Compressible fluid flows driven by stochastic forcing

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joint work with

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A (classical) result of Bensoussan and Temam

(Incompressible) Navier-Stokes system

$$\mathrm{d}\mathbf{u} + \left[\mathrm{div}_x \Big(\mathbf{u} \times \mathbf{u} \Big) + \nabla_x \Pi \right] \ \mathrm{d}t = \mu \Delta \mathbf{u} \ \mathrm{d}t + \mathrm{d}w$$

or

$$\partial_t \mathbf{u}(\omega) + \mathrm{div}_x \Big(\mathbf{u}(\omega) \times \mathbf{u}(\omega) \Big) + \nabla_x \Pi = \mu \Delta \mathbf{u}(\omega) + \partial_t \mathbf{w}(\omega)$$

Initial conditions

$$\mathbf{u}(0, \cdot, \omega) = \mathbf{u}_0(\omega)$$

$$\omega \in \mathcal{O} = \mathcal{O}(\mathcal{O}, \mathcal{B}, \mu)$$

$\mathcal{B}$ family of Borel sets, μ regular probability measure

Abstract result of J. von Neumann

Theorem

Let X, Y be two separable Banach spaces and Λ a multivalued mapping of X into closed non-void subsets of Y with closed graph. Then Λ admits a section that is universally Radon measurable, meaning there exists a mapping

$$\sigma : X \to Y$$

such that $\sigma(x) \in \Lambda(x)$ for any $x \in X$, and σ is measurable for any Radon measure defined on the family of Borel sets in X.

Compressible Navier-Stokes system

Equation of continuity

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u}) \ \mathrm{d}t = 0$$

Momentum equation

$$\mathrm{d}(\varrho \mathbf{u}) + \Big(\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) \Big) \ \mathrm{d}t = \mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{u}))\mathrm{d}t + \varrho \mathrm{d}\mathbf{w}$$

Newton's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Pressure law

$$p(\varrho) \approx a \varrho^\gamma, \ \ a > 0, a\gamma > 3/2$$

Boundary and initial data

$\Omega \subset R^3$ a regular bounded domain

No-slip

$$\mathbf{u}|_{\partial\Omega} = 0$$

Initial data

$$\varrho(0,\cdot,\omega) = \varrho_0(\omega), \ (\varrho\mathbf{u})(0,\cdot,\omega) = (\varrho\mathbf{u})_0(\omega)$$

Problems with variable density

- H. Fujita Yashima, Equations de Navier-Stokes non homogènes et applications, Tesi di Perfezionamento Scuola Normale Superiore, Pisa, 1992
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- E. Tornatore, H. Fujita Yashima, One-dimensional stochastic equations for a viscous barotropic gas
  Ric. Mat. **46** (1998), 255-283

Weak formulation

Renormalized equation of continuity

$$\int_0^T \int_\Omega \left(\Big(\varrho + b(\varrho)\Big)\partial_t \varphi + \Big(\varrho + b(\varrho)\Big)\mathbf{u}\cdot\nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \Big(b'(\varrho)\varrho - b(\varrho)\Big)\mathrm{div}_x \mathbf{u}\Big)\varphi \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \Big(\varrho_0 + b(\varrho_0)\Big)\varphi(0,\cdot) \, \mathrm{d}x$$

for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, and any $b \in C_c^\infty[0,\infty)$

Momentum equation

$$\int_0^T \int_\Omega \Big(\varrho(\mathbf{u} - \mathbf{w}) \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho)\mathrm{div}_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$\int_0^T \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi(x) + \varrho \mathbf{u} \cdot \nabla_x (\mathbf{w} \cdot \varphi) \Big) \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\varrho \mathbf{u})_0 \cdot \varphi(0,\cdot) \, \mathrm{d}x$$

for all $\varphi \in C_c^\infty([0,T) \times \Omega; R^3)$

Energy inequality

Energy inequality

$$
-\int_0^T \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{w}|^2 + P(\varrho) \right) \ \mathrm{d}x \ \partial_t \psi \ \mathrm{d}t
$$

$$
+ \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \psi \ \mathrm{d}t
$$

$$
\leq \psi(0) \int_\Omega \left(\frac{1}{2} \frac{|(\varrho \mathbf{u})_0|^2}{\varrho_0} + P(\varrho_0) \right) \ \mathrm{d}x
$$

$$
+ \int_0^T \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{w} - \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{w} - p(\varrho) \mathrm{div}_x \mathbf{w} \ \mathrm{d}x \ \psi \ \mathrm{d}t
$$

$$
+ \int_0^T \int_\Omega \frac{1}{2} \varrho \mathbf{u} \cdot \nabla_x |\mathbf{w}|^2 \Big) \ \mathrm{d}x \ \psi \ \mathrm{d}t
$$

for any $\psi \in C_c^\infty[0, T)$, $\psi \geq 0$

Sequential stability

Suppose

$$\mathrm{ess} \sup_{t \in (0,T)} \|\mathbf{w}_n(t, \cdot)\|_{W^{1,\infty}(\Omega;R^3)} \leq c, \mathbf{w}_n \to \mathbf{w} \text{ in } L^1(0, T; W^{1,1}(\Omega; R^3))$$

$$\varrho_n(0, \cdot) \to \varrho_0 \text{ in } L^1(\Omega)$$

Sequential stability

$$\varrho_n \to \varrho \text{ in } C_{\rm weak}([0, T]; L^\gamma(\Omega)) \text{ and in } \boxed{L^1((0, T) \times \Omega)},$$

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2(0, T; W_0^{1,2}(\Omega; R^3)),$$

$$\varrho_n(\mathbf{u}_n - \mathbf{w}_n) \to \varrho(\mathbf{u} - \mathbf{w}) \text{ in } C_{\rm weak}([0, T]; L^{2\gamma/(\gamma+1)}(\Omega; R^3)),$$

Effective viscous flux identity

Effective viscous flux identity (Lions):

$$\overline{p(\varrho)b(\varrho)} - \left(\frac{4}{3}\nu + \lambda\right)\overline{b(\varrho)\mathrm{div}_x\mathbf{u}} = \overline{p(\varrho)}\ \overline{b(\varrho)} - \left(\frac{4}{3}\nu + \lambda\right)\overline{b(\varrho)}\mathrm{div}_x\mathbf{u}$$

Supposing, for simplicity, that we can take $b(\varrho) = \varrho$, we get

$$\partial_t \int_\Omega \overline{\varrho \log(\varrho)}\ \mathrm{d}x + \int_\Omega \overline{\varrho \mathrm{div}_x \mathbf{u}}\ \mathrm{d}x = 0$$

$$\partial_t \int_\Omega \varrho \log(\varrho)\ \mathrm{d}x + \int_\Omega \varrho \mathrm{div}_x \mathbf{u}\ \mathrm{d}x = 0$$

whence

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho)$$

(Very) formal proof

$$\partial_t(\varrho_n \mathbf{u}_n) + \mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n) = \Delta \mathbf{u}_n + \varrho_n \partial_t \mathbf{w}_n$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \overline{p(\varrho)} = \Delta \mathbf{u} + \varrho \partial_t \mathbf{w}$$

Step 1: Apply $\Delta^{-1}\mathrm{div}_x$

$$\Delta^{-1}\partial_t \mathrm{div}_x(\varrho_n \mathbf{u}_n) + \mathrm{div}_x \Delta^{-1}\mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + p(\varrho_n) = \mathrm{div}_x \mathbf{u}_n + \Delta^{-1}\mathrm{div}_x(\varrho_n \partial_t \mathbf{w}_n)$$

$$\Delta^{-1}\partial_t \mathrm{div}_x(\varrho \mathbf{u}) + \mathrm{div}_x \Delta^{-1}\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \overline{p(\varrho)} = \mathrm{div}_x \mathbf{u} + \Delta^{-1}\mathrm{div}_x(\varrho \partial_t \mathbf{w})$$

Step 2: Multiply by $b(\varrho_n)$, $\overline{b(\varrho)}$

$$\Big(p(\varrho_n) - \mathrm{div}_x \mathbf{u}_n \Big) b(\varrho_n)$$

$$= -b(\varrho_n) \Delta^{-1} \partial_t \mathrm{div}_x (\varrho_n \mathbf{u}_n) - b(\varrho_n) \mathrm{div}_x \Delta^{-1} \mathrm{div}_x (\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n)$$

$$+ b(\varrho_n) \Delta^{-1} \mathrm{div}_x (\varrho_n \partial_t \mathbf{w}_n)$$

$$\Big(\overline{p(\varrho)} - \mathrm{div}_x \mathbf{u} \Big) \overline{b(\varrho)}$$

$$= -\overline{b(\varrho)} \Delta^{-1} \partial_t \mathrm{div}_x (\varrho \mathbf{u}) - \overline{b(\varrho)} \mathrm{div}_x \Delta^{-1} \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u})$$

$$+ \overline{b(\varrho)} \Delta^{-1} \mathrm{div}_x (\varrho \partial_t \mathbf{w})$$

Step 3: Replace terms by their equivalents modulo compact perturbation

$$\Big(p(\varrho_n) - \mathrm{div}_x \mathbf{u}_n\Big) b(\varrho_n)$$

$$= \boxed{\mathbf{u}_n b(\varrho_n) \cdot \nabla_x \Delta^{-1} \mathrm{div}_x(\varrho_n \mathbf{u}_n) - (\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) : \nabla_x \Delta^{-1} \nabla_x b(\varrho_n)}$$

$$+ b(\varrho_n) \mathrm{div}_x \Delta^{-1} \mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{w}_n) - b(\varrho_n) \mathbf{u}_n \cdot \nabla_x \Delta^{-1} \mathrm{div}_x(\varrho_n \mathbf{w}_n)$$

$$\Big(\overline{p(\varrho)} - \mathrm{div}_x \mathbf{u}\Big) \overline{b(\varrho)}$$

$$= \boxed{\mathbf{u} \overline{b(\varrho)} \cdot \nabla_x \Delta^{-1} \mathrm{div}_x(\varrho \mathbf{u}) - (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla_x \Delta^{-1} \nabla_x \overline{b(\varrho)}}$$

$$+ \overline{b(\varrho)} \mathrm{div}_x \Delta^{-1} \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{w}) - \overline{b(\varrho)} \mathbf{u} \cdot \nabla_x \Delta^{-1} \mathrm{div}_x(\varrho \mathbf{w})$$

Compactness lemma

$$\mathcal{R}_{i,j} = \partial_{x_i} \Delta^{-1} \partial_{x_j}$$

A variant of Div-Curl lemma

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^p$$

$$\mathbf{w}_n \to \mathbf{w} \text{ weakly in } L^q$$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$$

Then

$$v_n^i \mathcal{R}_{i,j}[w_n^j] - w_n^i \mathcal{R}_{i,j}[v_n^j] \to v^i \mathcal{R}_{i,j}[w^j] - w^i \mathcal{R}_{i,j}[v^j] \text{ weakly in } L^r$$

(summation convention)

Commutator lemma

Commutator lemma

$$\phi \in W^{1,r}(R^N),\ \mathbf{V} \in L^p(R^N; R^N)$$

$$1 < r < N,\ 1 < p < \infty,\ \frac{1}{r} + \frac{1}{p} < 1 + \frac{1}{N}$$

Then

$$\left\| \mathcal{R}_{i,j}[\phi V_j] - \phi \mathcal{R}_{i,j}[V_j \right\|_{W^{\beta,s}} \leq c \left\| \phi \right\|_{W^{1,r}} \left\| \mathbf{V} \right\|_{L^p},\ i = 1, \dots, N$$

(summation convention)

for certain $\beta > 0$, $s > 1$