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COMPLETELY DISSOCIATIVE GROUPOIDS

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Abstract. In a groupoid, consider arbitrarily parenthesized expressions on the k variables $x_0, x_1, \dots x_{k-1}$ where each x_i appears once and all variables appear in order of their indices. We call these expressions k-ary formal products, and denote the set containing all of them by $F^\sigma(k)$. If $u, v \in F^\sigma(k)$ are distinct, the statement that u and v are equal for all values of $x_0, x_1, \dots x_{k-1}$ is a generalized associative law.

Among other results, we show that many small groupoids are completely dissociative, meaning that no generalized associative law holds in them. These include the two groupoids on $\{0, 1\}$ where the groupoid operation is implication and NAND, respectively.

Keywords: groupoid, dissociative groupoid, generalized associative groupoid, formal product, reverse Polish notation (rPn)

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1. Introduction

Our earlier paper, [3], begins an investigation of groupoids $\mathcal{G} := \langle G; \star\rangle$ in which the binary operation $\star\colon\ G \times G \to G$ fails to be associative; that is, those $\mathcal{G}$ for which there exists an ordered triple $g_0g_1g_2 \in G^3$ with $(g_0 \star g_1) \star g_2 \neq g_0 \star (g_1 \star g_2)$. The failure of a triple to associate may induce differing products of longer strings too.

We study generalized associativity in $\mathcal{G}$, using product formulas $\mathbf{u}$. A k-ary $\mathbf{u}$ is on the k variables $x_0x_1 \dots x_{k-1}$, each appearing in $\mathbf{u}$ exactly once and in the order of its index. We focus on those $\mathcal{G}$ for which, given k-ary formulas $\mathbf{u} \neq \mathbf{v}$, there is a string $\vec{g} := g_0g_1 \dots g_{k-1} \in G^k$ such that $\vec{g}\mathbf{u}^\star \neq \vec{g}\mathbf{v}^\star$, where $\vec{g}\mathbf{u}^\star \in G$ denotes the product under $\star$ which is computed after g_i replaces x_i in $\mathbf{u}$ for each i. Such $\mathcal{G}$ we call *k-dissociative.* We call $\mathcal{G}$ *completely dissociative* if it is k-dissociative for all $k \geqslant 3$.

In §3 a general result enables us to prove many $\mathcal{G}$ to be completely dissociative. Six of the sixteen groupoids on the set $2 := \{0, 1\}$ are completely dissociative. Among them are those expressing the logical operators implication and NAND. The proof that NAND is completely dissociative uses boolean algebra. Some groupoids on the sets $3 := \{0, 1, 2\}$ and $4 := \{0, 1, 2, 3\}$ are completely dissociative. There are infinite completely dissociative groupoids as well.

In §4, Birkhoff's Theorem in universal algebra leads us to study *primitive* completely dissociative groupoids. These are the $\mathcal{G}$ that are minimal completely dissociative groupoids in the variety they generate. We show that a number of finite $\mathcal{G}$ are primitive, illustrating general techniques in the process.

§5 discusses representations for $k \geqslant 3$ of k-ary operations $\varphi\colon G^k \to G$ via a string $\vec{\beta} := \beta_0\beta_1 \ldots \beta_{k-2}$ of binary operations $\beta_j\colon G^2 \to G$. There exist $\varphi\colon G^k \to G$ that are unrepresentable thus if and only if $2 \leqslant |G| < \infty$.

Recalling [3], we summarize our terminology in §2.

2. Our language

Henceforth $\omega := \{0, 1, 2, \ldots\}$ and $\mathbb{N} := \{1, 2, 3, \ldots\}$. When $n \in \mathbb{N}$ then n also denotes the set $\{0, 1, \ldots, n-1\}$.

For $\{k, n\} \subseteq \mathbb{N}$ we write n^k to designate the set of all k-tuples of elements in n, and n^ω denotes the set of all infinite sequences $j_0j_1j_2\ldots$ whose terms are elements in n. Obviously the number of k-ary operations, $\varphi\colon n^k \to n$ on the set n, is equal to the integer n^{n^k}. The most familiar are for $k = 2$; namely, the n^{n^2} distinct *binary operations* on the set n.

As in [3], we often employ reverse Polish notation, rPn, for the k-ary operations $\varphi\colon G^k \to G$, putting operation symbols after what they are applied to.

The paper [3] discusses the set $F^\sigma(k)$ of all "formal k-products", which we reintroduce in Definition 2.1, below. Our "formal products" are special instances of what, in the more encompassing language of universal algebra, are called "terms". An example might clarify our intent:

Consider the formal 5-product $\mathbf{u} := x_0x_1 \bullet x_2x_3 \bullet \bullet x_4 \bullet \in F^\sigma(5)$. (Which is $((x_0 \bullet x_1) \bullet (x_2 \bullet x_3)) \bullet x_4$ in infix notation.)

When G is a set, and if $\vec{\beta}$ is a 4-tuple $\beta_0\beta_1\beta_2\beta_3$ with $\beta_i\colon G^2 \to G$ for every $i \in 4$, then by our comments after Definition 2.2 below, the equalities

$$\mathbf{u}^{\vec{\beta}} := x_0x_1 \bullet x_2x_3 \bullet \bullet x_4 \bullet^{\vec{\beta}} := x_0x_1\beta_0x_2x_3\beta_1\beta_2x_4\beta_3$$

present a 5-ary operation $\mathbf{u}^{\vec{\beta}}\colon G^5 \to G$ on G.

If the finite sequence of binary operations has $\beta_i = \star$ for all relevant i, then we may write $\mathbf{v}^{\vec{\beta}}$ more simply as $\mathbf{v}^\star$.

Here is a synopsis and modification of terminology introduced in [3]:

Viewing a *word* $\mathbf{w}$ as a (finite or infinite) sequence, we say that $\mathbf{s}$ is a *subword* of $\mathbf{w}$ iff $\mathbf{s}$ is a subsequence of $\mathbf{w}$. A subword of $\mathbf{w}$ whose letters occur consecutively in $\mathbf{w}$ we call a *segment* of $\mathbf{w}$; some authors use the word "block" to designate a segment in a one-letter alphabet.

An initial segment of $\mathbf{w}$ we call a *prefix* of $\mathbf{w}$; a terminal segment of $\mathbf{w}$ we call a *suffix* of $\mathbf{w}$. Of course a nonempty suffix of $\mathbf{w}$ is infinite if and only if $\mathbf{w}$ is itself infinite.

Finite words we usually call *tuples.* But from now on, infinite words will always be called *sequences*, and our "sequences" will always be infinite.

Henceforth $\vec{x} := x_0x_1x_2\ldots$ denotes an infinite sequence of distinct variables x_i, and $\bullet$ denotes an operator symbol. Let $\Sigma := \{\bullet, x_0, x_1, x_2, \ldots\}$. Then Σ^* denotes the set of finite words each of whose letters is an element in the infinite alphabet Σ.

Definition 2.1. Let $\mathbf{u} \in \Sigma^*$, and let $k \in \mathbb{N}$. Then $\mathbf{u}$ is said to be a *formal k-product* iff it consists of $x_0x_1\ldots x_{k-1}$ with $k-1$ many $\bullet$ symbols inserted in it to make a valid rPn expression.

The expression $F^\sigma(k)$ denotes the set of all formal k-products, and the set of all formal products is then $F^\sigma := \bigcup\{F^\sigma(k)\colon\ k \in \mathbb{N}\}$.

Given $\mathbf{b} \in F^\sigma(j)$, we let $\mathbf{b_i}$ denote the element in Σ^* obtained by replacing in $\mathbf{b}$ the letter x_t with the letter x_{i+t} for each $t \in \omega$. This enables us to define the binary operation $\odot\colon\ F^\sigma \times F^\sigma \to F^\sigma$ by $\mathbf{ab}\odot := \mathbf{ab_i}\bullet$ where $\mathbf{a} \in F^\sigma(i)$. It is routine to verify that every $\mathbf{w} \in F^\sigma \setminus \{x_0\}$ has a unique factorization under $\odot$ into a product $\mathbf{w} = \mathbf{ps}\odot$ of two elements in F^σ.

Definition 2.2. Let $\{n, k\} \subseteq \mathbb{N}$, and let $\mathbf{u} \in F^\sigma(k)$. For each $j \in k-1$ let $\beta_j\colon\ n^2 \to n$, and let $\vec{\beta}$ be the $(k-1)$-tuple $\beta_0\beta_1\ldots\beta_{k-2}$. Then $\mathbf{u}^{\vec{\beta}}$ denotes the word in $\{\beta_0, \beta_1, \ldots \beta_{k-2}, x_0, x_1, x_2, \ldots\}^*$ obtained by substituting the operation symbol β_j for the jth occurrence of the letter $\bullet$ in the word $\mathbf{u}$, for each $j \in k-1$. We call the words $\mathbf{u}^{\vec{\beta}}$ *formal k-ary products* in $\vec{\beta}$.

For a given $\vec{\beta}$, we also write $\mathbf{u}^{\vec{\beta}}$ for the operation on n that takes $g_0g_1\ldots g_{k-1} \in G^k$ to $\vec{g}\mathbf{u}^{\vec{\beta}} \in G$ obtained by replacing x_0 with g_0, x_1 with g_1 and so on in $\mathbf{u}^{\vec{\beta}}$. In this case, we say that $\mathbf{u}$ *represents φ via $\vec{\beta}$.*

For $k \in \mathbb{N}$ and $\mathbf{u} \in F^\sigma(k)$ and $\vec{g} \in G^\omega$, note that $\vec{g}\mathbf{u}^{\vec{\beta}}$ is determined by the length-k prefix of $\vec{g}$. The "extra" terms in $\vec{g}$ simplify our notation.

Two additional conventions: When $y \in G$ and $\vec{g} := g_0g_1g_2\ldots \in G^\omega$ is the sequence such that $g_t = y$ for all $t \in \omega$ then we may write $\vec{g}$ instead as $\vec{y}$, where $\vec{y} := yyy\ldots$ That is, when $y \in G$ then $\vec{y} := yyy\ldots \in G^\omega$.

For $\langle G; \star\rangle$ a groupoid, when $\vec{g} = g_0g_1g_1\ldots \in G^\omega$ and $m \in \omega$ then $\vec{g}_m$ denotes the infinite suffix $g_mg_{m+1}g_{m+2}\ldots$ of $\vec{g}$. Thus $\vec{g}\mathbf{u_m}^\star = \vec{g}_m\mathbf{u}^\star$.

For $\mathcal{G} := \langle G; \star\rangle$ a groupoid, we say that $\vec{g}$ *separates* $\mathbf{u}$ from $\mathbf{v}$ in $\mathcal{G}$ iff $\vec{g}\mathbf{u}^\star \neq \vec{g}\mathbf{v}^\star$.

Definition 3.1. $\langle G; \star\rangle$ is said to be *k-dissociative* iff every pair $\mathbf{u} \neq \mathbf{v}$ of elements in $F^\sigma(k)$ can be separated by a sequence in G^ω. We call $\langle G; \star\rangle$ *completely dissociative* iff $\langle G; \star\rangle$ is k-dissociative for every $k \geqslant 3$.

(In [3], we instead used the term "completely free".) The 3-dissociative groupoids are those which are not semigroups.

Another way to view this is as follows. Our formal products also correspond to specific kinds of *terms*; namely, those where the variables $x_0, x_1, x_2, \ldots x_{k-1}$ each appear exactly once, and in the order given here. Expressions such as $\mathbf{u}^\star$ we call *formal products interpreted* in $\mathcal{G}$ (or by $\star$).

We say that $\mathcal{G}$ satisfies the *identity* $u \approx v$ iff $\mathbf{u}^\star = \mathbf{v}^\star$ in $\mathcal{G}$. If $u \neq v$, then $u \approx v$ is a *nontrivial identity.* So we have that $\mathcal{G}$ is k-dissociative if and only if no nontrivial k-ary identity is satisfied between formal products interpreted in $\mathcal{G}$.

We begin with a few remarks. For every integer $k \geqslant 3$, notice that:

(1) $\mathcal{G}$ is k-dissociative if, and only if, $|F^\sigma(k)/\mathcal{G}| = |F^\sigma(k)|$ and each $\approx_{\mathcal{G}}$ equivalence class is a singleton $[\mathbf{v}]_{\mathcal{G}} = \{\mathbf{v}\} \subseteq F^\sigma(k)$.

(2) Both isomorphism and anti-isomorphism respect k-dissociativity.

(3) $\mathcal{G}$ is k-dissociative if $\mathcal{G}$ has a k-dissociative subgroupoid.

(4) If a component groupoid of a cartesian product groupoid is k-dissociative then the product groupoid is also k-dissociative.

(5) If $\mathcal{G}$ has a k-dissociative homomorphic image, $\mathcal{G}$ is k-dissociative.

While the five statements above can all be verified directly, the last four are also consequences of Birkhoff's Theorem in Universal Algebra. This will be discussed in more detail in §4.

Question 3.1. For each $k \geqslant 3$ is there a k-dissociative groupoid which is not $(k+1)$-dissociative?

Theorem 3.1. *If $\langle G; \star\rangle$ is k-dissociative, then $\langle G; \star\rangle$ is j-dissociative for all $j \in \{3, 4, \ldots, k-1\}$.*

Proof. We assume $k \geqslant 3$, since otherwise the assertion is vacuous. To prove the contrapositive, suppose that $3 \leqslant j < k$ and that $\mathbf{a}^\star = \mathbf{b}^\star$ for some formal j-products $\mathbf{a} \neq \mathbf{b}$. Then of course $\{\mathbf{a}x_j \ldots x_k \bullet \ldots \bullet, \mathbf{b}x_j \ldots x_k \bullet \ldots \bullet\} \subseteq F^\sigma(k+1)$ and $\mathbf{a}x_j \ldots x_k \bullet \ldots \bullet \neq \mathbf{b}x_j \ldots x_k \bullet \ldots \bullet$,. Let $\vec{g} \in G^\omega$ be arbitrary. Since $\vec{g}\mathbf{a}^\star = \vec{g}\,\mathbf{b}^\star$, we have $\vec{g}\mathbf{a}x_j \ldots x_k \bullet \ldots \bullet^\star = \vec{g}\mathbf{a}^\star g_j \ldots g_k \star \ldots \star = \vec{g}\mathbf{b}^\star g_j \ldots g_k \star \ldots \star = \vec{g}\mathbf{b}x_j \ldots x_k \bullet \ldots \bullet^\star$. Thus $\mathbf{a}x_j \ldots x_k \bullet \ldots \bullet^\star = \mathbf{b}x_j \ldots x_k \bullet \ldots \bullet^\star$. $\square$

We already have an example of an infinite completely dissociative groupoid. An easy induction shows that $\langle F^\sigma, \odot\rangle$ is completely dissociative.

Conjecture 3.13 in [3] fails. Of the 16 binary operation tables on set 2, eight are of semigroups. We call the tables themselves "concrete" semigroups.

We name "concrete" groupoids via a natural nomenclature that specifies binary operation tables on each set $n \in \mathbb{N}$, creating a dictionary, $\underline{\mathcal{G}}(n)$ of groupoids $n_j := \langle n; \star_j\rangle$, with $|\underline{\mathcal{G}}(n)| = n^{n^2}$. Viewed in base n, j encodes the table of $\star_j$. We let j be the sum over all $i, k \in n$ of $(i \star_j k) n^{n^2-in-k-1}$. For example, 2_{13} has its operation $\star = \star_{13}$ defined by $0 \star 0 = 1$, $0 \star 1 = 1$, $1 \star 0 = 0$ and $1 \star 1 = 1$, since 13 is 1101 in base 2. Interpreting 0 as "false" and 1 as "true", $\star_{13}$ is $\Rightarrow$, and we call 2_{13} the "implication groupoid".

$\underline{\mathcal{G}}(2) := \{2_0, 2_1, 2_2, \ldots, 2_{15}\}$ is the family of all $2^{2^2} = 16$ distinct such tables on the universe $2 := \{0, 1\}$. The groupoid 2_j is a semigroup if and only if $j \in \{0, 1, 3, 5, 6, 7, 9, 15\}$. Our computer verifies that neither 2_{10} nor 2_{12} is 4-dissociative, although not one of the eight triples $\langle a, b, c\rangle \in 2^3$ associates in 2_{10} or in 2_{12}. Theorem 3.6 will establish that the remaining six $2_j \in \underline{\mathcal{G}}(2)$ are completely dissociative.

Similarities in our early inductive proofs of the complete dissociativity of a few groupoids suggested a comprehensive fact, viz Theorem 3.2, below. We need additional terminology in order to present this theorem.

When a j-tuple $\vec{r} \in G^j$ occurs m times consecutively in a sequence $\vec{g} \in G^\omega$, we write the resulting mj-tuple segment of $\vec{g}$ as $\vec{r}^m$. This generalizes our notation y^m, which denotes the m-tuple $yy \ldots y \in G^m$ when $y \in G$.

For $\vec{g} \in G^\omega$ where $\mathcal{G} := \langle G; \star\rangle$ is a groupoid, and for $S \subseteq G$, we say that $\vec{g}$ *yields* S iff $\vec{g}\mathbf{u}^\star \in S$ for every $\mathbf{u} \in F^\sigma$.

Given $U \subseteq F^\sigma$, we say $\vec{g}$ yields S *on* U iff $\vec{g}\mathbf{u}^\star \in S$ for all $\mathbf{u} \in U$.

The set S is called *yieldable* iff there is some $\vec{g}$ which yields S. If $\vec{g}$ yields $\{a\}$ then we say that $\vec{g}$ yields a and that a is yieldable.

For $i \in \mathbb{N}$ and $\mathbf{u} \in F^\sigma$, we call $\mathbf{u}$ an *i-split* iff its unique factorization in $\langle F^\sigma; \odot\rangle$ is $\mathbf{u} = \mathbf{ab}\odot$ with $\mathbf{a} \in F^\sigma(i)$.

Theorem 3.2. *Let $\mathcal{G} := \langle G; \star\rangle$ be a groupoid, let $T \subseteq G$ with $|T| \geqslant 2$, and let the following three conditions hold.*

i) Left Separation: *If $\{x, y\} \subseteq T$ with $x \neq y$ then there is a yieldable $L_{x,y} \subseteq G$ with $sx\star \neq s'y\star$ and $\{sx\star, s'y\star\} \subseteq T$ for all $s, s' \in L_{x,y}$.*

ii) Right Separation: *If $\{x, y\} \subseteq T$ and $x \neq y$, there is a yieldable $R_{x,y} \subseteq G$ with $xs\star \neq ys'\star$ and $\{xs\star, ys'\star\} \subseteq T$ for all $s, s' \in R_{x,y}$.*

iii) Split Separation: *For all* $\{i,j,k\} \subseteq \mathbb{N}$ *with* $i < j \leqslant k$, *there are nonempty disjoint subsets* A *and* B *of* T, *and a* $\vec{g} \in G^{\omega}$ *that yields* A *on the set of all* i*-splits in* $F^{\sigma}(k+1)$ *and that yields* B *on the set of all* j*-splits in* $F^{\sigma}(k+1)$.

Then $\mathcal{G}$ *is completely dissociative.*

Proof. Note that every groupoid is trivially k-dissociative for $k \in \{1,2\}$. For $k \geqslant 3$, we will prove by induction that whenever $w, w' \in F^{\sigma}(k)$ are not equal that there exists $\vec{g} \in G^k$ so that $\vec{g}w^{\star}$ and $\vec{g}w'^{\star}$ are distinct elements of T. Note that the 1-split $x_0x_1x_2\bullet\bullet$ and the 2-split $x_0x_1\bullet x_2\bullet$ are the only two elements in $F^{\sigma}(3)$. By Split Separation, there exist $A \subseteq T$ and $B \subseteq T$ and $\vec{g} \in G^{\omega}$ with $A \cap B = \emptyset$, and such that $a = \vec{g}x_0x_1x_2\bullet\bullet^{\star}$ for some $a \in A$ and $b = \vec{g}x_0x_1\bullet x_2\bullet^{\star}$ for some $b \in B$. So $\mathcal{G}$ is k-dissociative if $k = 3$. The basis is done.

Now choose $k \in \{3,4,5,\ldots\}$, and suppose that $\mathcal{G}$ is v-dissociative for every $v \in \{1,2,\ldots,k\}$. Pick any $\{\mathbf{w},\mathbf{w}'\} \subseteq F^{\sigma}(k+1)$ with $\mathbf{w} \neq \mathbf{w}'$. These formal $(k+1)$-products have unique factorizations $\mathbf{w} = \mathbf{ab}\odot$ and $\mathbf{w}' = \mathbf{a}'\mathbf{b}'\odot$ in the groupoid $\langle F^{\sigma};\odot\rangle$, where $\langle \mathbf{a},\mathbf{a}'\rangle \in F^{\sigma}(i) \times F^{\sigma}(i')$ for some $\{i,i'\} \subseteq \{1,2,\ldots,k\}$. Without loss of generality, take it that $i \leqslant i'$.

If $i < i'$ then by Split Separation there exist disjoint subsets A and B of T, and a sequence $\vec{g} \in G^{\omega}$, such that $\vec{g}\mathbf{w}^{\star} \in A$ while $\vec{g}\mathbf{w}'^{\star} \in B$, whence $\vec{g}\mathbf{w}^{\star} \neq \vec{g}\mathbf{w}'^{\star}$, and so $\mathbf{w}^{\star} \neq \mathbf{w}'^{\star}$. Therefore we may take it that $i = i'$. Since $\mathbf{w} \neq \mathbf{w}'$, either $\mathbf{a} \neq \mathbf{a}'$ or $\mathbf{b} \neq \mathbf{b}'$.

Case $\mathbf{a} \neq \mathbf{a}'$: By the inductive hypothesis, there exists $\vec{g} \in G^{\omega}$ with $\vec{g}\mathbf{a}^{\star} \neq \vec{g}\mathbf{a}'^{\star}$ and $\{\vec{g}\mathbf{a}^{\star}, \vec{g}\mathbf{a}'^{\star}\} \subseteq T$. Let $x := \vec{g}\mathbf{a}^{\star}$ and $x' := \vec{g}\mathbf{a}'^{\star}$. By Right Separation there exists $\vec{h} \in G^{\omega}$ such that $xs\star$ and $x's'\star$ are distinct elements in T, where $s := \vec{h}\mathbf{b}^{\star}$ and $s' := \vec{h}\mathbf{b}'^{\star}$. We may suppose $\vec{g_i} = \vec{h}$. Thus $\vec{g}\mathbf{w}^{\star} = \vec{g}\mathbf{ab}\odot^{\star} = \vec{g}\mathbf{ab_i}\bullet^{\star} = \vec{g}\mathbf{a}^{\star}\vec{g}\mathbf{b_i}^{\star}\star = \vec{g}\mathbf{a}^{\star}\vec{g_i}\mathbf{b}^{\star}\star = xs\star \neq x's'\star = \vec{g}\mathbf{a}'^{\star}\vec{g_i}\mathbf{b}'^{\star}\star = \vec{g}\mathbf{a}'^{\star}\vec{g}\mathbf{b}_{\mathbf{i}}'^{\star}\star = \vec{g}\mathbf{a}'\mathbf{b}_{\mathbf{i}}'\bullet^{\star} = \vec{g}\mathbf{a}'\mathbf{b}'\odot^{\star} = \vec{g}\mathbf{w}'^{\star}$, and so $\mathbf{w}^{\star} \neq \mathbf{w}'^{\star}$.

Case $\mathbf{b} \neq \mathbf{b}'$: The proof is similar to that of the previous case, but uses Left Separation.

Thus $\mathcal{G}$ is $(k+1)$-dissociative. So $\mathcal{G}$ is completely dissociative. □

Although Theorem 3.2 is quite general, it is often used in a simple way. For instance, if $a \in G$ is idempotent then surely a is yieldable, for we can let $\vec{g} = a^{\omega}$. Likewise, if $x \mapsto xa\star$ is a permutation of T then Right Separation is shown by setting $R_{x,y} := \{a\}$ for all $x \neq y$. And, if $\star$ is commutative then Right Separation is equivalent to Left Separation. We also often have that $T = G$.

Some situations arise repeatedly when we argue that $\vec{g}$ yields a particular set. Note that if H is a subgroupoid of $\mathcal{G}$, and if $g_0g_1\ldots g_{k-1} \in H^k$ for some $k \in \mathbb{N}$, then $\vec{g}$ yields H. Another situation arises when $\{a,b\}$ is a 2-element subgroupoid of $\mathcal{G}$ with $aa\star = a$ and with $ab\star = ba\star = bb\star = b$; that is, if the set $\{a,b\}$ *forms* *a semilattice* and if $a^\omega \neq \vec{g} \in \{a,b\}^\omega$, then $\vec{g}$ yields b, because b is an absorptive element in subgroupoid $\{a,b\}$.

As our first example of the use of Theorem 3.2, we prove the following.

Theorem 3.3. *The groupoid $\mathcal{B}$, depicted below, is completely dissociative.*

P r o o f. The following table defines the groupoid $\mathcal{B} := \langle 4; \star\rangle$:

$\star$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	2	3	1
2	3	2	3	2
3	1	2	1	3

$\mathcal{B}$

To apply Theorem 3.2, let $T := 4$. The idempotent 0 is yieldable. In fact, 0 is an identity. Moreover, Left and Right Separation are equivalent, since $\mathcal{B}$ is abelian. So, for each $x \neq y$ we let $L_{x,y} = R_{x,y} = \{0\}$.

For Split Separation, let $1 \leqslant i < j \leqslant k$, and take $\vec{g} := 0^{i-1}120^{j-i-1}30^{k-j}\vec{g}_{k+1}$. We let $A := \{1\}$ and $B := \{3\}$. For an i-split $\mathbf{w} = \mathbf{ab}\odot \in F^\sigma(k+1)$ that has $\langle \mathbf{a}, \mathbf{b}\rangle \in F^\sigma(i) \times F^\sigma(k+1-i)$, we compute that $\vec{g}\mathbf{w}^\star = \vec{g}\mathbf{ab}\odot^\star = \vec{g}\mathbf{ab_i}\bullet^\star = \vec{g}\mathbf{a}^\star\vec{g}\,\mathbf{b_i}^\star\star = \vec{g}\mathbf{a}^\star\vec{g}_i\mathbf{b}^\star\star = 0^{i-1}1\vec{g}_i\mathbf{a}^\star\vec{g}_i\mathbf{b}^\star\star = 1(\vec{g}_i\mathbf{b}^\star)\star = 1(20^{j-i-1}30^{k-j}\vec{g}_{k+1}\mathbf{b}^\star)\star = 123\star\star = 11\star = 1 \in A$. That is to say, $\vec{g}$ yields A on the set of all i-splits in $F^\sigma(k+1)$. Similarly $\vec{g}$ yields $12\star 3\star = 3 \in B$ on the set of all j-splits in $F^\sigma(k+1)$. So if $\mathbf{w}' \in F^\sigma(k+1)$ is a j-split then $\vec{g}\mathbf{w}^\star \neq \vec{g}\mathbf{w}'^\star$. That is, $\vec{g}$ separates $\mathbf{w}$ and $\mathbf{w}'$. $\square$

The groupoid $\mathcal{B}$ is interesting because of its subgroupoid $\{1,2,3\}$, which is isomorphic to the groupoid $CI3_{21}$ discussed in §4. The latter groupoid falls just short of complete dissociativity, and constitutes a natural example between semigroups and completely dissociative groupoids. It is curious that the removal from $\mathcal{B}$ of its identity element, 0, destroys complete dissociativity.

We now examine the implication groupoid, the concrete groupoid $2_{13} := \langle 2; \star_{13}\rangle$.

Lemma 3.1. *Let $\vec{g} \in 2^\omega$, let $k \in \mathbb{N}$, and let $\mathbf{u} \in F^\sigma(k)$. Then the groupoid $2_{13} := \langle 2; \star\rangle$ satisfies the following conditions.*

(1) *If $k \geqslant 1$ and $\vec{g} = 1^k\vec{g}_k$ then $\vec{g}\mathbf{u}^\star = 1$.*
(2) *If $k \geqslant 1$ and $\vec{g} = 1^{k-1}0\vec{g}_k$ then $\vec{g}\mathbf{u}^\star = 0$.*
(3) *If $k \geqslant 2$ and $j \leqslant k-2$ and $\vec{g} = 1^j01^{k-j-1}\vec{g}_k$ then $\vec{g}\mathbf{u}^\star = 1$.*

P r o o f. The claim 3.1.1 follows from the fact that 1 is an idempotent.

While 3.1.2 can be easily proved by induction, the key idea is that the last step in the evaluation of $\vec{g}\mathbf{u}^\star$ is $1\star 0 = 0$.

The basis of an inductive proof of 3.1.3 involves $k = 2$ and $j = 0$ and $\vec{g} = 01\vec{g}_2 \in 2^\omega$. For the only $\mathbf{u} \in F^\sigma(2)$ we then get $\vec{g}\mathbf{u}^\star = 01\star = 1$. Pick $k \geqslant 2$. Let $\mathbf{v} = \mathbf{ab}\odot \in F^\sigma(k+1)$ be an i-split with $i \leqslant k-1$. Let $\vec{g} = 1^j01^{k-j}\vec{g}_{k+1}$ with $0 \leqslant j < k$. Suppose, for all $t \in \{2, 3, \dots, k\}$ and $0 \leqslant s \leqslant t-2$, that $1^s01^{t-s-1}\vec{g}_t$ yields 1 on $F^\sigma(t)$. Now if $i \leqslant j$ then $\vec{g}\mathbf{a}^\star = \vec{1}\mathbf{a}^\star = 1$ by 3.4.1, and $\vec{g}_i\mathbf{b}^\star = 1^{j-i}01^{k-j}\vec{\mathbf{b}}^\star = 1$ by the inductive hypothesis, whence $\vec{g}\mathbf{v}^\star = 11\star = 1$. However, if $i > j$ then $\vec{g}_i\mathbf{b}^\star = 1$ by 3.1.1, whence $\vec{g}\mathbf{v}^\star = \vec{g}\mathbf{a}^\star\vec{g}_i\mathbf{b}^\star\star \in \{01\star, 11\star\} = \{1\}$. $\square$

Theorem 3.4. *The implication groupoid* 2_{13} *is completely dissociative.*

Proof. Take $T := \{0, 1\}$, and use Theorem 3.2. For Left Separation, let $L_{x,y} = \{1\}$ for all $x \neq y$; this set is yieldable by Lemma 3.1.1. For Right Separation, let $R_{x,y} = \{0\}$ for all $x \neq y$; this set is yieldable by 3.1.2.

To show Split Separation, let $1 \leqslant i < j \leqslant k$, let $\vec{g} := 1^{i-1}01^{k-i}0\vec{g}_{k+1}$, let $\mathbf{ab}\odot \in F^\sigma(k+1)$ be an i-split, and let $\mathbf{a}'\mathbf{b}'\odot \in F^\sigma(k+1)$ be a j-split. By Lemma 3.1 we get that $\vec{g}\mathbf{ab}\odot^\star = (1^{i-1}0\vec{g}_i\mathbf{a}^\star)(1^{k-i}0\vec{g}_{k+1}\mathbf{b}^\star)\star = 00\star = 1$, while on the other hand $\vec{g}\mathbf{a}'\mathbf{b}'\odot^\star = (1^{i-1}01^{j-i}\mathbf{a}^\star)(1^{k-j}0\mathbf{b}^\star)\star = 10\star = 0$. $\square$

To finish our determination of the completely dissociative 2_j, we offer Theorem 3.5. However, since there seem to be no sequences $\vec{g} \in 2^\omega$ that reliably yield sets we need in order to apply Theorem 3.2 to its proof, we sought and found in boolean algebra an alternative sort of proof.

Theorem 3.5. *The NAND groupoid* 2_{14} *is completely dissociative.*

Proof. For reference, the table of 2_{14} is given below. (Here $\star := \star_{14}$.)

$\star$	**0**	**1**
0	1	1
1	1	0

2_{14}

The binary operation $\star$ of 2_{14} is equivalent to an expression in the standard boolean algebra on $2 := \{0, 1\}$. The binary operations of this boolean algebra are *join* or *sum*, written $\vee$, and *meet* or *product*, written $\wedge$, and its unary operation is *complement*, written $'$. As is usual, we write $\vee$ and $\wedge$ as infix operations. With this notation, we have that $xy\star = x' \vee y'$ for all $\langle x, y\rangle \in 2 \times 2$, so $\star$ is what is commonly called the NAND operation.

Our proof will proceed via boolean algebra expressions that are equivalent to formal products. These expressions will be reduced to a standard form similar to disjunctive normal form. The following terminology is due mainly to W. V. Quine; viz [7] or Chapter XIV of [8]. However, our presentation will be self-contained.

Expressions will be built up out of variables; a *literal* will be either a single variable x_i or its complement x_i'. A *fundamental formula* is either a single literal or a conjunction of literals with no repeated variables. A formula Φ is *normal* if it is either fundamental or a disjunction of fundamental formulas. In the latter case, the fundamental formulas are *clauses* of Φ.

A formula Θ is said to *imply* a formula Φ iff every uniform assignment of values to the variables in the formulas that makes Θ equal to 1 also makes Φ equal to 1; we then call Θ an *implicant* of Φ. A *prime implicant* of Φ is a fundamental formula that implies Φ, but fails to do so if any of its literals is removed.

These formulas are also called "Sum of Product" or SoP forms. We focus upon a special SoP form, called the *complete sum* form. Given a formula Φ which is equivalent neither to 0 nor to 1, its complete sum form is defined to be the disjunction of its prime implicants. (A formula equivalent to 0 has no implicants; a formula equivalent to 1 has an "empty product" as its sole prime implicant. We avoid these *trivial* cases.) It is easy to recognize the implicants of a nontrivial formula, and the prime implicants are clearly identifiable. The complete sum form of a nontrivial formula is unique, up to the order of clauses and of literals within clauses.

For example, in the formula $\Phi := (x \wedge y) \vee (x \wedge y') \vee z \vee (x' \wedge y \wedge z)$, each of its four clauses $x \wedge y$, $x \wedge y'$, z and $x' \wedge y \wedge z$ are implicants of Φ, as are such fundamental formulas such as $x \wedge z$ and $x' \wedge y' \wedge z$. The clause z is a prime implicant of Φ, but $x \wedge y$ and $x \wedge y'$ are not—they can be *combined* into the fundamental formula x. The final clause $x' \wedge y \wedge z$ also fails to be a prime implicant; it is *subsumed* by z, and thus can be *deleted*. So the prime implicants of Φ are x and z, and the complete sum form of Φ is $x \vee z$. Quine attributes this process of combining and deleting clauses to Samson and Mills, and presents a proof that it always yields our complete sum form of a nontrivial formula. (Quine calls our complete sum form of a formula "the alternation of its prime implicants".) His proof is sometimes called *Quine's Theorem*; it states that a formula is in complete sum form if and only if no clauses can be combined or deleted. It could be used to simplify the proofs of Claims 2 and 3, below.

Claim 1: If $\mathbf{u} \in F^{\sigma}(k)$ then there exist $\vec{g} \in 2^{\omega}$ such that $\vec{g}\mathbf{u}^{\star} = 0$ and $\vec{r} \in 2^{\omega}$ such that $\vec{r}\mathbf{u}^{\star} = 1$. The evaluation in 2_{14} of each $\mathbf{u} \in F^{\sigma}(k)$ depends on all its variables x_i for $i \in k$.

The claim is obvious for $k = 1$. If it holds for $\{\mathbf{u}, \mathbf{v}\} \subseteq F^{\sigma}$, it holds for $\mathbf{u}\mathbf{v}\odot$. So induction establishes Claim 1, none of our $\mathbf{w} \in F^{\sigma}$ are trivial, and we can restrict our focus to the complete sum form of $\mathbf{w}$.

Claim 2: If $p = st\star \in F^{\sigma,\star}$ then the complete sum form of p is equal to the join of the complete sum forms of s' and t'.

To prove this, first note that s, t and p are nontrivial by Claim 1. So s' and t' also are nontrivial, and the complete sum forms of p, of s, of s', of t, and of t' all exist. Also, s' and t' have no variables in common.

Let r be an implicant of $p = s' \vee t'$. Then r is an implicant either of s' or of t'; for, if not, then values can be assigned to variables so that r is 1 while both s' and t' are 0, whence p will also be 0, contradicting the hypothesis that r is an implicant of p. But if r is a prime implicant of p, then r cannot be an implicant of both s' and t'; for, if so, then the removal from r of the literals of variables in t would yield a shorter implicant of s' and hence of p. Therefore the prime implicants of p are already prime implicants either of s' or of t'. Claim 2 is proved.

Consider some $p := \mathbf{u}^\star = st\star = s' \vee t'$ with $\mathbf{u}^\star \in F^{\sigma,\star}(k)$. For this p, define the binary relation ϱ on $\{x_i\colon\ i \in k\}$ by: $x_i \varrho x_j$ iff literals of x_i and of x_j appear together in some clause of the complete sum form of p. Claim 1 implies that ϱ is reflexive, and ϱ is symmetric by construction. Thus the transitive closure of ϱ is an equivalence relation E_p on $\{x_i\colon\ i \in k\}$.

Claim 3: If $p = st\star \in F^{\sigma,\star}(k)$ for any $k \geqslant 2$, then E_p has exactly two equivalence classes.

Let $p = st\star = s' \vee t'$. Claim 2 implies that E_p does not relate variables in s' with variables in t'. So it remains only to show that all of the variables in s', say, are related to each other by E_p. This is immediate if s is a single literal. So we may take it that $s = v' \vee w'$, for formal products v and w interpreted in 2_{14}. By DeMorgan's Law, $s' = v \wedge w$.

We show that the prime implicants of $v \wedge w$ are precisely the formulas of the form $m \wedge n$, where m is a prime implicant of v, and n is a prime implicant of w: Let q be an implicant of $v \wedge w$. Then q is an implicant both of v and of w. Thus q must be an implicant of some prime implicant m of v and some prime implicant n of w. So q must be an implicant of $m \wedge n$. This shows that every prime implicant of $v \wedge w$ must be some $m \wedge n$. But none of the $m \wedge n$ can imply another; for, suppose $m_0 \wedge n_0$ were an implicant of $m_1 \wedge n_1$. Then $m_0 \wedge n_0$ is an implicant of m_1. No variables in n_0 appear in m_1; so we can remove their literals, getting that m_0 implies m_1. As prime implicants of v, they are equal. Similarly $n_0 = n_1$. We infer the assertion opening this paragraph.

Now let x be a variable of v, and y a variable of w. Claim 1 implies that v depends on x. So x must appear in some prime implicant m of v. Similarly, y appears in some prime implicant n of w. Thus both x and y appear in $m \wedge n$, which is a prime implicant of s by the previous paragraph. Therefore every variable of v is related by ϱ to every variable of w. So E_p relates all variables in $s' = v \wedge w$. Claim 3 follows.

For $\{\mathbf{u}, \mathbf{v}\} \subseteq F^\sigma(k)$, it is obvious that $\mathbf{u} = \mathbf{v} \Rightarrow \mathbf{u}^\star = \mathbf{v}^\star$. The converse is obvious for $k = 1$. This is the basis step of an induction on k.

Let $k \in \mathbb{N}$; suppose for all $j \in \{1, 2, \ldots, k\}$ that $\mathbf{u}^\star = \mathbf{v}^\star \Rightarrow \mathbf{u} = \mathbf{v}$ when $\{\mathbf{u}, \mathbf{v}\} \in F^\sigma(j)$. Let $\{\mathbf{u}, \mathbf{v}\} \subseteq F^\sigma(k+1)$. Suppose that $\mathbf{u}^\star = \mathbf{v}^\star$.

As above, in Boolean language we write $\mathbf{u}^\star = p_{\mathbf{u}} = s_{\mathbf{u}} t_{\mathbf{u}} \star = s'_{\mathbf{u}} \vee t'_{\mathbf{u}}$ and $\mathbf{v}^\star = p_{\mathbf{v}} = s_{\mathbf{v}} t_{\mathbf{v}} \star = s'_{\mathbf{v}} \vee t'_{\mathbf{v}}$. From $p_{\mathbf{u}} = p_{\mathbf{v}}$ we get by Claim 3 that $s_{\mathbf{u}}$ has the same variables as $s_{\mathbf{v}}$ and that $t_{\mathbf{u}}$ has the same variables has $t_{\mathbf{v}}$. If on 2_{14} it happens both that $s_{\mathbf{u}} = s_{\mathbf{v}}$ and that $t_{\mathbf{u}} = t_{\mathbf{v}}$ then by the inductive hypothesis the corresponding factors of $\mathbf{u}$ and $\mathbf{v}$ in $\langle F^\sigma; \odot \rangle$ also are equal, and therefore $\mathbf{u} = \mathbf{v}$ as alleged.

Without loss of generality, pretend that there is an assignment of values to the variables in $s_{\mathbf{u}}$ which gives $s_{\mathbf{u}}$ the value 1 while $s_{\mathbf{v}}$ gets the value 0. Then, by Claim 1 there is an assignment of values to the variables in $t_{\mathbf{u}}$ which gives $t_{\mathbf{u}}$ the value 1. It follows for these independent value assignments to the elements in $\{x_0, x_1, \ldots, x_k\}$ that $p_{\mathbf{u}}$ gets the value $11\star = 1' \vee 1' = 0$ while $p_{\mathbf{v}}$ gets the value $0 t_{\mathbf{v}} \star = 0' \vee t'_{\mathbf{v}} = 1$, contrary to the hypothesis that $\mathbf{u}^\star = \mathbf{v}^\star$. □

Theorem 3.6. *The concrete groupoid 2_j is completely dissociative if and only if $j \in \{2, 4, 8, 11, 13, 14\}$.*

P r o o f. We write $\mathcal{A} \asymp \mathcal{B}$ iff the groupoid $\mathcal{A}$ is either anti-isomorphic or isomorphic to $\mathcal{B}$. Plainly $\asymp$ is an equivalence relation on $\underline{\mathcal{G}}(2)$. The $\asymp$ equivalence classes of the eight $2_j \in \underline{\mathcal{G}}(2)$ which are non-semigroups are: $\{2_2, 2_4, 2_{11}, 2_{13}\}$, $\{2_8, 2_{14}\}$, and $\{2_{10}, 2_{12}\}$.

Theorem 3.4 gives us that 2_{13} is completely dissociative, and 2_{14} is completely dissociative by Theorem 3.6. In 2_{10}, the value of an expression depends only on the value of its final input. Thus $wx \star y \star z \star$ and $wxyz \star \star\star$ always produce the same value. Therefore 2_{10} fails to be 4-dissociative, and consequently 2_{10} is not completely dissociative. □

Theorem 3.7. *There are at least seventeen completely dissociative $\langle 3; \star \rangle$.*

P r o o f. We show first via Theorem 3.2 that the groupoid $\mathcal{D}$, depicted below, is completely dissociative. During our argument, we will note table entries which we never use, entailing that $\mathcal{D}$ is but one of at least seventeen completely dissociative groupoids $3_j \in \underline{\mathcal{G}}(3)$.

$\star$	**0**	**1**	**2**
0	0	1	0
1	1	1	0
2	0	0	2

$\mathcal{D}$

We note parenthetically the values of $\star$ to which our argument resorts. Let $T := \{0, 1\}$. The idempotent element 0 is is yieldable. (This uses $00\star = 0$.) Since $\star$ is commutative, Left Separation is equivalent to Right Separation. Let $L_{x,y} := R_{x,y} := \{0\}$ for all $\langle x, y\rangle \in 2^2$, (using $00\star = 0$ and $01\star = 1 = 10\star$.)

For Split Separation, note that $1^p 0^q \vec{g}_{p+q} \in 3^\omega$ yields 1 when $\{p, q\} \subseteq \mathbb{N}$, since $\{0, 1\}$ forms a semilattice, (never using the value of $01\star$.) Similarly $0^p 2^q \vec{g}_{p+q}$ yields 0 since $\{0, 2\}$ forms a semilattice, (not using the value of $20\star$.) Now suppose $1 \leqslant i < j \leqslant k$. Let $\vec{g} := 1^i 0^{j-i} 2^{k-j+1} \vec{g}_{k+1}$. If $\mathbf{ab}\odot$ is an i-split then $\vec{g}\mathbf{ab}\odot^\star = (1^i \vec{g}_i \mathbf{a}^\star)(0^{j-i} 2^{k-j+1} \vec{g}_{k+1} \mathbf{b}^\star)\star = 10\star = 1$, (using cited facts and that 1 is idempotent.) If $\mathbf{a}'\mathbf{b}'\odot$ is a j-split, then $\vec{g}\mathbf{a}'\mathbf{b}'\odot^\star = (1^i 0^{j-i} \vec{g}_j \mathbf{a}'^\star)(2^{k-j+1} \vec{g}_{k+1} \mathbf{b}'^\star)\star = 12\star = 0$, (using cited facts, that 2 is idempotent, and that $12\star = 0$.) So $\mathcal{D}$ is completely dissociative.

The values of $20\star$ and $21\star$ were never used in the argument above. So, we can change $\mathcal{D}$ to make eight other completely dissociative groupoids with $\{0\} \neq \{20\star, 21\star\}$. Since $\mathcal{D}$ is abelian, we could instead have used $\vec{g} = 2^i 0^{j-i} 1^{k-j+1} \vec{g}_{k+1}$ to show Split Separation—and never have used the values of $02\star$ and $12\star$ of $\mathcal{D}$. Thus we can make eight other completely dissociative groupoids by changing those values in $\mathcal{D}$. $\square$

Most of our proofs may be analyzed in the manner above, and slightly modified to show that additional groupoids are completely dissociative.

We have seen six 2-element completely dissociative groupoids, and know that any groupoid with a completely dissociative subgroupoid is completely dissociative. So it is plausible that "most" groupoids are completely dissociative, and we ask the following.

Question 3.2. For any given n, let $CD(n)$ be the number of concrete groupoids on n that are completely dissociative. What is $CD(n)/n^{n^2}$, in the limit as n goes to infinity?

4. Primitive completely dissociative groupoids

By the *variety* $\mathbf{V}(\mathcal{G})$ generated by a groupoid $\mathcal{G}$ we mean the closure of $\{\mathcal{G}\}$ under the formation of homomorphic images, subgroupoids and product groupoids. We will show later that $\mathcal{G}$ must be completely dissociative if any groupoid in $\mathbf{V}(\mathcal{G})$ is. Thus, of special interest are the completely dissociative groupoids not forced to be completely dissociative because smaller groupoids are.

We say that a finite completely dissociative groupoid $\mathcal{P}$ is *primitive* iff no smaller groupoid in $\mathbf{V}(\mathcal{P})$ is completely dissociative.

Observe that all of the 2-element completely dissociative groupoids are primitive, since the trivial groupoid is a semigroup. We will establish the primitiveness of many other small completely dissociative groupoids.

Question 4.1. Is there a primitive completely dissociative groupoid n_j for each integer $n \geqslant 2$?

To proceed with our study of primitive completely dissociative groupoids, we will need a little material from universal algebra. For background, we refer the reader to [5], which is a good beginning text and reference.

Our principal tool will be Birkhoff's Theorem, which first appeared in [1] and is carefully developed also in [5]. Before stating it, we should first review some terminology. Everything will be stated for groupoids, although it naturally generalizes to arbitrary algebras.

By a *term* we mean an expression built up from variables using the groupoid operation symbol. Since our explanations deal only with small terms, we will use infix notation for them in this section. An *identity* is an equality between terms that is true for all relevant values of the variables. It is customary to use $\approx$ to show that terms are equal in an identity. We say that an identity *holds* in a groupoid iff it is (always) true there, and that an identity holds in a class of groupoids iff it holds in each member of the class. Alternatively, we can say that a groupoid *satisfies* an identity. The Associative Law, $x \bullet (y \bullet z) \approx (x \bullet y) \bullet z$, is an example of an identity.

A groupoid is 3-dissociative if and only if the Associative Law does not hold in it, and a groupoid is completely dissociative if and only if all of the generalizations of the Associative Law fail to hold there.

A *variety* is a class of groupoids that is closed under homomorphic images, subgroupoids and (Cartesian) products of elements in that class. If Ω is a set of identities, then the *models* of Ω are precisely the groupoids for which all of the identities in Ω hold. We can now state Birkhoff's Theorem:

Theorem 4.1. *A class of groupoids is a variety if and only if it is the class of models of a set of identities.*

We need a related result, which also is due to Birkhoff.

Theorem 4.2. *If $\mathcal{G}$ is an groupoid, then the variety $\mathbf{V}(\mathcal{G})$ generated by $\mathcal{G}$ is equal to the class of models of the set of all identities holding in $\mathcal{G}$.*

Thus, to show for a groupoid $\mathcal{H}$ that $\mathcal{H} \notin \mathbf{V}(\mathcal{G})$, it suffices to produce an identity that holds in $\mathcal{G}$ but does not hold in $\mathcal{H}$. So, if $\mathcal{G}$ is not completely dissociative, some generalized associative law is an identity of $\mathcal{G}$. By Theorem 4.2, such an identity holds in every groupoid in $\mathbf{V}(\mathcal{G})$. So $\mathcal{G}$ is completely dissociative if $\mathbf{V}(\mathcal{G})$ contains a primitive groupoid.

For finite $\mathcal{G}$, the converse implication holds. If an n_j is completely dissociative then $\mathbf{V}(n_j)$ contains a completely dissociative M of minimum size. Since $\mathbf{V}(M) \subseteq \mathbf{V}(n_j)$, any such M is perforce primitive.

Thus knowing the primitive completely dissociative groupoids could give us information about the class of all completely dissociative groupoids.

As already noted, the six concrete completely dissociative groupoids in $\underline{\mathcal{G}}(2)$ are primitive. What about the groupoid $\mathcal{D}$ treated in Theorem 3.7?

Theorem 4.3. *$\mathcal{D}$ is a primitive completely dissociative groupoid.*

Proof. We have by 3.7 that $\mathcal{D}$ is completely dissociative. Observe that $\mathcal{D}$ satisfies the Idempotent and Commutative laws. Thus every groupoid in $\mathbf{V}(\mathcal{D})$ satisfies them too. But the only 2-element groupoids where these laws hold are the semigroups 2_7 and 2_1. Thus $\mathbf{V}(\mathcal{D})$ contains no completely dissociative groupoids which are smaller than $\mathcal{D}$, and so $\mathcal{D}$ is primitive. □

We suspect that the other 16 groupoids that were proved completely dissociative in Theorem 3.7 are primitive as well.

As another example, we will outline a proof that the groupoid $\mathcal{B}$ of Theorem 3.3 is primitive. The groupoid $\mathcal{B}$ satisfies the commutative and idempotent laws. So all of the groupoids in the variety $\mathbf{V}(\mathcal{B})$ also satisfy them. To this end, we investigate the 3-element groupoids that are commutative and idempotent.

So, consider the groupoids 3_t that conform to the binary operation table(s) $CI3_\alpha$, below, with $\langle a, b, c\rangle \in 3^3$ where $3 := \{0, 1, 2\}$.

$\star$	**0**	**1**	**2**
0	0	a	b
1	a	1	c
2	b	c	2

$CI3_\alpha$

$CI3_\alpha$ is our acronym for "Commutative Idempotent 3-element groupoid number α", where $\alpha = 9a + 3b + c$.

Many of the twenty-seven $CI3_\alpha$ are isomorphic to each other under permutations of the set 3. The isomorphism classes are:

i) $CI3_0 \cong CI3_{13} \cong CI3_{26}$
ii) $CI3_1 \cong CI3_2 \cong CI3_8 \cong CI3_{10} \cong CI3_{16} \cong CI3_{17}$
iii) $CI3_3 \cong CI3_{12} \cong CI3_{18} \cong CI3_{22} \cong CI3_{23} \cong CI3_{24}$
iv) $CI3_4 \cong CI3_6 \cong CI3_9 \cong CI3_{14} \cong CI3_{20} \cong CI3_{25}$
v) $CI3_5 \cong CI3_{15} \cong CI3_{19}$
vi) $CI3_7 \cong CI3_{11}$

vii) $CI3_{21}$

The groupoid $\mathcal{D}$ from §3 is in this list, as are three new completely dissociative groupoids. We examined each isomorphism class, and will summarize our results. Details will be left to the reader. None of the completely dissociative $CI3_\alpha$ are elements of $\mathbf{V}(\mathcal{B})$, since the identity β fails in each of them, where β is:

$$((x \star y) \star z) \star z \approx ((x \star y) \star (x \star z)) \star (x \star z)$$

One may verify that β holds in $\mathcal{B}$, and hence in $\mathbf{V}(\mathcal{B})$. Where it matters, we will indicate how the identity β fails.

i) $CI3_0 \cong CI3_{13} \cong CI3_{26}$

$CI3_0$ is a semigroup.

ii) $CI3_1 \cong CI3_2 \cong CI3_8 \cong CI3_{10} \cong CI3_{16} \cong CI3_{17}$

The groupoid $CI3_1$ is a semigroup.

iii) $CI3_3 \cong CI3_{12} \cong CI3_{18} \cong CI3_{22} \cong CI3_{23} \cong CI3_{24}$

$\star$	**0**	**1**	**2**
0	0	0	1
1	0	1	0
2	1	0	2

$CI3_3$

The groupoid $CI3_3$ is completely dissociative. We verify this, using Theorem 3.2. Let $T = 3 := \{0,1,2\}$. We always set $L_{x,y} = R_{x,y} = \{2\}$, giving Left and Right Separation.

For Split Separation, let $\vec{g} = 0^i 1^{j-i} 2^{k-j+1} \vec{g}_{k+1} \in 3^\omega$. If $\mathbf{ab}\odot \in F^\sigma(k+1)$ is an i-split, one finds that $\vec{g}\mathbf{ab}\odot^\star \in \{00\star, 01\star\} = \{0\}$. If $\mathbf{a}'\mathbf{b}'\odot \in F^\sigma(k+1)$ is a j-split, then $\vec{g}\mathbf{a}'\mathbf{b}'^\star = 0 \star 2 = 1$. Therefore $CI3_3$ is completely dissociative, as alleged.

Our aim is to prove that $\mathcal{B}$ is primitive. Since $CI3_3$ is completely dissociative and smaller than $\mathcal{B}$, we wish to show that $CI3_3 \notin \mathbf{V}(\mathcal{B})$. The identity β holds in $\mathcal{B}$. Thus it suffices to show that β fails in $CI3_3$. So let $\langle x,y,z\rangle := \langle 0,2,1\rangle$, and observe that then $((x \star y) \star z) \star z = ((0 \star 2) \star 1) \star 1 = 1 \neq 0 = ((0 \star 2) \star (0 \star 1)) \star (0 \star 1) = ((x \star y) \star (x \star z)) \star (x \star z)$, as desired.

iv) $CI3_4 \cong CI3_6 \cong CI3_9 \cong CI3_{14} \cong CI3_{20} \cong CI3_{25}$

Since $CI3_9 = \mathcal{D}$, which was proven in Theorem 3.7 to be completely dissociative, it remains to show that β fails in $\mathcal{D}$. Letting $\langle x,y,z\rangle = \langle 1,2,0\rangle$, accomplishes this.

v) $CI3_5 \cong CI3_{15} \cong CI3_{19}$

The groupoid $CI3_5$ is completely dissociative. Let $T = 3$, and let $L_{x,y} = R_{x,y} = \{1\}$ for all $\langle x,y\rangle \in 3^2$.

If $\mathbf{ab}\odot$ and $\mathbf{a}'\mathbf{b}'\odot$ are an i-split and a j-split, respectively, we let $\vec{g} := 01^{j-2}2^{k-j+2}$ $\vec{g}_{k+1}$. One obtains that $\vec{g}\mathbf{ab}\odot^\star = 0 \star 2 = 1$, while $\vec{g}\mathbf{a}'\mathbf{b}'\odot^\star = (0 \star 2) \star 2 = 2$, showing Split Separation.

Letting $\langle x, y, z\rangle = \langle 2, 0, 1\rangle$ shows that the identity β fails in $CI3_5$.

vi) $CI3_7 \cong CI3_{11}$

$CI3_7$ is completely dissociative. Let $T := \{0, 2\}$, and take $L_{x,y} = R_{x,y} = \{0\}$ for all $x \neq y$ in T.

To show Split Separation, let $\vec{g} := 0^i 1^{j-i} 2^{k-j+1}$. If $\mathbf{ab}\odot$ is an i-split and $\mathbf{a}'\mathbf{b}'\odot$ a j-split, then $\vec{g}\mathbf{ab}\odot^\star = 0 \star 1 = 0$ and $\vec{g}\mathbf{a}'\mathbf{b}'\odot^\star = 0 \star 2 = 2$.

Letting $\langle x, y, z\rangle := \langle 2, 1, 0\rangle$ shows β fails in $CI3_7$.

vii) $CI3_{21}$

This algebra is not completly dissociative. In the 3-element field, one has that $x \star y$ is equal to $2(x + y)$, so the generalized associative law $((v \star w) \star (x \star y)) \star z \approx v \star ((w \star x) \star (y \star z))$ holds.

Theorem 4.4. *Groupoids which are isomorphic to $CI3_3$, to $CI3_4$, to $CI3_5$, or to $CI3_7$, are primitive completely dissociative groupoids.*

P r o o f. The argument is identical to that in Theorem 4.3. $\square$

Theorem 4.5. *$\mathcal{B}$ is a primitive completely dissociative groupoid.*

P r o o f. The groupoid $\mathcal{B}$, of Theorem 4.3, satisfies the Idempotent and Commutative laws. Thus every groupoid in $\mathbf{V}(\mathcal{B})$ satisfies them too. But the only 2-element groupoids where these laws hold are the semilattices, 2_7 and 2_1, both of which are semigroups.

The idempotent commutative 3-element groupoids were studied above. The isomorphism classes of those which are completely dissociative—those represented by $CI3_3$, by $CI3_4$, by $CI3_5$, and by $CI3_7$—have no elements in common with $\mathbf{V}(\mathcal{B})$, since the identity β holds in none of them, but does hold in $\mathcal{B}$.

We have established that there exist no completely dissociative groupoids in $\mathbf{V}(\mathcal{B})$ that are smaller than $\mathcal{B}$. Therefore $\mathcal{B}$ is primitive. $\square$

P r o b l e m 4.1. Characterize primitive completely dissociative groupoids.

5. Making k-ary functions by composing binary functions

In this section we show that some functions $\varphi\colon n^k \to n$ are unrepresentable as any $\mathbf{u}^{\vec{\beta}}$. The simplest situation, where $n = 2$ and $k = 3$, is the more demanding.

Lemma 5.1. *There exists a 3-ary operation $\varphi\colon\ 2^3 \to 2$ such that for no ordered pair $\vec{\beta} := \langle \beta_0, \beta_1 \rangle$ of binary operations $\beta_i\colon\ 2^2 \to 2$ is the function $x_0x_1x_2\varphi$ equal to either $x_0x_1\beta_0x_2\beta_1$ or $x_0x_1x_2\beta_0\beta_1$.*

P r o o f. Define $\varphi\colon\ 2^3 \to 2$ by $000\varphi = 010\varphi = 011\varphi = 110\varphi = 111\varphi = 0$, and $001\varphi = 100\varphi = 101\varphi = 1$.

The argument consists of four main cases. Two of the cases show that $x_0x_1x_2\varphi \neq x_0x_1\beta_0x_2\beta_1$ while the other two show that $x_0x_1x_2\varphi \neq x_0x_1x_2\beta_0\beta_1$.

We detail only one case; it will suffice to reveal the nature of our argument.

C a s e: $00\beta_0 := 1$ and $\mathbf{v}^{\langle \beta_0, \beta_1 \rangle} := x_0x_1\beta_0x_2\beta_1$.

We show that there is no pair $\langle \beta_0, \beta_1 \rangle$ of binary operations on 2 for which $\varphi = \mathbf{v}^{\langle \beta_0, \beta_1 \rangle}$. This involves our proceeding step by step through the construction, of the functions β_0 and β_1, which is mandated by the φ specified above and the initial condition $00\beta_0 := 1$, until we ram into a wall.

From $10\beta_1 =: 00\beta_0 0\beta_1 = 000\varphi := 0$, we infer that $10\beta_1 = 0$. Also, $11\beta_1 = 00\beta_0 1\beta_1 = 001\varphi := 1$, and so $11\beta_1 = 1$.

$01\beta_0 0\beta_1 = 010\varphi := 0$ provides two possibilities: $01\beta_0 = 0$ or $01\beta_0 = 1$. If $01\beta_0 = 1$ then $11\beta_1 = 01\beta_0 1\beta_1 = 011\varphi := 0$, contrary to our prior observation that $11\beta_1 = 1$. Therefore $01\beta_0 = 0$.

Next, $01\beta_1 = 01\beta_0 1\beta_1 = 011\varphi := 0$ whence $01\beta_1 = 0$. By $10\beta_0 0\beta_1 = 100\varphi := 1$ we are again offered two possibilities: $10\beta_0 = 0$ or $10\beta_0 = 1$. But if $10\beta_0 = 1$ then $10\beta_1 = 10\beta_0 0\beta_1 = 100\varphi := 1$, contrary to our earlier inference that $10\beta_1 = 0$. Therefore, $10\beta_0 = 0$.

Finally, $0 = 01\beta_1 = 10\beta_0 1\beta_1 = 101\varphi := 1$, and we hit the wall.

We omit the similar second case, which shows that $x_0x_1x_2\varphi \neq x_0x_1\beta_0x_2\beta_1$ when $00\beta_0 = 0$. Likewise $x_0x_1x_1\varphi = x_0x_1x_2\beta_0\beta_1$ is impossible. $\square$

We used a case-ridden argument to prove Lemma 5.1 because there are twice as many formal 3-ary products interpreted by some duple of binary operations $2^2 \to 2$ as there are 3-ary operations on the set $2 := \{0, 1\}$. However, when either $n \geqslant 3$ or $k \geqslant 4$, a straightforward counting argument enables us easily to show that the result established for $\langle n, k \rangle = \langle 2, 3 \rangle$ extends to every pair $\langle n, k \rangle$ of integers with $n \geqslant 2$ and $k \geqslant 3$.

Theorem 5.1. *For $n \geqslant 2$ and $k \geqslant 3$ integers, there exists a k-ary operation $\varphi\colon\ n^k \to n$ such that $\varphi \neq \mathbf{u}^{\vec{\beta}}$ for every $\mathbf{u} \in F^{\sigma}(k)$ and for every $(k-1)$-tuple $\vec{\beta} := \beta_0\beta_1 \ldots \beta_{k-2}$ of binary operations $\beta_i\colon\ n^2 \to n$.*

P r o o f. Since Lemma 5.1 establishes our claim for the case $\langle n, k \rangle = \langle 2, 3 \rangle$, we may take it that either $n \geqslant 3$ or $k \geqslant 4$.

From [6] or [9] we have that $|F^{\sigma,\vec{\beta}}(k)| = C_{k-1}$ for every $(k-1)$-tuple $\vec{\beta} := \beta_0\beta_1 \ldots \beta_{k-2}$ of binary operations $\beta_j\colon n^2 \to n$, where

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad \text{is the } n\text{-th Catalan number.}$$

Since there are $n^{n^2(k-1)}$ such $\vec{\beta}$, it follows that the number $\Phi(n,k)$ of formal k-ary products interpreted by some such $\vec{\beta}$ is

$$\Phi(n,k) = n^{n^2(k-1)} C_{k-1} = n^{n^2(k-1)} \frac{(2k-2)!}{(k-1)!k!}.$$

Thus the ratio $R(n,k)$ of the number n^{n^k} of distinct k-ary operations on n to the number of distinct interpreted formal k-products is

$$R(n,k) = \frac{n^{n^k}}{\Phi(n,k)} = n^{n^k - n^2(k-1)} \frac{(k-1)!k!}{(2k-2)!}.$$

Notice that $R(n,k) > 1$ for every pair $\langle n,k\rangle$ of integers such that either $n \geqslant 3$ while $k \geqslant 3$ or $n \geqslant 2$ while $k \geqslant 4$. $\square$

It is reasonable to wonder whether enlarging our tool kit of building-block operations on n enables the construction of all operations of given arities larger than the arities of permitted building blocks. In this light we ask

Question 5.1. For each $r \in \{3,4,5,\ldots\}$, is there an $n(r) \in \mathbb{N}$ such that, for each pair $\langle m,k\rangle$ of integers with $m \geqslant n(r)$ and $k \geqslant r$, there is some k-ary operation $\varphi\colon m^k \to m$ which it is impossible to "build" using a natural formal product construction generalizing F^σ by allowing j-ary operations on m with $j \in \{2,3,\ldots,r-1\}$ instead of using only binary operations?

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