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A ROLE OF THE COEFFICIENT OF THE DIFFERENTIAL TERM IN QUALITATIVE THEORY OF HALF-LINEAR EQUATIONS

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Abstract. The aim of this contribution is to study the role of the coefficient r in the qualitative theory of the equation $(r(t)\Phi(y^{\Delta}))^{\Delta}+p(t)\Phi(y^{\sigma})=0$, where $\Phi(u)=|u|^{\alpha-1}\operatorname{sgn} u$ with $\alpha>1$. We discuss sign and smoothness conditions posed on r, (non)availability of some transformations, and mainly we show how the behavior of r, along with the behavior of the graininess of the time scale, affect some comparison results and (non)oscillation criteria. At the same time we provide a survey of recent results acquired by sophisticated modifications of the Riccati type technique, which are supplemented by some new observations.

Keywords: half-linear dynamic equation, time scale, transformation, comparison theorem, oscillation criteria

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1. Introduction

Consider the half-linear dynamic equation

$$(r(t)\Phi(y^{\Delta}))^{\Delta}+p(t)\Phi(y^{\sigma})=0, \tag{1.1}$$

where $\Phi(u)=|u|^{\alpha-1}\operatorname{sgn} u$ with $\alpha>1$, $1/r(t)$ and $p(t)$ are rd-continuous functions defined on a time scale interval $[a,\infty)=\{t\in\mathbb{T}\colon t\geqslant a\}$, $\mathbb{T}$ being a time scale. If $\mathbb{T}=\mathbb{R}$ and $\alpha=2$, then (1.1) reduces to the well-known Sturm-Liouville linear differential equation

$$(r(t)y')'+p(t)y=0, \tag{1.2}$$

where $r>0$ and p are continuous.

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We assume that the reader is familiar with the notion of time scales. Thus, note just that σ, f^σ, μ, f^Δ, $\int_a^b f(s)\,\Delta s$, and $C_{\rm rd}$ stand for the forward jump operator, $f\circ\sigma$, the graininess, the delta derivative of f, the delta integral of f from a to b, and the class of rd-continuous functions, respectively. Recall that, for instance, $f^\Delta(t)=f'(t)$ when $\mathbb{T}=\mathbb{R}$, $f^\Delta(t)=\Delta f(t)$ when $\mathbb{T}=\mathbb{Z}$, and $f^\Delta(t)=D_qf(t)$ when $\mathbb{T}=\{q^k\colon k\in\mathbb{N}_0\}$ with $q>1$, where D_q denotes the Jackson derivative. See [7], which is the initiating paper of the time scale theory written by Hilger, and the monograph [3] by Bohner and Peterson containing a lot of information on time scale calculus. Time scale intervals will be denoted as the usual real intervals, and from the context it will always be clear whether the interval under consideration is real or of time scale type.

Basic qualitative properties of equation (1.1) can be found e.g. in [2], [10]; some of them are recalled also below. For a survey on oscillation of (1.2) see [17] and also [16]. The book [6] presents a comprehensive treatment of the qualitative theory of half-linear differential equations.

The paper contains the following sections: First we deal with sign and smoothness conditions which are/have to be usually posed on r. In Section 3 we discuss (non)availability of transformations which are related to equation (1.1). One of the most important transformations, namely the Riccati type one, is described in details in Section 4. The role of the coefficient r in the oscillation theory of (1.1), which is played there along with the graininess of the time scale, is demonstrated on two types of results: Power type comparison theorems in Section 5 and Hille-Nehari type (non)oscillation criteria in Section 6. The paper is concluded by summarizing the observations which follow from our results.

2. Sign and smoothness conditions on r

Even though it is not the main objective of this paper, we first focus on a sign condition on r. As shown in [10], the mere condition $r(t)\neq 0$ (along with a sufficient smoothness, which reduces to the usual continuity in the case $\mathbb{T}=\mathbb{R}$) guarantees validity of classical results for (1.1), like the existence and uniqueness of IVP or Sturmian type theorems; see also, e.g., [1] for the linear case. Notice however that the concept of generalized zero has to be adjusted in the following way, when r is allowed to change its sign: A solution y of (1.1) has a *generalized zero at* t if $y(t)=0$. A solution y of (1.1) has a *generalized zero in* $(t,\sigma(t))$ if $r(t)y(t)y^\sigma(t)<0$; sometimes, the interval $[t,\sigma(t)]$ is considered instead of $(t,\sigma(t))$. Why does r occur in the definition of a generalized zero? For simplicity, let us work here with linear difference equations. First consider the Fibonacci recurrence relation $y(t+2)-y(t+1)-y(t)=0$. This equation has two linearly independent solutions $u(t)=\big((1+\sqrt{5})/2\big)^t$ and

$v(t) = \left((1-\sqrt{5})/2\right)^t$. We see that u is positive while v changes its sign at every $(t, t+1)$. Since this relation is in fact a second order linear difference equation, it makes sense to ask: Is a Sturm type separation theorem violated? Let us write the Fibonacci relation in the form (1.1), i.e., $\Delta((-1)^{t+1}\Delta y(t)) + (-1)^{t+1} y(t+1) = 0$; then we get, in particular, $r(t) \not> 0$. With the new definition of generalized zeros we see that both the solutions oscillate, the equation is oscillatory, and the separation result works well. We also see that without allowing r to change its sign, important examples of second order equations would not be included in the general theory. But we still do not know where from r comes into the definition of a generalized zero. A second order linear difference equation $\Delta(r(t)\Delta y(t)) + p(t) \times y(t+1) = 0$ can be viewed as an Euler-Lagrange equation associated with the quadratic functional $\mathcal{F}(\xi) = \sum_{t=a}^{b-1} [r(t)(\Delta\xi(t))^2 - p(t)(\xi(t+1))^2]$. Under certain assumptions, by means of the Picone type identity, we can write the functional in the form $\mathcal{F}(\xi) = A + \sum_{t=a}^{b-1} B^2(t)C(t)$, where $A = 0$ provided $\xi(a) = 0 = \xi(b)$, $B^2(t)$ is clearly nonnegative, and $\operatorname{sgn} C(t) = \operatorname{sgn}(r(t)y(t)y(t+1))$, with y being a solution of the equation. Now we can see that if we want, as usual, the disconjugacy of the equation to imply the positivity of the associated functional over the set of admissible sequences, we have to assume $r(t)y(t)y(t+1) > 0$ to be not a generalized zero of y. Similar arguments work also in the general case for equation (1.1). Recall that a nontrivial *solution* y of (1.1) (and thus also the *equation* itself, in view of the separation result) is called *oscillatory* if it has infinitely many (isolated) zeros. Otherwise, a *solution/equation* is said to be *nonoscillatory*. Finally, note that two nonproportional solutions of (1.1) cannot have a common zero, but may have a common generalized zero, and this may happen also when $r(t) > 0$.

Next we will discuss the smoothness condition on r. We consider positive r's, if not said otherwise. One may wonder why we assume $1/r \in C_{\mathrm{rd}}$ and not $r \in C_{\mathrm{rd}}$ as is usual. There are at least two reasons. First, since we want to assume conditions in terms of $\int r^{1/(1-\alpha)}(s)\Delta s$, we need an integrability of $r^{1/(1-\alpha)}$ to be guaranteed. Note that $r \in C_{\mathrm{rd}}$ does not imply $1/r \in C_{\mathrm{rd}}$ in contrast to the usual continuity. Indeed, for $r \in C_{\mathrm{rd}}$, at a left-dense $t_0 \in \mathbb{T}$, it may happen that $\lim_{t \to t_0-} r(t) = 0$ and $r(t_0) > 0$ (the author thanks R. Šimon Hilscher for drawing his attention to such possible behavior). The second reason is again related to the just described behavior, but seems to be more serious. It goes back even to the basic theory of linear "formally self-adjoint" dynamic equations of the form $(r(t)y^\Delta)^\Delta + p(t)y^\sigma = 0$, see e.g. [3], which are usually considered under the assumptions $r, p \in C_{\mathrm{rd}}$ with $r \neq 0$ or $r > 0$. To show solvability of such an equation we rewrite it as a first order system, and then we use the existence theory for systems which utilizes, in particular, rd-continuity of the right hand side.

But, in our particular case, one of the system coefficients has the form $1/r$, which need not be rd-continuous. Thus, not $r \in C_{\mathrm{rd}}$, but $1/r \in C_{\mathrm{rd}}$ has to be naturally required (no matter if $r > 0$ or $r \neq 0$ is assumed). The reader may see a parallel with the usual differential equations case, where the assumptions of continuity of r and p is relaxed to the local Lebesgue integrability of $1/r$ and p. A similar observation holds also for half-linear and some other similar second order dynamic equations. In fact, this reasoning shows that the assumption $r \in C_{\mathrm{rd}}$ should be corrected to $1/r \in C_{\mathrm{rd}}$ in dozens of existing works (including the author's ones) which deal with such types of second order dynamic equations. See also [8], where the condition $\inf_{t\in[a,b]} |r(t)| > 0$ for all $b \in [a, \infty)$ was introduced when deriving existence results for the equation $(r(t)y^{\Delta})^{\Delta} + f(t, x^{\sigma}) = 0$. This condition, under the assumption $r \in C_{\mathrm{rd}}$, is actually equivalent to $1/r \in C_{\mathrm{rd}}$. Finally, note that by a solution of (1.1) we mean a function y such that y and $r\Phi(y^{\Delta})$ are rd-continuously delta differentiable, and y satisfies (1.1).

3. Transformations

It is known that most of the results concerning oscillation or asymptotic behavior of (1.2) can be derived from those for the simpler equation $y'' + q(t)y = 0$, through suitable transformations (see, for instance, [17]). However, the transformations which are needed either use a chain rule for computation of derivatives of composed functions, a tool which is not at disposal in a proper form in time scale calculus, or are heavily dependent on the linearity of the solution space, and thus do not apply to the half-linear case. These transformations can be used also in more general cases than (1.2), but only partially. Indeed, the transformation of the independent variable $s = \int_a^t r^{1/(1-\alpha)}(s)\,\mathrm{d}s$, $x(s) = y(t)$ transforms the *differential* equation $(r(t)\Phi(y'))' + p(t)\Phi(y) = 0$ into the equation $(\mathrm{d}/\mathrm{d}s)(\Phi(\mathrm{d}x/\mathrm{d}s)) + \tilde{p}(s)\Phi(x) = 0$. Moreover, if $\int^{\infty} r^{1/(1-\alpha)}(s)\,\mathrm{d}s = \infty$, then an unbounded interval $[a, \infty)$ is transformed into the interval $[0, \infty)$, which is of the same form as $[a, \infty)$. Such a transformation however does not work in a general time scale case. Further, with the assumption $\int^{\infty} r^{-1}(s)\,\Delta s < \infty$, the change of the dependent variable $y(t) = u(t)\int_t^{\infty} r^{-1}(s)\,\Delta s$ transforms the *linear* dynamic equation $(r(t)y^{\Delta})^{\Delta} + p(t)y^{\sigma} = 0$ into the equation $(\tilde{r}(t)u^{\Delta})^{\Delta} + \tilde{p}(t)u^{\sigma} = 0$, where $\int^{\infty} \tilde{r}^{-1}(s)\,\Delta s = \infty$. Such a transformation however does not work in the general half-linear case. Another tool useful in studying properties of solutions of (1.2) is the reciprocity principle, a transformation which somehow interchanges the role of the coefficients in (1.2): The substitution $u = ry'$ yields $((1/p(t))u')' + (1/r(t))u = 0$. This transformation extends to the half-linear case, however in a time scale case it requires commutativity of the delta-derivative and the jump operator. This property of commutativity holds only for some special (having a constant graininess) time scales and, moreover, does not always give the desired type of result even in the continuous case. All these observations show that in the time scale setting, equation (1.1) is a significant generalization of the equation $(\Phi(y^\Delta))^\Delta + p(t)\Phi(y^\sigma) = 0$, and in studying nonlinear equation (1.1) with $r(t) > 0$ we have to distinguish the cases $\int^\infty r^{1/(1-\alpha)}(s)\,\Delta s = \infty$ and $\int^\infty r^{1/(1-\alpha)}(s)\,\Delta s < \infty$. Moreover, the approaches in the individual cases of divergence or convergence of the delta integral of $r^{1/(1-\alpha)}$ can often substantially differ.

Another advantage of considering a general coefficient r in (1.1) lies in the fact that the equation with a damped term $(a(t)\Phi(y^\Delta))^\Delta + b(t)\Phi(y^\Delta) + c(t)\Phi(y^\sigma) = 0$ can be written as (1.1); this can be done via multiplying the damped equation by a suitable expression.

From the linear continuous theory at least two useful transformations are known, namely the *Prüfer* one and the *Riccati* one, see e.g. [6], [17]. The Prüfer transformation is based on expressing a solution and its quasiderivative in polar coordinates. While in the continuous half-linear case this was proved to be very useful in many situations, see [6], this does not seem to be the case for general equation (1.1). In [4], a Prüfer transformation was introduced for linear dynamic equations (and even for symplectic dynamic systems) and some applications were presented. An extension of this transformation to equation (1.1) has not been introduced yet, and there are serious reasons to doubt about its possible wide applicability. The Riccati transformation is however a different case: It has been shown to be an extremely useful tool also in general situations related to ours. Since it plays a crucial role in the proofs of our main results, it will be described with more details in the next section.

4. Riccati transformation and ramifications

The basic relation between equation (1.1) and an associated Riccati type relation along with its modifications are summarized in the next theorem. We point out that the basic statements do not require any additional conditions, while their improvements need certain assumptions; in particular, we distinguish the divergence and convergence of $\int^\infty r^{1/(1-\alpha)}(s)\,\Delta s$, where $r(t) > 0$. We introduce the notation $\mathcal{S}(w,r) = \lim_{\lambda\to\mu} w\lambda^{-1}(1 - r\Phi(\Phi^{-1}(r) + \lambda\Phi^{-1}(w))^{-1})$, where β stands for the conjugate number of α, i.e., $1/\alpha + 1/\beta = 1$, Φ^{-1} stands for the inverse of Φ, $R_D(t) = R_D(t,a) = \int_a^t r^{1-\beta}(s)\,\Delta s$, and $R_C(t) = \int_t^\infty r^{1-\beta}(s)\,\Delta s$.

Theorem 4.1 ([9], [10], [11]). *The following statements are equivalent:*

(i) *Equation* (1.1) *is nonoscillatory.*

(ii) *The inequality* $w^{\Delta}(t)+p(t)+\mathcal{S}(w,r)(t)\leqslant 0$ *(the statement holds also when the inequality is replaced by equality) has a solution* w *such that* $\Phi^{-1}(r(t))+\mu(t)\Phi^{-1}(w(t))>0$ *for large* t.

(iii) *The inequality* $w(t)\geqslant\int_t^{\infty}p(s)\,\Delta s+\int_t^{\infty}\mathcal{S}(w,r)(s)\,\Delta s$ *(the statement holds also when the inequality is replaced by equality) has a (positive) solution* w *for large* t *provided* $r(t)>0$, $\int^{\infty}r^{1-\beta}(s)\,\Delta s=\infty$, *and* $\int_t^{\infty}p(s)\,\Delta s\geqslant 0$ $(\not\equiv 0)$ *for large* t. *If, in addition,* $p(t)\geqslant 0$, *then* $w(t)\leqslant R_D^{1-\alpha}(t,t_0)$ *for large* t, *say* $t>t_0$.

(iv) *The inequality* $R_C^{\alpha}(t)w(t)\geqslant\int_t^{\infty}p(s)(R_C^{\sigma}(s))^{\alpha}\Delta s+\int_t^{\infty}\mathcal{S}(w,r)(s)(R_C^{\sigma}(s))^{\alpha}\Delta s-\int_t^{\infty}w(s)(R_C^{\alpha}(s))^{\Delta}\Delta s$ *(the statement holds also when the inequality is replaced by equality) has a solution* w *such that* $w(t)\geqslant -R_C^{1-\alpha}(t)$ *for large* t *and* $\int^{\infty}p(s)(R_C^{\sigma}(s))^{\alpha}\Delta s$ *converges provided* $r(t)>0$, $\int^{\infty}r^{1-\beta}(s)\,\Delta s<\infty$, *and* $p(t)\geqslant 0$.

(v) *There exists* $t_0\in[a,\infty)$ *such that* $\lim_{k\to\infty}\varphi_k(t)=\varphi(t)$ *for* $t\geqslant t_0$ *provided* $r(t)>0$, $\int^{\infty}r^{1-\beta}(s)\,\Delta s=\infty$, *and* $\int_t^{\infty}p(s)\,\Delta s\geqslant 0$ $(\not\equiv 0)$ *for large* t, *where* $\varphi_0(t)=\int_t^{\infty}p(s)\,\Delta s$ *and* $\varphi_k(t)=\varphi_0(t)+\int_t^{\infty}\mathcal{S}(\varphi_{k-1},r)(s)\,\Delta s$, $k=1,2,\ldots$.

(vi) *There exists* $t_0\in[a,\infty)$ *such that* $\lim_{k\to\infty}\psi_k(t)=\psi(t)$ *for* $t\geqslant t_0$ *provided* $r(t)>0$, $\int^{\infty}r^{1-\beta}(s)\,\Delta s<\infty$, *and* $p(t)\geqslant 0$, *where* $\psi_0(t)=-R_C^{1-\alpha}(t)$ *and* $\psi_k(t)=R_C^{-\alpha}(t)\int_t^{\infty}p(s)(R_C^{\sigma}(s))^{\alpha}\Delta s+R_C^{-\alpha}(t)\int_t^{\infty}[-\psi_{k-1}(s)(R_C^{\alpha}(s))^{\Delta}+\mathcal{S}(\psi_{k-1},r)(s)(R_C^{\sigma}(s))^{\alpha}]\,\Delta s$, $k=1,2,\ldots$.

In some of the above statements, a sign condition posed on p can be relaxed, but our aim is different. A particularly important role in the statements is played by the mapping $\mathcal{S}$, which occurs in the Riccati type relations: In spite of its complexity, it proves nice properties (e.g. monotonicity) known from the special cases (linear or continuous). On the other hand, different forms of $\mathcal{S}$ in different settings cause the discrepancies between, for example, the discrete and the continuous theory. Finally, notice that sometimes it is useful to understand $\mathcal{S}$ also as a function of the nonlinearity, i.e., $\mathcal{S}=\mathcal{S}(w,r,\alpha)$, see the next section.

5. Comparison theorems

Along with (1.1) consider the equation

$$(r(t)\Phi_{\delta}(y^{\Delta}))^{\Delta}+p(t)\Phi_{\delta}(y^{\sigma})=0, \tag{5.1}$$

where $\Phi_{\delta}(u)=|u|^{\delta-1}\operatorname{sgn}u$ with $\delta>1$. First we present a basic comparison statement, where the cases of the divergence and of the convergence of $\int^{\infty}r^{1-\beta}(s)\,\Delta s$ are distinguished. Important comments, particularly those related to the role of the coefficient r, and improvements are given immediately, in Remark 5.1.

Theorem 5.1 [14]. *Let either*

(a) $r(t) > 0$, $\int^{\infty} r^{1-\beta}(s)\,\Delta s = \infty$, $\int_t^{\infty} p(s)\,\Delta s \geqslant 0$ ($\not\equiv 0$) *for large* t, *and*

$$\liminf_{t\to\infty} r(t) > 0, \tag{5.2}$$

or

(b) $r(t) > 0$, $\int^{\infty} r^{1-\beta}(s)\,\Delta s < \infty$, $p(t) \geqslant 0$ *for large* t, *and*

$$r^{1-\beta}(t)/R_C(t) \leqslant 1 \quad \text{for large } t. \tag{5.3}$$

If $\delta \leqslant \alpha$ *and* (1.1) *is nonoscillatory, then* (5.1) *is nonoscillatory.*

Remark 5.1. (i) If $\mu(t) \geqslant 1$ for large t, then all additional conditions in (a) and in (b) can be omitted, and the following simply holds: If $\delta \leqslant \alpha$ and (1.1) is nonoscillatory, then (5.1) is nonoscillatory.

(ii) Assume that $\mu(t) \not\geqslant 1$ for large t. If $p(t) \geqslant 0$ in (a), then (5.2) may be replaced by the weaker condition

$$\limsup_{t\to\infty} r^{1-\beta}(t)/R_D(t) < z_0^{(\delta-1)(\beta-1)}, \tag{5.4}$$

where z_0 is the positive root of $f_h(z) := \lim\limits_{\lambda\to h} [(1+\lambda z)\ln(1+\lambda z)\ln z]/\lambda = 0$ with $h \in [0,1)$ such that $h \leqslant \mu(t)$ for large t. Similarly, in (b), condition (5.3) can be relaxed to

$$r^{1-\beta}(t)/R_C(t) \leqslant \tilde{z}_0^{(\delta-1)(\beta-1)} \tag{5.5}$$

for large t, where $\tilde{z}_0$ is the positive root of $\tilde{f}_h(z) := \lim\limits_{\lambda\to h} [(\lambda z - 1)\ln(1-\lambda z) - \lambda z \ln z]/\lambda = 0$ with $\mu(t) \leqslant h < 1$.

(iii) There is an interesting question whether additional conditions like (5.4) or (5.5) can be omitted when $\mu(t) \not\geqslant 1$. In general, the answer is no. Moreover, in the continuous case, optimality of those conditions has been shown, see [14]: If $\mathbb{T} = \mathbb{R}$, then $z_0 = \tilde{z}_0 = \mathrm{e}$, where e is the Euler number, and this e is the best possible in (5.4) and (5.5), i.e., it cannot be increased. We conjecture that a similar optimality result can be shown also on $\mathbb{T} = h\mathbb{Z}$ with $h \in (0,1)$ where the below presented Hille-Nehari type criteria will find an application.

(iv) This remark concerns the case $\mathbb{T} = \mathbb{R}$. If $\int^{\infty} r^{1-\beta}(s)\,\mathrm{d}s = \infty$, then the differential equation (1.1) can be transformed by means of a transformation of the independent variable into an equation of the same form, but with the coefficient in the differential term being identically equal to 1, see Section 3. Hence, additional conditions, like (5.2) or (5.3) or (5.4) or (5.5) are trivially fulfilled. But we have to pay for it by the following undesired property: The coefficient in the second term of the resulting equation becomes dependent on α. Nevertheless, even in such a case we cannot infer that a statement equivalent to our one might be obtained, by using a different method.

(v) The proof of the theorem is based on the relations between (i)–(iv) of Theorem 4.1, see [14]. The main idea lies in the fact that the only term of the associated Riccati type equation $w^\Delta + p(t) + \mathcal{S}(w, r, \alpha) = 0$ depending on the nonlinearity is $\mathcal{S}$, and, moreover, the function $x \mapsto \mathcal{S}(w, r, x)$ is increasing under certain additional conditions. The need of those additional conditions then causes the occurrence of conditions like (5.2), (5.3), (5.4), and (5.5). To be more precise, the desired monotonicity property of $\mathcal{S}$ is guaranteed by the nonnegativity of f_h when $w > 0$, or of $\tilde{f}_h$ when $w < 0$. The optimality of the constant e mentioned in the item (iii) of this remark is shown by a suitable example involving a generalized Euler type differential equation.

(vi) In the case $\int^\infty r^{1-\beta}(s)\,\Delta s < \infty$, a different approach, based on the reciprocity principle, can be used. But the Riccati type transformation is used as well, in a way similar to that in the proof of Theorem 5.1 (a). Notice that the "key" condition $\delta \geqslant \alpha$ is the opposite in comparison with that in Theorem 5.1. However, the resulting equation is different from (5.1). Moreover, because of the method, the theorem holds only on time scales with a constant graininess (which guarantees commutativity of the σ-operator and the Δ-operator). The statement reads as follows ([14]): Let $\mu(t) \equiv h \geqslant 0$ and $p(t) > 0$ for large t. Assume that $r(t) > 0$, $\int^\infty r^{1-\beta}(s)\,\Delta s < \infty$, and $\int^\infty p(s)\,\Delta s = \infty$. If $\delta \geqslant \alpha$ and (1.1) is nonoscillatory, then the equation $\left((r^\sigma(t))^{(1-\beta)(1-\delta)}\Phi_\delta(x^\Delta)\right)^\Delta + p^\sigma(t)\Phi_\delta(x^\sigma) = 0$ is nonoscillatory. Note that the condition $\int^\infty p(s)\,\Delta s = \infty$ is no restriction, since the convergence of this integral implies the existence of a nonoscillatory solution to the latter equation as can be shown by means of the Schauder fixed point theorem, see [9].

6. (Non)oscillation criteria

First we introduce the notation, where the subscript D corresponds to the divergence of $\int^\infty r^{1-\beta}(s)\,\Delta s$, while C corresponds to its convergence:

$$\begin{aligned}\mathfrak{R}_i(\lambda)(t) &:= \lambda(t) r^{1-\beta}(t)/R_i(t)\\ M_*(i) &:= \liminf_{t\to\infty} \mathfrak{R}_i(\mu)(t), \quad M^*(i) := \limsup_{t\to\infty} \mathfrak{R}_i(\mu)(t),\end{aligned}$$

where $i \in \{D, C\}$,

$$\gamma_i(x) := \begin{cases} \lim\limits_{t\to x} \left(\dfrac{(t+1)^{\frac{\alpha-1}{\alpha}} - 1}{t}\right)^{\alpha} \dfrac{t}{(t+1)^{\alpha-1} - 1}, & x \in [0,\infty) \cup \{\infty\}, \text{ for } i = D, \\ \lim\limits_{t\to x} \left(\dfrac{1 - (1-t)^{\frac{\alpha-1}{\alpha}}}{t}\right)^{\alpha} (1-t), & x \in [0,1], \text{ for } i = C, \end{cases}$$

$$\mathcal{A}_i(t) := \begin{cases} R_D^{\alpha-1}(t) \displaystyle\int_t^{\infty} p(s)\,\Delta s & \text{for } i = D, \\ R_C^{-1}(t) \displaystyle\int_t^{\infty} (R_C^{\sigma}(s))^{\alpha} p(s)\,\Delta s & \text{for } i = C. \end{cases}$$

Next we give criteria which extend the classical Hille-Nehari results with the critical oscillation constant $1/4$ (see, e.g., [16], [17]).

Theorem 6.1 ([13], [15]). *Let either*

(a) $r(t) > 0$, $\int^{\infty} r^{1-\beta}(s)\,\Delta s = \infty$, $\int_t^{\infty} p(s)\,\Delta s \geqslant 0$ $(\not\equiv 0)$ *for large* t,

or

(b) $r(t) > 0$, $\int^{\infty} r^{1-\beta}(s)\,\Delta s < \infty$, *and* $p(t) \geqslant 0$ *for large* t.

If $\liminf\limits_{t\to\infty} \mathcal{A}_i(t) > \gamma_i(M_*(i))$ *for* $i = D$ *or* $i = C$ *according to whether* (a) *or* (b) *occurs, then* (1.1) *is oscillatory. If* $\limsup\limits_{t\to\infty} \mathcal{A}_i(t) < \gamma_i(M^*(i))$ *for* $i = D$ *or* $i = C$ *according to whether* (a) *or* (b) *occurs, then* (1.1) *is nonoscillatory.*

Remark 6.1. (i) We have $M_*(D), M^*(D) \in [0,\infty) \cup \{\infty\}$, $x \mapsto \gamma_D(x)$ is decreasing on $[0,\infty) \cup \{\infty\}$, $M_*(C), M^*(C) \in [0,1]$, and $x \mapsto \gamma_C(x)$ is decreasing on $[0,1]$.

(ii) If $M := M_*(D) = M^*(D)$, then $\gamma_D(M)$ is the critical constant satisfying

$$\gamma_D(M) = \begin{cases} \dfrac{1}{\alpha}\left(\dfrac{\alpha-1}{\alpha}\right)^{\alpha-1} \overset{\alpha=2}{=} \dfrac{1}{4} & \text{if } M = 0, \\ \left(\dfrac{(M+1)^{\frac{\alpha-1}{\alpha}} - 1}{M}\right)^{\alpha} \dfrac{M}{(M+1)^{\alpha-1} - 1} \overset{\alpha=2}{=} \dfrac{1}{\left(\sqrt{M+1}+1\right)^2} & \text{if } 0 < M < \infty, \\ 0 & \text{if } M = \infty. \end{cases}$$

If $N := M_*(C) = M^*(C)$, then $\gamma_C(N)$ is the critical constant satisfying

$$\gamma_C(N) = \begin{cases} \beta^{-\alpha} \overset{\alpha=2}{=} \dfrac{1}{4} & \text{if } N = 0, \\ (1-N)\left(\dfrac{1-(1-N)^{\frac{\alpha-1}{\alpha}}}{N}\right)^{\alpha} \overset{\alpha=2}{=} \dfrac{1-N}{\left(\sqrt{1-N}+1\right)^2} & \text{if } 0 < N < 1, \\ 0 & \text{if } N = 1. \end{cases}$$

Thus we see that the critical constant is not invariant with respect to time scales and may be strictly less than the constant known from the continuous theory. In addition to the graininess, the coefficient r affects its value. The critical constant can be different from the continuous like constant even in the well explored difference equations case, e.g., with $r(t) = 2^{-t}$. If $\alpha = 2$, then the results match the ones known from the linear theory, see [12] and also the q-calculus case in [5]. If, for example, $r(t) \equiv 1$, then the q-calculus type results ($\mathbb{T} = q^{\mathbb{N}}$) correspond with the case $0 < M = q - 1 < \infty$. With the choice $\alpha = 2$ and $\mathbb{T} = \mathbb{R}$ we obtain the classical Hille-Nehari results, see [16], [17].

(iii) The above criteria have wide applications (see [13], [15]): For example, oscillation of generalized Euler type dynamic equations can be fully analyzed, and the strong and the conditional oscillation can be described. Kneser type criteria with the sharp constant can be established. An optimality/nonoptimality of the constants involved in some existing Hardy and Wirtinger type inequalities can be revealed. Some new inequalities of Hardy type involving the best possible constants can be derived.

(iv) The proof of Theorem 6.1 is based on the function sequence technique, i.e., on the equivalence (i) $\Leftrightarrow$ (v) of Theorem 4.1 in the case $\int^{\infty} r^{1-\beta}(s)\,\Delta s = \infty$ and on the equivalence (i) $\Leftrightarrow$ (vi) of Theorem 4.1 in the case $\int^{\infty} r^{1-\beta}(s)\,\Delta s < \infty$. The details can be found in [13] and [15], respectively. Just note that the constant $\gamma_D(M)$ arises when solving the algebraic problem

$$x = \gamma_D(M) + \lim_{t\to M} \frac{x}{1-(1+t)^{1-\alpha}}\Big(1 - \frac{1}{(1+x^{\beta-1}t)^{\alpha-1}}\Big).$$

Roughly speaking, the complicated expression in this problem (the one involving lim) takes its form from the integrand of the formula for $\{\varphi_k\}$, thus depends on $\mathcal{S}$, where we take into account also the limit behavior of $\mathfrak{R}_D(\mu)$. Similarly we obtain the constant $\gamma_C(N)$ which is somehow related to the form of the latter integrand of the formula for $\{\psi_k\}$ taking into account the limit behavior of $\mathfrak{R}_C(\mu)$.

(v) Using the case (a) of Theorem 6.1 with $\alpha = 2$ and a transformation of dependent variable, see Section 3, in [12, Theorem 3.3] we established the linear version of the case (b) of Theorem 6.1, where the expression $\mathfrak{R}_C(\lambda)(t)$ is replaced by the expression $\widetilde{\mathfrak{R}}_C(\lambda)(t) = \lambda(t)r^{-1}(t)/R_C^{\sigma}(t)$ and the function $\gamma_C(x)$ is replaced by $\tilde{\gamma}_C(x) = \lim_{t\to x}(\sqrt{t+1}+1)^{-2}$. We have $\mathfrak{R}_C(\mu) = 1/(1+1/\widetilde{\mathfrak{R}}_C(\mu))$ for $\mu > 0$ and $\gamma_C \circ \vartheta = \tilde{\gamma}_C$ with $\vartheta(x) = 1/(1+1/x)$. It follows that Theorem 6.1 with (b) reduces to [12, Theorem 3.3] for $\alpha = 2$. In particular, if there exists $\widetilde{N} = \lim_{t\to\infty} \widetilde{\mathfrak{R}}_C(\mu)(t)$, then $\widetilde{N} \in [0,\infty) \cup \{\infty\}$, $N = \lim_{t\to\infty} \mathfrak{R}_C(\mu)(t)$ exists, and $\gamma_C(N) = \tilde{\gamma}_C(\widetilde{N})$. Note that also for our general case, i.e., $\alpha > 1$, the critical constant $\gamma(N)$ can be expressed in terms of $\widetilde{\mathfrak{R}}_C$, $\widetilde{N}$ and $\tilde{\gamma}$, and reads $\tilde{\gamma}_C(\widetilde{N}) = \lim_{t\to\widetilde{N}}(((t+1)^{\frac{\alpha-1}{\alpha}}-1)/t)^{\alpha} = (\gamma_C \circ \vartheta)(\widetilde{N}) = \gamma_C(N)$, where $\widetilde{N} = \lim_{t\to\infty} \widetilde{\mathfrak{R}}_C(\lambda)(t)$ with $\widetilde{\mathfrak{R}}_C(\lambda)(t) = \lambda(t)r^{1-\beta}(t)/R_C^\sigma(t)$. Recall however that the desired transformation of the dependent variable is not available in the nonlinear case, see Section 3.

7. Concluding remarks

From the above results we can observe the following interesting facts:

(i) The coefficient r may play an important role in the qualitative theory of half-linear dynamic equations which are in a "self-adjoint" form, where r stands as the leading coefficient.

(ii) The role of the coefficient r may not be known from some classical cases.

(iii) The role of the coefficient r is closely related to the behavior of the graininess of the time scale.

(iv) In general, we cannot say that "continuous" results are always "simpler" and require less additional assumptions than their discrete counterparts or vice versa. A "big" graininess is more "favorable" to the above comparison type results, while a "small" graininess "simplifies" the (non)oscillation criteria from Section 6. Indeed, if $\mu(t) \geqslant 1$ eventually, then Theorem 5.1 does not require additional conditions like (a) or (b). If, in Theorem 6.1, $r(t) \equiv 1$, for example, then $\mu(t) = o(t)$ guarantees that the critical oscillation constants look (simply) like in the continuous case.

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