h\rangle = 0\}$$

the space tangent to Z at x. Note that by virtue of 36

$$D^2\psi(x)h = D^2 d(x)h\,, \quad \forall\, x \in \partial Z,\, h \in T(x)\,. \tag{37}$$

We will assume that Z is uniformly convex in the sense that

$$\exists\, \gamma_0 > 0 \text{ such that } \left\langle D^2 d(x)h, h\right\rangle \ge \gamma_0 |h|^2\,, \quad \forall\, x \in \partial Z,\, h \in T(x)\,. \tag{38}$$

Lemma 3.9. *Condition 38 is equivalent to*

$$\exists\, \gamma > 0 \text{ such that } \left\langle D^2\psi(x)h, h\right\rangle \ge \gamma |h|^2\,, \quad \forall\, x \in \partial Z,\, h \in \mathbb{R}^m\,. \tag{39}$$

Making the constants smaller if necessary we see that by continuity, 38 (resp. 39) remain true in a small outer neighbourhood of ∂Z.

Proof. Due to 37, 39 implies 38. For the converse, let $x \in \partial Z$, let $h = h_N \nabla d(x) + h_T$ be the orthogonal decomposition of $h \in \mathbb{R}^m$. Since $|\nabla d(x)| = 1$ and $D^2 d(x)\nabla d(x) = 0$ by virtue of Lemma 3.8, we get from 36 that

$$\left\langle D^2\psi(x)h, h\right\rangle = h_N^2 + (d(x)+1)\left\langle D^2 d(x)h_T, h_T\right\rangle \geq h_N^2 + \gamma_0|h_T|^2 \geq \min\{1,\gamma_0\}|h|^2 .$$

The assertion now follows from the continuity of $D^2\psi$. □

Note that, for example, $D^2\psi(x) = I$ for $x \in \partial Z$ if Z is the unit ball.

The derivatives of $\Psi(x) = \rho(\psi(x))$ are given by

$$\nabla\Psi(x) = \rho'(\psi(x))\nabla\psi(x) \tag{40}$$

$$D^2\Psi(x)h = \rho''(\psi(x))\left\langle \nabla\psi(x), h\right\rangle \nabla\psi(x) + \rho'(\psi(x))D^2\psi(x)h . \tag{41}$$

3.5. **Estimates for the adjoints in the regularized problem.** Testing 26 with $p/|p|$ and 27 with $q/|q|$, respectively, we get, writing $D^2\Psi(t)$ instead of $D^2\Psi(z_\varepsilon(t))$,

$$|p(t)| \leq c\Big(1 + \int_t^T |p(s)| + |q(s)|\,\mathrm{d}s\Big) \tag{42}$$

$$|q(t)| + \frac{1}{\varepsilon}\int_t^T \frac{\left\langle D^2\Psi(s)q(s), q(s)\right\rangle}{|q(s)|}\,\mathrm{d}s \leq c\Big(1 + \int_t^T |p(s)| + |q(s)|\,\mathrm{d}s\Big), \tag{43}$$

and thus, since $D^2\Psi$ is positive semidefinite, the solutions p_ε, q_ε of 26, 27 satisfy

$$\|p_\varepsilon\|_\infty \leq C , \quad \|q_\varepsilon\|_\infty \leq C . \tag{44}$$

This implies, using the equation 27 for p_ε,

$$\|\dot p_\varepsilon\|_r \leq C(1 + \|\ell_\varepsilon^y\|_r), \quad 1 \leq r \leq \infty . \tag{45}$$

The estimate for q_ε is more involved. Let us abbreviate the values of ρ, ψ, Ψ and their derivatives along the ε-trajectory as $\rho(t) = \rho(\psi(z_\varepsilon(t)))$, $\nabla\psi(t) = \nabla\psi(z_\varepsilon(t))$ and so on. We rewrite 27 as

$$-\dot q = -\frac{1}{\varepsilon}D^2\Psi(t)q + r_\varepsilon(t), \tag{46}$$

where

$$r_\varepsilon(t) = D_\varepsilon(t)^T p(t) + D_\varepsilon(t)^T S^T q(t) + \ell_\varepsilon^z(t) . \tag{47}$$

From 41 we get

$$-\dot q + \frac{1}{\varepsilon}\rho''(t)\left\langle \nabla\psi(t), q\right\rangle \nabla\psi(t) + \frac{1}{\varepsilon}\rho'(t)D^2\psi(t)q = r_\varepsilon(t) . \tag{48}$$

Note that the terms involving $1/\varepsilon$ can be nonzero only where $z_\varepsilon(t) \notin Z$. Multiplication with $q/|q|$ yields

$$-\frac{\mathrm{d}}{\mathrm{d}t}|q| + \frac{1}{\varepsilon}\rho''(t)\left\langle \nabla\psi(t), q\right\rangle^2 + \frac{1}{\varepsilon}\rho'(t)\frac{\left\langle D^2\psi(t)q, q\right\rangle}{|q|} = \frac{\left\langle r_\varepsilon(t), q\right\rangle}{|q|} .$$

The second term on the left is nonnegative, and $\left\langle D^2\psi(t)q, q\right\rangle \geq \gamma|q|^2$ by Lemma 3.9. Integration over any interval $[s,t] \subset [0,T]$ yields

$$|q_\varepsilon(s)| - |q_\varepsilon(t)| + \frac{\gamma}{\varepsilon}\int_s^t \rho'(\tau)|q_\varepsilon(\tau)|\,\mathrm{d}\tau \leq \int_s^t |r_\varepsilon(\tau)|\,\mathrm{d}\tau \leq C , \tag{49}$$

and thus we obtain the first a priori estimate

$$\frac{1}{\varepsilon}\int_0^T \rho'(\tau)|q_\varepsilon(\tau)|\,\mathrm{d}\tau \le C\,. \tag{50}$$

We now introduce the projection of q_ε onto the "approximate" normal direction,

$$q_\varepsilon^N(t) = \langle q_\varepsilon(t), \nabla\psi(z_\varepsilon(t))\rangle\ .$$

Its derivative becomes (again, we abbreviate)

$$\dot q_\varepsilon^N = \langle \dot q_\varepsilon, \nabla\psi(t)\rangle + \left\langle q_\varepsilon, D^2\psi(t)\dot z_\varepsilon\right\rangle\ .$$

We insert $\dot q_\varepsilon$ from 48 and $\dot z_\varepsilon$ from 18 and obtain, after some computation

$$-\dot q_\varepsilon^N + \frac{1}{\varepsilon}\rho''(t)q_\varepsilon^N|\nabla\psi(t)|^2 = \tilde r_\varepsilon(t)\,, \tag{51}$$

where

$$\tilde r_\varepsilon(t) = -\left\langle q_\varepsilon(t), D^2\psi(t)S(f(t)+Bu_\varepsilon(t))\right\rangle + \langle r_\varepsilon(t), \nabla\psi(t)\rangle\ .$$

From the previous estimates we know that $\|\tilde r_\varepsilon\|_1 \le C$ uniformly in ε. We test 51 over any interval $[s,t]\subset[0,T]$ with the sign of q_ε^N and obtain

$$|q_\varepsilon^N(s)| - |q_\varepsilon^N(t)| + \frac{1}{\varepsilon}\int_s^t \rho''(\tau)|q_\varepsilon^N(\tau)||\nabla\psi(\tau)|^2\,\mathrm{d}\tau \le \int_s^t |\tilde r_\varepsilon(\tau)|\,\mathrm{d}\tau \le C\,. \tag{52}$$

Since $q_\varepsilon^N(T)=0$ and $|\nabla\psi(\tau)|\ge 1$ whenever $\rho''(\tau)\ne 0$, we get the second a priori estimate

$$\frac{1}{\varepsilon}\int_0^T \rho''(t)|q_\varepsilon^N(t)||\nabla\psi(t)|\,\mathrm{d}t \le C\,. \tag{53}$$

Because of 50 and 53, all terms in 48 are bounded in L^1 uniformly in ε. Therefore

$$\int_0^T |\dot q_\varepsilon(t)|\,\mathrm{d}t \le C \tag{54}$$

uniformly in ε.

4. **Passage to the limit.** From Theorem 3.3 we know already that $u_\varepsilon \to u_*$ strongly in L^2 and $(y_\varepsilon, z_\varepsilon)\to(y_*,z_*)$ uniformly, where u_* is a given optimal control and (y_*,z_*) is the corresponding solution of the dynamics 1 and 3 of problem (P). Moreover, the a priori bounds 21 and 23 imply that for some subsequence

$$\dot y_\varepsilon \to \dot y_*\,,\quad \dot z_\varepsilon \to \dot z_*\,,\quad \text{weakly in } L^2. \tag{55}$$

Due to the a priori bound 44 and 45, for some subsequence and some $p\in H^1(0,T)$,

$$\dot p_\varepsilon \to \dot p \quad \text{weakly in } L^2\,,\quad p_\varepsilon \to p \quad \text{uniformly.} \tag{56}$$

Due to the a priori bound 54, for some subsequence and some $q\in BV[0,T]$,

$$q_\varepsilon \to q \quad \text{pointwise,}\quad \mathrm{Var}(q) \le \liminf_{\varepsilon\to 0}\mathrm{Var}(q_\varepsilon)\,. \tag{57}$$

Alternatively we may interpret $\dot q_\varepsilon \in L^1$ as an element of the dual of $C[0,T]$. The a priori bound 54 implies by Alaoglu's compactness theorem that

$$\dot q_\varepsilon \to \mathrm{d}q \quad \text{weak star} \tag{58}$$

for some subsequence and some $\mathrm{d}q\in C[0,T]^*$ which we interpret as a signed regular Borel measure. It is an exercise in real analysis to show that the function q is related to the measure $\mathrm{d}q$ by the formulas

$$q(t-) - q(s+) = \mathrm{d}q((s,t))\,,\quad q(t+) - q(s-) = \mathrm{d}q([s,t])\,, \tag{59}$$

valid for $[s,t] \subset [0,1]$. Here, by $q(t+)$ and $q(t-)$ we denote the right and left hand limits of q at t, respectively, with the convention $q(T+) = q(T)$ and $q(0-) = q(0)$.

Due to the convergence properties above, we may pass to the limit in the maximum condition 30 to obtain a.e. in t

$$\left\langle B^T p(t) + B^T S^T q(t) + \partial_u L(t, y_*(t), z_*(t)), w - u_*(t) \right\rangle \geq 0\,, \quad \forall w \in \Omega\,, \tag{60}$$

and in the adjoint equation for p_ε, so p solves

$$-\dot{p} = A(t)^T p + A(t)^T S^T q + \ell^y(t)\,, \quad p(T) = 0\,, \tag{61}$$

where $A(t) = \partial_y f(t, y_*(t), z_*(t))$ and $\ell^y(t) = \partial_y L(t, y_*(t), z_*(t), u_*(t))$.

We now discuss the properties of the limit q of q_ε. For convenience, let us repeat the equation for q_ε,

$$-\dot{q_\varepsilon} + \frac{1}{\varepsilon}\rho''(t)\left\langle \nabla\psi(t), q_\varepsilon \right\rangle \nabla\psi(t) + \frac{1}{\varepsilon}\rho'(t) D^2\psi(t) q_\varepsilon = r_\varepsilon(t)\,. \tag{62}$$

First, we consider the part I_0 of $[0,T]$ where z_* lies in the interior of Z. As z_* is continuous, I_0 is open and thus can be represented as a disjoint union of at most countably many open intervals. Let (a,b) be such an interval. On any compact subinterval $[s,t]$ of (a,b) the functions $\rho'(\psi(z_\varepsilon(\cdot)))$ and $\rho''(\psi(z_\varepsilon(\cdot)))$ vanish for small ε, due to the uniform convergence of z_ε to z_*. Therefore,

$$q_\varepsilon(s) - q_\varepsilon(t) = \int_s^t r_\varepsilon(\tau)\,\mathrm{d}\tau = \int_s^t D_\varepsilon(\tau)^T p_\varepsilon(\tau) + D_\varepsilon(\tau) S^T q_\varepsilon(\tau) + \ell_\varepsilon^z(\tau)\,\mathrm{d}\tau\,,$$

whenever ε is small enough. Letting $\varepsilon \to 0$, the following lemma is proved.

Lemma 4.1. *We have $q \in H^1(a,b)$ for any open subinterval (a,b) of I_0, and*

$$-\dot{q} = D(t)^T p + D(t)^T S^T q + \ell^z(t)\,, \tag{63}$$

where $D(t) = \partial_z f(t, y_*(t), z_*(t))$ and $\ell^z(t) = \partial_z L(t, y_*(t), z_*(t), u_*(t))$. In particular, q is absolutely continuous on I_0.

Next, we analyze the behaviour of the penalty terms on I_∂. Since

$$\dot{v}_\varepsilon(t) - \dot{z}_\varepsilon(t) = \frac{1}{\varepsilon}\nabla\Psi(z_\varepsilon(t)) = \frac{1}{\varepsilon}\rho'(\psi(z_\varepsilon(t)))\nabla\psi(z_\varepsilon(t))\,,$$

and since $|\nabla\psi(x)| \geq 1$ whenever $\rho'(x) \neq 0$, we obtain from 15 the following result.

Lemma 4.2. *For $\varepsilon \to 0$, we have that*

$$\begin{aligned}
t &\mapsto \frac{1}{\varepsilon}\rho'(\psi(z_\varepsilon(t)))\nabla\psi(z_\varepsilon(t)) && \textit{converges to} \quad \dot{v}_* - \dot{z_*} = \dot{\xi} = |\dot{\xi}| n\,,\\
t &\mapsto \frac{1}{\varepsilon}\rho'(\psi(z_\varepsilon(t)))|\nabla\psi(z_\varepsilon(t))|^2 && \textit{converges to} \quad \left\langle \dot{\xi}, n \right\rangle = |\dot{\xi}|\,,\\
t &\mapsto \frac{1}{\varepsilon}\rho'(\psi(z_\varepsilon(t))) && \textit{converges to} \quad |\dot{\xi}|
\end{aligned}$$

weakly in $L^2(0,T)$. Moreover, for any interval $[s,t] \subset [0,T]$,

$$\frac{1}{\varepsilon}\int_s^t \rho'(\psi(z_\varepsilon(\tau))) D^2\psi(z_\varepsilon(\tau)) q_\varepsilon(\tau)\,\mathrm{d}\tau \to \int_s^t |\dot{\xi}(\tau)| D^2\psi(z_*(\tau)) q(\tau)\,\mathrm{d}\tau\,. \tag{64}$$

Recall that $\dot{\xi} = 0$ on I_0.

Let us define

$$q^N(t) = \langle q(t), \nabla\psi(z_*(t))\rangle \,, \quad t \in [0,T] \,. \tag{65}$$

On boundary parts of the optimal trajectory, q^N is just the projection of q onto the normal direction

$$q^N(t) = \langle q(t), n(t)\rangle \,, \quad t \in I_\partial \,. \tag{66}$$

For the approximate normal projection $q_\varepsilon^N(t) = \langle q_\varepsilon(t), \nabla\psi(z_\varepsilon(t))\rangle$ considered above we have

$$q_\varepsilon^N \to q^N \quad \text{pointwise on } [0,T] \,. \tag{67}$$

From the a priori estimate for q_ε^N, we obtain a complementarity condition.

Lemma 4.3. *We have*

$$\left\langle q, \dot\xi \right\rangle = q^N |\dot\xi| = 0 \quad \textit{a.e. on } I_\partial \,. \tag{68}$$

Proof. Since ρ'' is nondecreasing and $\rho'(1) = 0$, we have $\rho'(x) \le (x-1)\rho''(x)$ for all $x \in \mathbb{R}$. We get

$$\begin{aligned} 0 &\le \frac{1}{\varepsilon} \int_{I_\partial} \rho'(\psi(z_\varepsilon(t)))|\nabla\psi(z_\varepsilon(t))|^2 |q_\varepsilon^N(t)| \, \mathrm{d}t \\ &\le \frac{1}{\varepsilon} \int_{I_\partial} (\psi(z_\varepsilon(t)) - 1)\rho''(\psi(z_\varepsilon(t)))|\nabla\psi(z_\varepsilon(t))|^2 |q_\varepsilon^N(t)| \, \mathrm{d}t \\ &\le C \sup_{t\in I_\partial} |\psi(z_\varepsilon(t)) - 1| \,, \end{aligned}$$

where the latter inequality follows from the a priori estimate 53. Since $\psi(z_*(t)) = 1$ on I_∂, the integrals converge to zero as $\varepsilon \to 0$. As $q_\varepsilon^N \to q^N$ pointwise and the q_ε^N are uniformly bounded, we conclude from Lemma 4.2 that

$$\int_{I_\partial} |\dot\xi(t)||q^N(t)| \, \mathrm{d}t = 0$$

which proves the assertion. □

We now investigate the jumps of the adjoint. While the component $p \in H^1$ does not jump, the component q does in general. It turns out that the absolute values $|q|$ and $|q^N|$ can only jump downward, in reverse time.

Lemma 4.4. *For any $t \in [0,T]$ we have $|q(t-)| \le |q(t+)|$ and $|q^N(t-)| \le |q^N(t+)|$. In particular, $q(T-) = 0$ and $q^N(T-) = 0$.*

Proof. From 49 and 52 we see that, for any $s < t < \sigma$,

$$|q_\varepsilon(s)| - |q_\varepsilon(\sigma)| \le \int_s^\sigma |r_\varepsilon(\tau)| \, \mathrm{d}\tau \,, \quad |q_\varepsilon^N(s)| - |q_\varepsilon^N(\sigma)| \le \int_s^\sigma |\tilde r_\varepsilon(\tau)| \, \mathrm{d}\tau \,.$$

As $\|r_\varepsilon\|_2 \le C$ and $\|\tilde r_\varepsilon\|_2 \le C$ uniformly in ε, letting $\varepsilon \to 0$ gives

$$|q(s)| - |q(\sigma)| \le \int_s^\sigma m(\tau) \, \mathrm{d}\tau \,, \quad |q^N(s)| - |q^N(\sigma)| \le \int_s^\sigma \tilde m(\tau) \, \mathrm{d}\tau$$

for some functions $m, \tilde m \in L^2$. Letting $s \uparrow t$ and $\sigma \downarrow t$ we obtain the assertion, with the corresponding modifications for the case $t = 0$ and $t = T$, noting that $q(T) = 0$. □

We now turn to the limit behaviour of the term

$$\mu_\varepsilon(t) = \frac{1}{\varepsilon}\rho''(\psi(z_\varepsilon(t)))q_\varepsilon^N(t)\nabla\psi(z_\varepsilon(t)) \tag{69}$$

in 62. Since $\|\mu_\varepsilon\|_1 \le C$ by virtue of 53,

$$\int_0^T \langle\varphi,\mu_\varepsilon\rangle \,\mathrm{d}t \to \int_0^T \langle\varphi,\mathrm{d}\mu\rangle \ , \quad \text{for all } \varphi\in C[0,T], \tag{70}$$

holds for some measure $\mathrm{d}\mu$, a weak star limit point of $\{\mu_\varepsilon\}$ in the dual of $C[0,T]$. Since for all φ with compact support in I_0 the integral on the left vanishes for small ε,

$$\operatorname{supp}(\mathrm{d}\mu)\subset I_\partial\,. \tag{71}$$

In addition, the form of μ_ε suggests that $\mathrm{d}\mu$ is concentrated on the normal direction.

Lemma 4.5. *Let φ be continuous on $[0,T]$ such that $\langle\varphi(t),n(t)\rangle = 0$ for all $t\in I_\partial$. Then*

$$0=\int_0^T \langle\varphi,\mathrm{d}\mu\rangle = \int_{I_\partial}\langle\varphi,\mathrm{d}\mu\rangle\ . \tag{72}$$

Consequently,

$$\mathrm{d}\mu = \langle n,\mathrm{d}\mu\rangle\, n\,. \tag{73}$$

Proof. We have, due to the a priori estimate 53,

$$\begin{aligned}\int_{I_\partial} |\,\langle\varphi,\mu_\varepsilon\rangle\,|\,\mathrm{d}t &= \int_{I_\partial}\frac{1}{\varepsilon}\rho''(t)|q_\varepsilon^N(t)||\,\langle\nabla\psi(z_\varepsilon(t))-\nabla\psi(z_*(t)),\varphi(t)\rangle\,|\,\mathrm{d}t\\ &\le C\|\varphi\|_\infty\|z_\varepsilon - z_*\|_\infty\,.\end{aligned}$$

Set $I_\eta = \{t : t\in I_0, \operatorname{dist}(t,I_\partial)<\eta\}$. Then, for any $\eta>0$, $I_0\setminus I_\eta$ is a compact subset of I_0, and therefore $\mu_\varepsilon = 0$ on $I_0\setminus I_\eta$ whenever ε is small enough. Finally, for I_η we observe that, for any fixed φ,

$$\sup_{t\in I_\eta} |\,\langle\nabla\psi(z_\varepsilon(t)),\varphi(t)\rangle\,| \le \alpha(\eta) + C\|z_\varepsilon - z_*\|_\infty$$

for some function $\alpha(\eta)$ which tends to 0 as $\eta\to 0$. Thus,

$$\int_{I_\eta} |\,\langle\varphi,\mu_\varepsilon\rangle\,|\,\mathrm{d}t \le C(\alpha(\eta)+\|z_\varepsilon - z_*\|_\infty)\,.$$

Therefore

$$\limsup_{\varepsilon\to 0}\int_0^T |\,\langle\varphi,\mu_\varepsilon\rangle\,|\,\mathrm{d}t \le C\alpha(\eta)\,.$$

Letting $\eta\to 0$ we obtain 72. Applying 72 to $\varphi - \langle\varphi,n\rangle\, n$ we obtain 73. □

In the equation for q_ε we may pass to the limit in the dual space of $C[0,T]$ as follows.

Lemma 4.6. *For any $\varphi\in C[0,T]$ we have*

$$\int_0^T -\langle\varphi,\mathrm{d}q\rangle + \int_{I_\partial}\langle\varphi,n\rangle\,\langle n,\mathrm{d}\mu\rangle = \int_0^T \langle\varphi(t),g(t)\rangle\,\mathrm{d}t\,, \tag{74}$$

where

$$g(t) = -|\dot\xi(t)|D^2\psi(z_*(t))q(t) + D(t)^T p(t) + D(t)^T S^T q(t) + \ell^z(t)\,. \tag{75}$$

Moreover, for any $\varphi \in C(I_\partial)$, *we have*

$$\int_{I_\partial} -\langle \varphi, \mathrm{d}q \rangle + \int_{I_\partial} \langle \varphi, n \rangle \langle n, \mathrm{d}\mu \rangle = \int_{I_\partial} \langle \varphi(t), g(t) \rangle \, \mathrm{d}t \,. \tag{76}$$

Note that since $\dot{\xi} = 0$ on I_0, Lemma 4.1 is recovered if in 74 we choose test functions with compact support in I_0.

Proof. Testing 62, the equation for q_ε, with $\varphi \in C[0,T]$ we get

$$\int_0^T -\langle \varphi(t), \dot{q}_\varepsilon(t) \rangle \, \mathrm{d}t + \int_0^T \langle \varphi(t), \mu_\varepsilon(t) \rangle \, \mathrm{d}t + \int_0^T \left\langle \varphi(t), \frac{1}{\varepsilon} \rho'(t) D^2 \psi(t) q_\varepsilon(t) \right\rangle \mathrm{d}t$$
$$= \int_0^T \langle \varphi(t), r_\varepsilon(t) \rangle \, \mathrm{d}t \,.$$

We pass to the limit $\varepsilon \to 0$, taking into account 64, 47, 63 and Lemma 4.5. Thus 74 is proved. To prove 76 for a given $\varphi \in C(I_\partial)$, let $\tilde{\varphi} \in C[0,T]$ be an extension of φ and set $\varphi_k(x) = \max\{0, 1 - kd(x, I_\partial)\} \tilde{\varphi}(x)$. Inserting φ_k into 74 and letting $k \to \infty$ we obtain 76, since $\varphi_k(x) \to 0$ for all $x \in I_0$ and $\|\varphi_k\|_\infty \le C$, so the integrals over I_0 vanish in the limit $k \to \infty$. □

Lemma 4.7. *For any* $\varphi \in C[0,T]$ *we have*

$$-\int_{I_\partial} \big\langle \varphi - \langle \varphi, n \rangle \, n, \mathrm{d}q \big\rangle = \int_{I_\partial} \big\langle \varphi - \langle \varphi, n \rangle \, n, g \big\rangle \, \mathrm{d}t \,, \tag{77}$$

where g *is given in* 75.

Proof. Setting $\varphi = \chi n$ in 76 with scalar-valued $\chi \in C(I_\partial)$, we obtain

$$-\int_{I_\partial} \chi \, \langle n, \mathrm{d}q \rangle + \int_{I_\partial} \chi \, \langle n, \mathrm{d}\mu \rangle = \int_{I_\partial} \chi \, \langle n, g \rangle \, \mathrm{d}t \,,$$

so we have $\langle n, \mathrm{d}\mu \rangle = \langle n, \mathrm{d}q \rangle + \langle n, g \rangle \, \mathrm{d}t$ for the measures on I_∂. Replacing $\langle n, \mathrm{d}\mu \rangle$ in 76 accordingly yields the assertion. □

A nonzero jump $q(t+) - q(t-)$ of the adjoint at some point $t \in I_\partial$ corresponds to a Dirac contribution $(q(t+) - q(t-))\delta_t$ in the measure $\mathrm{d}q$. As an immediate consequence of the previous lemma, we see that such jumps can occur only in the normal direction.

Lemma 4.8. *For all* $t \in I_\partial$ *we have*

$$q(t-) - q(t+) = (q^N(t-) - q^N(t+)) n(t) \,. \tag{78}$$

Proof. Let $t \in I_\partial$ be given. By 77 we have, for any test function φ,

$$\big\langle \varphi(t) - \langle \varphi(t), n(t) \rangle \, n(t), q(t-) - q(t+) \big\rangle = 0 \,.$$

Choosing $\varphi(t) = c$ with arbitrary $c \perp n(t)$ we see that $q(t-) - q(t+) = \alpha n(t)$ for some scalar α, therefore $q^N(t-) - q^N(t+) = \alpha$ and 78 follows. □

Up to now, we have not made any structural assumption concerning the optimal trajectory. To proceed further, we assume the **regularity condition**

$$\dot{\xi}(t) \ne 0 \quad \text{a.e. in } I_\partial \,. \tag{79}$$

Since $\dot{v}_*(t) = \dot{z}_*(t) + \dot{\xi}(t)$ represents the unique decomposition of $\dot{v}_*(t)$ w.r.t the tangent cone (here a half-space) and the normal cone (here the outer normal half-line) to Z at $z_*(t)$, 79 is equivalent to saying that the optimal input $v_*(t)$ points into the open outer half-space (the set-theoretic complement of the tangent cone) at $z_*(t)$, so the set Z actually restricts the movement at almost all times $t \in I_\partial$.

In this case, the complementarity condition of Lemma 4.3 can be sharpened.

Lemma 4.9. *Let the regularity condition 79 hold. Then we have*

$$q^N(t) = \langle q(t), n(t) \rangle = 0\,, \quad \text{for a.e. } t \in I_\partial\,. \tag{80}$$

In particular, $q^N(t+) = 0$ *if* $(t, t+\eta) \subset I_\partial$ *for some* $\eta > 0$, *and* $q^N(t-) = 0$ *if* $(t-\eta, t) \subset I_\partial$ *for some* $\eta > 0$.

Lemma 4.10. *Let the regularity condition 79 hold. Then* $q \in H^1(a,b)$ *for any open subinterval* (a,b) *of* I_∂, *and solves, a.e. in* (a,b),

$$-\,\dot q = \langle q, \dot n(t)\rangle\, n(t) + g(t) - \langle g(t), n(t)\rangle\, n(t)\,, \tag{81}$$

where

$$g(t) = -|\dot\xi(t)| D^2 d(z_*(t)) q(t) + D(t)^T p(t) + D(t)^T S^T q(t) + \ell^z(t)\,. \tag{82}$$

Proof. Note first that, due to 80 and 37, $|\dot\xi(t)|D^2 d(z_*(t))q(t) = |\dot\xi(t)|D^2\psi(z_*(t))q(t)$ a.e. in $(0,T)$, so 82 and 75 coincide if 79 holds. Due to Lemma 4.9, partial integration yields

$$\int_s^t \langle n, \mathrm{d}q\rangle + \int_s^t \langle q, \dot n\rangle \;\mathrm{d}\tau = 0$$

for all $[s,t] \subset (a,b)$, therefore $-\langle n, \mathrm{d}q\rangle = \langle q, \dot n\rangle$ as measures on (a,b). For $\varphi \in C[0,T]$ with compact support in (a,b), Lemma 4.7 now gives

$$\int_a^b -\langle \varphi, \mathrm{d}q\rangle = \int_a^b \langle \varphi, n\rangle \langle q, \dot n\rangle \;\mathrm{d}t + \int_a^b \langle \varphi - \langle \varphi, n\rangle\, n, g\rangle \;\mathrm{d}t\,.$$

Since φ is arbitrary, the assertion follows. ☐

Lemma 4.11. *Let* $t \in I_\partial$. *If* $q^N(t+) = 0$, *then* q *is continuous at* t. *If* $q^N(t-) = 0$, *then* $q(t-) = q(t+) - \langle q(t+), n(t)\rangle\, n(t)$.

Proof. Both assertions follow from Lemma 4.8, in the first case because $q^N(t-) = 0$ by Lemma 4.4. ☐

5. **Optimality conditions for problem (P).** In this section, we summarize the optimality conditions proved so far in the form of a theorem. We consider an optimal control $u_* \in L^2(0,T;\mathbb{R}^d)$ with corresponding states $y_* \in H^1(0,T;\mathbb{R}^n)$ and $v_*, z_* \in H^1(0,T;\mathbb{R}^m)$, let $\xi = v_* - z_*$. Let

$$I_0 = \{t : z_*(t) \in \mathrm{int}(Z)\}\,, \quad I_\partial = \{t : z_*(t) \in \partial Z\}$$

denote the interior resp. the boundary part of the optimal trajectory. A time $t \in (0,T)$ is called a $\boldsymbol{(0,\partial)}$**-switching time** if $(t-\varepsilon, t) \subset I_0$ and $(t, t+\varepsilon) \subset I_\partial$ for some $\varepsilon > 0$, and it is called a $\boldsymbol{(\partial,0)}$**-switching time** if $(t-\varepsilon,t) \subset I_\partial$ and $(t, t+\varepsilon) \subset I_0$ for some $\varepsilon > 0$. The optimal trajectory is called **regular** if

$$\dot\xi(t) \neq 0 \quad \text{a.e. in } I_\partial\,. \tag{83}$$

For convenience of the reader, we summarize the assumptions made throughout the paper.

Hypothesis 5.1. (i) *The set $Z \subset \mathbb{R}^m$ is closed, convex, has nonempty interior and moreover is uniformly convex according to*

$$\exists \gamma_0 > 0 \text{ such that } \left\langle D^2 d(x)h, h\right\rangle \geq \gamma_0 |h|^2 , \quad \forall x \in \partial Z,\, h \in T(x) , \tag{84}$$

where $d(x)$ denotes the distance of x to Z and $T(x)$ the hyperplane tangent to Z at x.

(ii) *The functions f, $\partial_y f$, $\partial_z f$ as well as L, $\partial_y L$, $\partial_z L$ satisfy a Carathéodory condition.*

(iii) *We have*

$$|f(t, y, z)| \leq \alpha_0(t) + \alpha_1(|y| + |z|) \tag{85}$$

for some $\alpha_0 \in L^1$ and some constant $\alpha_1 > 0$, and

$$|L(t, 0, 0)| \leq \beta_4(t) \tag{86}$$

for some $\beta_4 \in L^1$.

(iv) *We have*

$$\|\partial_y f(t, y, z)\| + \|\partial_z f(t, y, z)\| \leq \alpha_2(t) \cdot \alpha_3(|y|, |z|) \tag{87}$$

for some $\alpha_2 \in L^\infty$ and some continuous function α_3, and

$$|\partial_y L(t, y, z)| + |\partial_z L(t, y, z)| \leq \beta_3(t) \cdot \beta_4(|y|, |z|) \tag{88}$$

for some $\beta_3 \in L^2$ and some continuous function β_4.

(v) *The matrix $E \in \mathbb{R}^{(d,d)}$ is symmetric and positive semidefinite. The set $\Omega \subset \mathbb{R}^d$ is closed and convex, and either it is bounded, or E is positive definite and L is bounded from below by a constant.*

Note that these assumptions in particular imply those of Hypothesis 2.2, and that (ii) – (iv) except possibly 85 are satisfied whenever f and L are C^1.

Theorem 5.2 (Main result)**.**
Let $u_* \in L^2(0, T; \mathbb{R}^d)$ be a solution of (P). Then there exists adjoints $p \in H^1(0, T; \mathbb{R}^n)$ and $q \in BV(0, T; \mathbb{R}^m)$ with the following properties. There holds the maximum condition

$$\left\langle B^T p(t) + B^T S^T q(t) + E u_*(t), w - u_*(t)\right\rangle \geq 0 , \quad \forall w \in \Omega , \tag{89}$$

for a.e. $t \in (0, T)$. On $(0, T)$, the adjoint p satisfies

$$-\dot{p} = \partial_y f(t, y_*(t), z_*(t))^T (p + S^T q) + \partial_y L(t, y_*(t), z_*(t), u_*(t)) , \quad p(T) = 0 . \tag{90}$$

On I_0, the adjoint q is absolutely continuous and satisfies

$$-\dot{q} = \partial_z f(t, y_*(t), z_*(t))^T (p + S^T q) + \partial_z L(t, y_*(t), z_*(t), u_*(t)) . \tag{91}$$

On I_∂, the adjoint q satisfies $\left\langle q, \dot{\xi}\right\rangle = 0$ a.e. and

$$-\int_{I_\partial} \left\langle \varphi - \langle \varphi, n\rangle n, \mathrm{d}q\right\rangle = \int_{I_\partial} \left\langle \varphi - \langle \varphi, n\rangle n, g\right\rangle \mathrm{d}t \tag{92}$$

for any $\varphi \in C([0, T]; \mathbb{R}^m)$, where $\mathrm{d}q$ denotes the measure associated with q and

$$\begin{aligned} g(t) = &- |\dot{\xi}(t)| D^2 \psi(z_*(t)) q + \partial_z f(t, y_*(t), z_*(t))^T (p + S^T q) \\ &+ \partial_z L(t, y_*(t), z_*(t), u_*(t)) . \end{aligned} \tag{93}$$

At the end point we have $q(T) = 0$. At every discontinuity point $t \in I_\partial$ of q, it holds

$$q(t-) - q(t+) = (q^N(t-) - q^N(t+)) n(t) , \quad q^N = \langle q, n\rangle . \tag{94}$$

Let moreover the regularity condition 83 hold. Then $q^N = \langle q, n\rangle = 0$ *a.e. on* I_∂. *On every subinterval* $(a,b) \subset I_\partial$, q *is in* $H^1(a,b;\mathbb{R}^m)$ *and satisfies*

$$-\dot{q} = \langle q, \dot{n}(t)\rangle\, n(t) + g(t) - \langle g(t), n(t)\rangle\, n(t)\,, \tag{95}$$

where

$$\begin{aligned} g(t) = &- |\dot{\xi}(t)| D^2 d(z_*(t)) q + \partial_z f(t, y_*(t), z_*(t))^T (p + S^T q) \\ &+ \partial_z L(t, y_*(t), z_*(t), u_*(t))\,. \end{aligned} \tag{96}$$

At a $(0,\partial)$*-switching point,* q *is continuous. At a* $(\partial, 0)$*-switching point* t, *we have*

$$q(t-) = q(t+) - \langle q(t+), n(t)\rangle\, n(t)\,. \tag{97}$$

Proof. We list the references for the individual statements. For 89 refer to 60, for 90 to 61, for 91 to Lemma 4.1, for 92 to Lemma 4.7, for 94 to Lemma 4.8, for 95 to Lemma 4.10, for 97 to Lemma 4.11. □

In the case where the optimal trajectory consists of a finite succession of interior and boundary arcs and moreover the regularity condition 83 holds, Theorem 5.2 shows that the adjoint q is piecewise absolutely continuous and provides explicit jump relations. In other cases, a more general behaviour of q is expected to occur.

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