126 (2001) MATHEMATICA BOHEMICA No. 2, 293–305

THE FAR-FIELD MODELLING OF TRANSONIC COMPRESSIBLE FLOWS

C. A. Coclici, Stuttgart, I. L. Sofronov, Moskva,
W. L. Wendland, Stuttgart

Dedicated to Professor Dr. J. Nečas on the occasion of his 70th birthday

Abstract. We present a method for the construction of artificial far-field boundary conditions for two- and three-dimensional exterior compressible viscous flows in aerodynamics. Since at some distance to the surrounded body (e.g. aeroplane, wing section, etc.) the convective forces are strongly dominant over the viscous ones, the viscosity effects are neglected there and the flow is assumed to be inviscid. Accordingly, we consider two different model zones leading to a decomposition of the original flow field into a bounded computational domain (near field) and a complementary outer region (far field). The governing equations as e.g. compressible Navier-Stokes equations are used in the near field, whereas the inviscid far field is modelled by Euler equations linearized about the free-stream flow. By treating the linear model analytically and numerically, we get artificial far-field boundary conditions for the (nonlinear) interior problem. In the two-dimensional case, the linearized Euler model can be handled by using complex analysis. Here, we present a heterogeneous coupling of the above-mentioned models and show some results for the flow around the NACA0012 airfoil. Potential theory is used for the three-dimensional case, leading also to non-local artificial far-field boundary conditions.

Keywords: artificial boundary and transmission conditions, compressible transonic flow, linearized Euler equations, integral equations with kernels of Cauchy type, potential theory, domain decomposition

MSC 2000: 35Q30, 35Q35, 35M20, 45E05, 65M55, 76N17, 76N20, 76H05, 76M40

1. Introduction

A sensitive issue in the numerical treatment of exterior flow problems is the accurate description of the flow field also far away from the travelling bodies. In various problems in CFD, it turns out that the influence of the far field on the flow properties in the immediate vicinity of the body is significant. Constant or characteristic boundary conditions involving constant values at infinity are simple and widely used in transonic flow computations. However, such conditions need to be imposed far enough from the object (a requirement which may lead to tremendeous computational costs, even when using unstructured meshes), and can also generate nondesirable reflections at the far-field boundary, see e.g. Hirsch [10].

A possible incorporation of the far-field behaviour, by maintaining the efficiency of the numerical discretization, is to replace in the far-field regions the (mostly expensive) complete model and the (rather inaccurate) constant flow by intermediate models, obtained e.g. by neglecting terms which are not relevant there and by linearization.

The far-field simplifications should be chosen in such a way that on the one hand the outer flow is still modelled accurately enough, while on the other hand, the computational times are reasonable. The coupling of different models leads to heterogeneous domain decomposition methods, efficiently used in the numerical treatment of exterior boundary value problems, see e.g. [8], [15], [11], [2], [1] for problems in aerodynamics.

The physical properties of compressible Navier-Stokes flows around 2D or 3D bodies, and a dimensional analysis of different quantities in the equations indicate that the viscous terms are of order of magnitude comparable with the convective terms only inside a thin region near to the body, whereas away from it, the shear stresses are extremely small and are strongly dominated by the convective part. Consequently, we neglect the viscous terms in some distance to the body and model the outer far-field domain with the Euler equations, linearized about the free-stream subsonic uniform background flow; this model can treat flows with wakes admitting non-zero vorticity and non-constant entropy at the outflow.

By solving the simplified model in the far field, we generate artificial far-field boundary conditions for the interior problem. The linearized Euler system is partly solved by using potential theory. The corresponding boundary conditions are based on integral representations of the solution, hence they are non-local. We present the construction of these conditions, which are exact with respect to the linearized model considered, and show results for the two-dimensional flow around the NACA0012 airfoil.

2. Compressible flow models

We introduce here two models for the description of the near-field region around some solid body, and of the corresponding far-field domain.

2.1. Mathematical modelling of the near field

Let $\Omega_a \subset \mathbb{R}^d$ $(d = 2,3)$ be a bounded domain representing a given body (as e.g. aeroplane, wing section, etc.). The conservation laws lead to the continuity equation, the *Navier-Stokes equations* and the energy equation written as the system in conservative form:

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{i=1}^{d} \frac{\partial \mathbf{f}_i(\mathbf{w})}{\partial x_i} = \sum_{i=1}^{d} \frac{\partial \mathbf{R}_i(\mathbf{w}, \nabla_{\mathbf{x}}\mathbf{w})}{\partial x_i} \quad \text{in} \quad \Omega_a^c \times (0,T). \tag{1}$$

Here $(0,T)$ is a time interval; $\mathbf{w} = (\varrho, \varrho\mathbf{v}, e)^\top$ collects the conservative variables with ϱ the density, $\mathbf{v} = (v_1, \dots, v_d)^\top$ the velocity vector, and e the total energy. The inviscid fluxes $\mathbf{f}_i$ and the viscous terms $\mathbf{R}_i$ are given by

$$\mathbf{f}_i(\mathbf{w}) = (\varrho v_i, \varrho v_i v_1 + \delta_{i1} p, \dots, \varrho v_i v_d + \delta_{id} p, (e+p)v_i)^\top, \tag{2}$$

$$\mathbf{R}_i(\mathbf{w}, \nabla_{\mathbf{x}}\mathbf{w}) = (0, \tau_{i1}, \dots, \tau_{id}, \tau_{i1}v_1 + \dots + \tau_{id}v_d + k\partial\theta/\partial x_i)^\top \tag{3}$$

for $i = 1, \dots, d$. The functions p, θ and τ_{ij} denote the pressure, the absolute temperature and the components of the viscous part of the stress tensor $\tau_{ij} = \lambda \operatorname{div} \mathbf{v}\, \delta_{ij} + \mu(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ for $i,j = 1, \dots, d$, respectively. Here, λ and μ are the viscosity coefficients and k is the heat conductivity. In addition, we employ the equations of state

$$p = R\,\varrho\,\theta \quad \text{and} \quad e = \varrho\Big(c_V\theta + \frac{|\mathbf{v}|^2}{2}\Big), \tag{4}$$

with the universal gas constant R and the specific heat c_V at constant volume, related by $R = (\kappa - 1)c_V$, where κ is Poisson's adiabatic constant.

Existence and uniqueness results for compressible viscous flows with the Navier-Stokes equations exist only for small data and under restrictive assumptions (as e.g. ideal gas in thermal equilibrium). Even for simpler models, as e.g. the full potential equation, existence of transonic solutions is still open, see Feistauer and Nečas [7]. However, we are interested here in the numerical treatment of compressible viscous flows.

Since the viscous terms are of order of magnitude comparable with the convective terms only inside Prandtl's boundary layer, and away from it, the viscous stress tensor becomes strongly dominated by the inertial forces, we neglect these terms in some distance to the body and decompose the exterior domain Ω_a^c into a bounded computational domain Ω_1, where we employ the full governing system of equations, and a corresponding far field $\Omega_2 := \mathbb{R}^d \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_a)$, where we assume the flow to be inviscid.

2.2. Mathematical modelling of the far field

We begin our analysis in the exterior Ω_2 with the 3D Euler equations

$$\left\{\begin{aligned} &\frac{\partial(\varrho v_1)}{\partial x_1}+\frac{\partial(\varrho v_2)}{\partial x_2}+\frac{\partial(\varrho v_3)}{\partial x_3}=0,\\ &\varrho\Big(v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}+v_3\frac{\partial v_1}{\partial x_3}\Big)+\frac{\partial p}{\partial x_1}=0,\\ &\varrho\Big(v_1\frac{\partial v_2}{\partial x_1}+v_2\frac{\partial v_2}{\partial x_2}+v_3\frac{\partial v_2}{\partial x_3}\Big)+\frac{\partial p}{\partial x_2}=0,\\ &\varrho\Big(v_1\frac{\partial v_3}{\partial x_1}+v_2\frac{\partial v_3}{\partial x_2}+v_3\frac{\partial v_3}{\partial x_3}\Big)+\frac{\partial p}{\partial x_3}=0,\\ &v_1\frac{\partial p}{\partial x_1}+v_2\frac{\partial p}{\partial x_2}+\kappa p\Big(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\Big)=0\end{aligned}\right. \tag{5}$$

for the primitive variables ϱ, v_1, v_2, v_3 and p. In Ω_2 we represent these as

$$\varrho=\varrho_\infty+\tilde{\varrho},\ v_1=v_\infty^{(1)}+\tilde{v}_1,\ v_2=v_\infty^{(2)}+\tilde{v}_2,\ v_3=v_\infty^{(3)}+\tilde{v}_3,\ p=p_\infty+\tilde{p}, \tag{6}$$

where $\mathbf{V}_\infty := (\varrho_\infty, v_\infty^{(1)}, v_\infty^{(2)}, v_\infty^{(3)}, p_\infty)^\top$ defines the free-stream subsonic flow. To simplify our analysis, we specify the coordinate system such that

$$\mathbf{v}_\infty \equiv (v_\infty^1, 0, 0)^\top \tag{7}$$

having in mind that the general case of non-zero $v_\infty^{(2)}$ and $v_\infty^{(3)}$ can be reduced to (7) by an elementary transformation. For the perturbation variables $\widetilde{\mathbf{V}} := (\tilde{\varrho}, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{p}\,)^\top$ we obtain via linearization of (5):

$$\begin{aligned} &v_\infty^{(1)}\frac{\partial\tilde{\varrho}}{\partial x_1}+\varrho_\infty\Big(\frac{\partial\tilde{v}_1}{\partial x_1}+\frac{\partial\tilde{v}_2}{\partial x_2}+\frac{\partial\tilde{v}_3}{\partial x_3}\Big)=0,\\ &\varrho_\infty v_\infty^{(1)}\frac{\partial\tilde{v}_1}{\partial x_1}+\frac{\partial\tilde{p}}{\partial x_1}=0,\quad \varrho_\infty v_\infty^{(1)}\frac{\partial\tilde{v}_2}{\partial x_1}+\frac{\partial\tilde{p}}{\partial x_2}=0,\quad \varrho_\infty v_\infty^{(1)}\frac{\partial\tilde{v}_3}{\partial x_1}+\frac{\partial\tilde{p}}{\partial x_3}=0,\\ &v_\infty^{(1)}\frac{\partial\tilde{p}}{\partial x_1}+\kappa p_\infty\Big(\frac{\partial\tilde{v}_1}{\partial x_1}+\frac{\partial\tilde{v}_2}{\partial x_2}+\frac{\partial\tilde{v}_3}{\partial x_3}\Big)=0.\end{aligned} \tag{8}$$

Here we neglect terms of the form $\tilde{h}_1\cdot\partial\tilde{h}_2/\partial x_j$, $\partial(\tilde{h}_1\tilde{h}_2)/\partial x_j$ $(j=1,\dots,d)$, which are small of quadratic order, where $\tilde{h}_1, \tilde{h}_2 \in \{\tilde{\varrho}, \tilde{v}_1, \tilde{v}_2, \tilde{p}\}$ represent perturbations of the flow variables. Hence, the linear model can describe inviscid flows characterized by small vorticity. Nevertheless, the linearized Euler equations are capable to model a "weakly" rotational, compressible flow, and their numerical treatment requires minimal computational cost. For subsonic free-stream flow, (8) is a system of composed type, having three real and two complex conjugate eigenvalues.

Let us denote by $a_\infty := \sqrt{\kappa\, p_\infty/\varrho_\infty}$ and $M_\infty := |\mathbf{v}_\infty|/a_\infty$ the local speed of sound and the Mach number at infinity, respectively. Now we introduce a new Cartesian system $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ by

(9) $$\bar{x}_1 = \tilde{x}_1/\sqrt{1-M_\infty^2}, \quad \bar{x}_2 = \tilde{x}_2, \quad \bar{x}_3 = \tilde{x}_3 \quad \text{(Prandtl transformation)},$$

and, in addition, new unknown functions (perturbations) by

(10)
$$\bar{\varrho} = \tilde{\varrho} \cdot \frac{\kappa p_\infty \sqrt{1-M_\infty^2}}{\varrho_\infty^2 v_\infty^{(1)}}, \; \bar{v}_1 = \tilde{v}_1\sqrt{1-M_\infty^2}, \; \bar{v}_2 = \tilde{v}_2, \; \bar{v}_3 = \tilde{v}_3, \; \bar{p} = \tilde{p} \cdot \frac{\sqrt{1-M_\infty^2}}{\varrho_\infty v_\infty^{(1)}}.$$

With these transformations, the system can be reformulated to an elliptic subsystem for $\bar{p}$, $\bar{v}_2$ and $\bar{v}_3$, and two transport equations for $\bar{\varrho} - \bar{p}$ and $\bar{v}_1 + \bar{p}$:

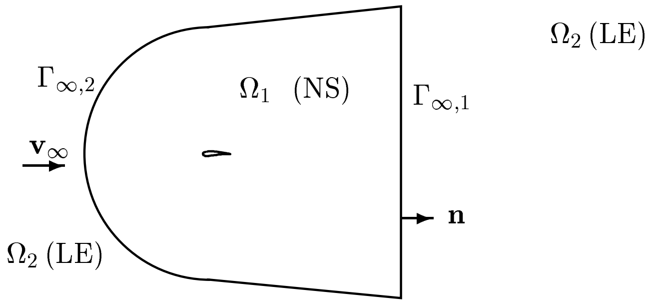
(11)
$$\begin{cases} \Big[\dfrac{\partial \bar{v}_2}{\partial \bar{x}_1} + \dfrac{\partial \bar{p}}{\partial \bar{x}_2}\Big](\bar{\mathbf{x}}) = 0, \\ \Big[\dfrac{\partial \bar{v}_3}{\partial \bar{x}_1} + \dfrac{\partial \bar{p}}{\partial \bar{x}_3}\Big](\bar{\mathbf{x}}) = 0, \\ \Big[-\dfrac{\partial \bar{p}}{\partial \bar{x}_1} + \dfrac{\partial \bar{v}_2}{\partial \bar{x}_2} + \dfrac{\partial \bar{v}_3}{\partial \bar{x}_3}\Big](\bar{\mathbf{x}}) = 0, \\ \dfrac{\partial}{\partial \bar{x}_1}(\bar{\varrho} - \bar{p})(\bar{\mathbf{x}}) = 0, \\ \dfrac{\partial}{\partial \bar{x}_1}(\bar{v}_1 + \bar{p})(\bar{\mathbf{x}}) = 0, \quad \text{for} \quad \bar{\mathbf{x}} \in \Omega_2. \end{cases}$$

We require the flow quantities to be bounded at infinity and to tend in the upstream region to the free-stream quantities. Therefore, we supplement the system (11) with the condition

(12) $$|\bar{\varrho}|, |\bar{v}_1|, |\bar{v}_2|, |\bar{v}_3|, |\bar{p}| < \infty \text{ for } |\bar{\mathbf{x}}| \to \infty \text{ and } \bar{\varrho}, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{p} \to 0 \text{ as } \bar{x}_1 \to -\infty.$$

3. Far-field boundary conditions for $2D$ flows

In the two-dimensional case, the artificial far-field boundary separating the model zones Ω_1 and Ω_2 is a simply closed curve Γ_∞, which we assume to be piecewise smooth. In the sequel, $\mathbf{n}$ denotes the unit normal vector on $\partial\Omega_1$ exterior to the interior domain Ω_1.



For a given velocity field $\mathbf{v}$, the input and the output parts of the boundary Γ_∞ are defined with respect to the far-field domain Ω_2 by

$$\begin{aligned}\Gamma_{\infty,\mathrm{in}} &:= \{\mathbf{x} \in \Gamma_\infty \mid (\mathbf{v}\cdot\mathbf{n})(\mathbf{x}) \geqslant 0\},\\ \Gamma_{\infty,\mathrm{out}} &:= \Gamma_\infty \setminus \Gamma_{\infty,\mathrm{in}}.\end{aligned}$$

In two space dimensions, the system (11) of the linearized Euler equations does not contain equation $(11)_2$, and equation $(11)_3$ takes the form

$$\Big[-\frac{\partial \bar p}{\partial \bar x_1} + \frac{\partial \bar v_2}{\partial \bar x_2}\Big](\bar{\mathbf{x}}) = 0 \quad \text{for} \quad \bar{\mathbf{x}} \in \Omega_2. \tag{13}$$

The elliptic subsystem, consisting here of equations $(11)_1$ and (13), is the Cauchy-Riemann system for $\bar p$ and $\bar v_2$ in Ω_2 and can be solved by using the complex Cauchy integral. Let $\psi(\zeta) = \bar p^0(\zeta) + \mathrm{i}\bar v_2^0(\zeta)$, $\zeta = \bar\xi_1 + \mathrm{i}\bar\xi_2 \in \Gamma_\infty$ be a given complex-valued function defined on Γ_∞. Then the function $\chi\colon \Omega_2 \to \mathbb{C}$, defined by the Cauchy potential

$$\chi(z) = \bar p(z) + \mathrm{i}\bar v_2(z) := \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_\infty} \frac{\psi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \quad \text{for} \quad z = \bar x_1 + \mathrm{i}\bar x_2 \in \Omega_2, \tag{14}$$

is a *holomorphic solution of the elliptic subsystem* $(11)_1$, (13). Since it is necessary to solve the linearized Euler model up to the boundary Γ_∞, we need the behavior of the Cauchy potential $\chi(z)$ as $z \in \Omega_2$ approaches a given point z_0 situated on Γ_∞.

Lemma 3.1. *Let $\psi\colon \Gamma_\infty \to \mathbb{C}$, $\psi \in C^1(\Gamma_\infty)$ be the trace of some solution to the Cauchy-Riemann system in the exterior domain Ω_2. Then, for arbitrary $z_0 \in \Gamma_\infty$, the value $\psi(z_0)$ satisfies the relation*

$$\psi(z_0) = \frac{1}{\pi\mathrm{i}} \lim_{\varepsilon\to 0} \oint_{\Gamma_\infty \wedge |\zeta - z_0| \geqslant \varepsilon > 0} \frac{\psi(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta, \tag{15}$$

and the Cauchy principal value exists.

Lemma 3.2 (Plemelj-Sokhotski). *Let $\psi\colon \Gamma_\infty \to \mathbb{C}$ be Hölder-continuous on Γ_∞. Then, for every $z_0 \in \Gamma_\infty$, the jump relation*

$$\lim_{\substack{z\to z_0\\ z\in\Omega_2}} \chi(z) = \lim_{\substack{z\to z_0\\ z\in\Omega_2}} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_\infty} \frac{\psi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}} \lim_{\varepsilon\to 0} \oint_{\Gamma_\infty \wedge |\zeta - z_0| \geqslant \varepsilon > 0} \frac{\psi(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta + \frac{1}{2}\psi(z_0) \tag{16}$$

holds.

Corollary. *If $\psi(\zeta)$ is the trace of some solution to the Cauchy-Riemann equations in Ω_2, then*

$$\lim_{\substack{z\to z_0\\ z\in\Omega_2}} \chi(z) = \psi(z_0) \quad \textit{for all} \quad z_0 \in \Gamma_\infty. \tag{17}$$

Hence, the trace of the Hilbert transform (14) *on Γ_∞ is a projection.*

This projection property will decisively be exploited for the numerical realization of the coupling [2]. It is valid also in the three-dimensional case; its interpretation will be given in detail in the next section.

For numerical treatment, we impose as transmission condition for the elliptic components $\bar{p}$ and $\bar{v}_2$ their *continuity* across Γ_∞:

$$\bar{v}_2^{LE} := \bar{v}_2^{NS} \quad \text{and} \quad \bar{p}^{LE} := \bar{p}^{NS}, \tag{18}$$

compute these quantities in the far field by using the Cauchy integral (14) (in the case of an overlapping scheme) or the Plemelj-Sokhotski formula to get their traces on Γ_∞ (in the case of a non-overlapping scheme).

As transmission conditions for the equations $(11)_{4,5}$, we require the *continuity* of those characteristic variables which correspond to characteristic directions "entering" the domain of hyperbolicity (see e.g. Hirsch [10], Quarteroni and Stolcis [11]). On the inflow boundary $\Gamma_{\infty,\mathrm{in}}$ we impose the *continuity of the Riemann invariants*: The Riemann invariants corresponding to the linearized Euler equations in Ω_2 are equated to their counterparts from the interior domain, i.e. to the Riemann invariants corresponding to the (nonlinear) 2D steady Euler equations in Ω_1 (for more details, see Coclici and Wendland [3]). On the output boundary $\Gamma_{\infty,\mathrm{out}}$, the characteristics are outgoing and therefore we impose here the *compatibility conditions*

$$\frac{\partial}{\partial \bar{x}_1}(\bar{\varrho} - \bar{p})(\bar{x}_1, \bar{x}_2) = 0 \quad \text{and} \quad \frac{\partial}{\partial \bar{x}_1}(\bar{v}_1 + \bar{p})(\bar{x}_1, \bar{x}_2) = 0. \tag{19}$$

We find the solution $(11)_{4,5}$ in the form

$$\bar{\varrho}(\bar{x}_1, \bar{x}_2) = \bar{p}(\bar{x}_1, \bar{x}_2) + c_1(\bar{x}_2) \quad \text{and} \quad \bar{v}_1(\bar{x}_1, \bar{x}_2) = -\bar{p}(\bar{x}_1, \bar{x}_2) + c_2(\bar{x}_2),$$

where $\bar{p} = \bar{p}^{LE}$ is already known and where, as a consequence of (12), the functions c_1, c_2 are equal to zero in the upstream domain.

As interface conditions from Ω_2 to Ω_1 we impose the *continuity of the normal fluxes* across Γ_∞. The normal inviscid flux of the linearized Euler equations in Ω_2 is set equal to the normal total flux for the Navier-Stokes system solution (see, e.g. Quarteroni and Stolcis [11]):

$$-\sum_{i=1}^{2} \mathbf{R}_i(\mathbf{w}, \nabla \mathbf{w}) n_i + \sum_{i=1}^{2} \mathbf{f}_i(\mathbf{w}) n_i = \sum_{i=1}^{2} \mathbf{f}_i^{LE}(\mathbf{w}) n_i \quad \text{across} \quad \Gamma_\infty. \tag{20}$$

Here, the flux functions $\mathbf{f}_i^{LE}$ are linearizations of the Euler fluxes $\mathbf{f}_i$ about the free-stream flow parameters $\mathbf{V}_\infty$.

For the numerical solution of the Navier-Stokes equations we use the operator-splitting FVM-FEM algorithm by M. Feistauer (Prague, see Feistauer et al. [6]) executed on a triangular mesh (Figure 1, left). We present results for flows around the NACA0012 airfoil at the free-stream Mach number $M_\infty = 0.85$, for $\mathrm{Re} = 5000$ and zero angle of attack. In Figures 1 (right) and 2 we show the isolines of the Mach number, of the entropy and of the vorticity field corresponding to the coupled solution.

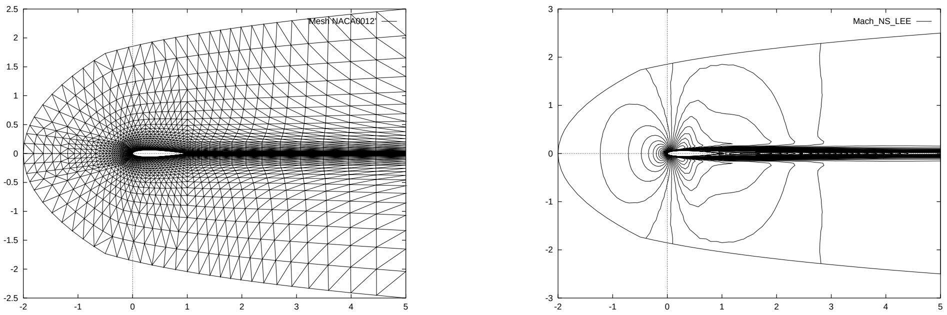


Fig. 1 Computational domain and isolines of the Mach number

4. Solution of the 3D linearized Euler system

In the three-dimensional case, the artificial far-field boundary $S_\infty := \partial\Omega_1 \cap \partial\Omega_2$ is a closed surface, which we assume to consist of a finite number of simply connected regular surface segments $S_{\infty,i}$, i.e. $S_\infty = \bigcup_i S_{\infty,i}$ with $S_{\infty,i} \in C^1$ for $i = 1, \ldots, m$.

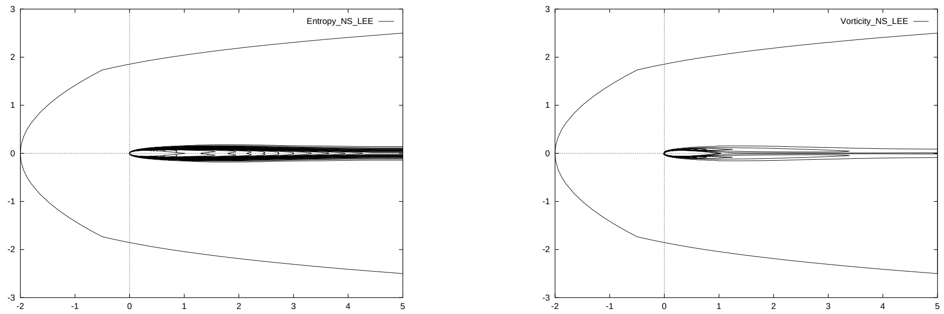


Fig. 2 Isolines of the entropy and vorticity for the coupled solution

We begin our analysis with the elliptic subsystem $(11)_{1,2,3}$ and collect the elliptic variables into the vector-valued function

$$\mathbf{E}(\bar{\mathbf{x}}) := (-\bar{p}, \bar{v}_2, \bar{v}_3)^\top(\bar{\mathbf{x}}) \quad \text{for} \quad \bar{\mathbf{x}} \in \Omega_2, \tag{21}$$

and add to $(11)_{1,2,3}$ the equation

$$\frac{\partial \bar{v}_3}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_3} = b \tag{22}$$

with an auxiliary function $b\colon \Omega_2 \to \mathbb{R}$. Then $(11)_{1,2,3}$, (22) take the form

$$\begin{cases} \operatorname{rot} \mathbf{E} = \mathbf{b}, \\ \operatorname{div} \mathbf{E} = 0 \quad \text{in} \quad \Omega_2 \end{cases} \tag{23}$$

with $\mathbf{b} := (b, 0, 0)^\top\colon \Omega_2 \to \mathbb{R}^3$ to be specified later on. From (12) we get the conditions at infinity

$$\lim_{|\bar{\mathbf{x}}|\to\infty} |\mathbf{E}(\bar{\mathbf{x}})| < \infty, \qquad \lim_{\bar{x}_1\to-\infty} \mathbf{E}(\bar{\mathbf{x}}) = \mathbf{0}. \tag{24}$$

In what follows, we are concerned with the solution of the exterior boundary value problem (23), (24). Since we have to model also the wake, let us introduce the cylindrical "shadow" domain $\Omega_W \subset \Omega_2$ by

$$\Omega_W := \{\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^\top \in \Omega_2;\ \bar{x}_1 > \bar{y}_1, \bar{x}_2 = \bar{y}_2, \bar{x}_3 = \bar{y}_3, \bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \bar{y}_3)^\top \in \Omega_1\}$$

and define the "outflow" part $\omega = S_\infty \cap \overline{\Omega}_W$ of the artificial boundary S_∞, see Figure 3.

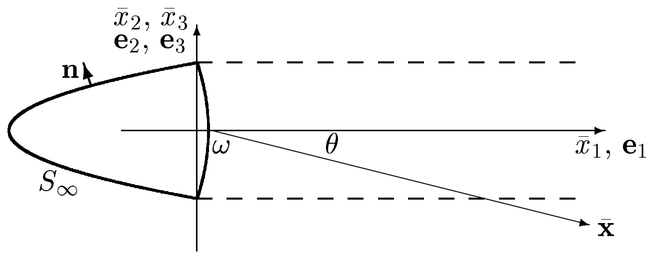


Fig. 3 Angle ϑ for an observation point $\bar{\mathbf{x}}$

Assuming that ω has a single-valued projection on the plane $\bar{x}_1 = 0$, we obtain

Lemma 4.1. *The function b, associated with the solution of the linearized Euler equations via* (22), *satisfies the relations*

$$b \equiv 0 \text{ in } \Omega_2 \setminus \Omega_W, \quad b \equiv b(\bar{x}_2, \bar{x}_3) \text{ in } \Omega_W, \quad \int_\omega \mathbf{b} \cdot \mathrm{d}\boldsymbol{\sigma} = 0. \tag{25}$$

Consequently, the specific properties of the right-hand side $\mathbf{b}$ in (23) restrict the set of admissible vector fields $\mathbf{E}$, which permits us to model the far-field behaviour. In order to proceed with our analysis, we introduce for arbitrary vector fields $\mathbf{f} \in \mathbf{C}(S_\infty)$ with $\mathbf{f}|_{S_{\infty,i}} \in \mathbf{C}^1(S_{\infty,i})$ the surface-curl quantity

$$(\mathbf{n} \times \nabla, \mathbf{f})(\bar{\mathbf{x}}) := \Big[\sum_{k=1}^{3} (\mathbf{n} \times \nabla_{\bar{\mathbf{x}}})_k\, f_k\Big](\bar{\mathbf{x}})$$

containing only the tangential derivatives of $\mathbf{f}$.

Let us consider an arbitrary vector field $\mathbf{a}\colon S_\infty \to \mathbb{R}^3$ with $\mathbf{a} \in \mathbf{C}(S_\infty)$ and $\mathbf{a}|_{S_{\infty,i}} \in \mathbf{C}^1(S_{\infty,i})$ $(i = 1, \dots, m)$, satisfying with $b_{\mathbf{n}} = b_{\mathbf{n}}(\mathbf{a}) := (\mathbf{n} \times \nabla, \mathbf{a})\colon S_\infty \to \mathbb{R}$ the conditions

$$\operatorname{supp}(b_{\mathbf{n}}) \subset \omega \quad \text{and} \quad \int_\omega b_{\mathbf{n}}\, \mathrm{d}\sigma = 0. \tag{26}$$

In the sequel we shall assume the general case $b_{\mathbf{n}} \not\equiv 0$ which corresponds to a non-zero wake. Denote by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ the basis vectors of the cartesian coordinate system. In dependence on $\mathbf{a}$, we define now the vector field $\mathbf{b}\colon \Omega_2 \to \mathbb{R}^3$ by

$$\mathbf{b}(\bar{\mathbf{x}}) := \begin{cases} \Big(\dfrac{b_{\mathbf{n}}(\bar{x}_1|_\omega, \bar{x}_2, \bar{x}_3)}{(\mathbf{n}, \mathbf{e}_1)}, 0, 0\Big)^\top & \text{for} \quad \bar{\mathbf{x}} \in \Omega_W, \\ \mathbf{0} & \text{for} \quad \bar{\mathbf{x}} \in \Omega_2 \setminus \Omega_W. \end{cases} \tag{27}$$

In what follows, we rename for brevity the variables $\bar{\mathbf{x}} \to \mathbf{x}$. By means of the vector-valued functions $\mathbf{a}\colon S_\infty \to \mathbb{R}^3$ and $\mathbf{b} = \mathbf{b}(\mathbf{a})\colon \Omega_2 \to \mathbb{R}^3$, we introduce the vector field $\mathbf{E} = \mathbf{E}(\mathbf{x})\colon \Omega_2 \to \mathbb{R}^3$ by

$$\begin{aligned} \mathbf{E}(\mathbf{x}) := & -\int_\omega \frac{1}{2}\Big(1 + \frac{x_1 - y_1}{|\mathbf{x}-\mathbf{y}|}\Big)(\mathbf{n}\times\nabla, \mathbf{a})\,\mathbf{e}_1 \times \nabla_{\mathbf{y}'}\varphi(\mathbf{x}'-\mathbf{y}')\,\mathrm{d}\sigma_{\mathbf{y}} \\ & -\int_{S_\infty} \Big\{[\mathbf{n}\times\mathbf{a}]\times\nabla_{\mathbf{y}}\Phi(\mathbf{x}-\mathbf{y}) + (\mathbf{n}\cdot\mathbf{a})\nabla_{\mathbf{y}}\Phi(\mathbf{x}-\mathbf{y})\Big\}\,\mathrm{d}S_{\mathbf{y}}, \end{aligned} \tag{28}$$

where $\Phi(\mathbf{x}-\mathbf{y}) := -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}$ and $\varphi(\mathbf{x}'-\mathbf{y}') := -\frac{1}{2\pi}\log|\mathbf{x}'-\mathbf{y}'|$ denote the fundamental solutions of the Laplacian in three and two dimensions, respectively, and where $\mathbf{x}' := (x_2, x_3) \in \mathbb{R}^2$ for $\mathbf{x} \in \mathbb{R}^3$. The solution of the linearized Euler system in the far-field region Ω_2 can now be characterized in the following theorems (see Sofronov and Wendland [14]).

Theorem 4.2. *For arbitrary* $\mathbf{a}$ *satisfying* (26)*, the function* $\mathbf{E}$ *evaluated via* (28) *is a solution of the system*

$$\begin{cases} \operatorname{rot}\mathbf{E} = \mathbf{b}, \\ \operatorname{div}\mathbf{E} = 0 \end{cases}$$

with $\mathbf{b}$ *defined by* (27). *Moreover, the following estimate is valid for* $\mathbf{E} \equiv (-\bar{p}, \bar{v}_2, \bar{v}_3)^\top$ *as* $|\mathbf{x}| \to \infty$ *(here* ϑ *denotes the angle between* $\mathbf{x}$ *and* $\mathbf{e}_1$*, see Figure* 3*):*

$$|\bar{p}| = \mathcal{O}(|\mathbf{x}|^{-2}) \quad \textit{and} \quad |\bar{v}_2|, |\bar{v}_3| = \mathcal{O}\Big(\frac{1}{1+|\mathbf{x}|^2\sin^2(\vartheta/2)}\Big). \tag{29}$$

Hence, (28) gives the representation of a solution to (23) in terms of some vector field $\mathbf{a}$ on S.

Theorem 4.3 [14]**.** *For any solution* $\mathbf{E}$ *of the problem* (23), (24)*, the formula* (28) *with* $\mathbf{a} = \mathbf{E}|_{S_\infty}$ *recovers* $\mathbf{E}$ *in the whole exterior domain* Ω_2*.*

Therefore, it follows similarly to the two-dimensional case that the operator in (28) is a *projection* and the components $\bar{p}$, $\bar{v}_2$ and $\bar{v}_3$ can be obtained by means of the values of these functions on S_∞.

The remaining two components $\bar{\varrho}$ and $\bar{v}_1$ are to be found by extending our two-dimensional analysis [1], [3] of the transport equations to three dimensions. If an overlapping domain decomposition is used, then the solution of the transport equations $(11)_{4,5}$ can be represented by

$$\bar{\varrho}(\bar{\mathbf{x}}) = \bar{p}(\bar{\mathbf{x}}) + g_1(\bar{x}_2, \bar{x}_3) \quad \text{and} \quad \bar{v}_1(\bar{\mathbf{x}}) = -\bar{p}(\bar{\mathbf{x}}) + g_2(\bar{x}_2, \bar{x}_3), \tag{30}$$

where $\bar{p}$ is already known and where, due to (24), g_1 and g_2 are non-zero only in the "shadow" (wake) domain Ω_W, i.e.

$$
(31) \qquad \begin{cases} g_1 \equiv 0 \text{ in } \Omega_2 \setminus \Omega_W, \\ g_1 \equiv (\bar{\varrho} - \bar{p})|_\omega \text{ in } \Omega_W, \end{cases} \quad \text{and} \quad \begin{cases} g_2 \equiv 0 \text{ in } \Omega_2 \setminus \Omega_W, \\ g_2 \equiv (\bar{v}_1 + \bar{p})|_\omega \text{ in } \Omega_W. \end{cases}
$$

We summarize the results obtained in this section as follows:

Theorem 4.4. *The general solution to* (11), (12) *(equivalently* (23), (24)*) in* Ω_2 *is expressed by* (28) *and* (30)*, where* $\mathbf{a}$ *is a function satisfying* (26) *and where the functions* g_1, g_2 *are defined in* (31). *If* $\mathbf{a}$ *in* (28) *is the trace of some solution* $\mathbf{E} = (-\bar{p}, \bar{v}_2, \bar{v}_3)$ *to* (11), (12) *(equivalently* (23), (24)*) then formula* (28) *recovers* $\mathbf{E}$ *in* Ω_2.

As in the 2D case, the flux condition (20) can be imposed in the opposite direction: $LE \to NS$. Using this transmission condition and the solution of the linearized Euler equations in the outer domain Ω_2, we obtain far-field boundary conditions at the artificial boundary S_∞. Details of computational aspects are discussed in [14].

References

[1] *C. A. Coclici*: Domain decomposition methods and far-field boundary conditions for two-dimensional compressible flows around airfoils. Doctoral Thesis, Math. Inst. A, University of Stuttgart, 1998.

[2] *C. A. Coclici, I. L. Sofronov, W. L. Wendland*: Far-field boundary conditions for 2D compressible viscous flows. In Numerical Modelling in Continuum Mechanics (M. Feistauer et al., eds.). vol. 1, Matfyzpress Univerzita Karlova Praha, 1997, pp. 212–220.

[3] *C. A. Coclici, W. L. Wendland*: Analysis of a heterogeneous domain decomposition for compressible viscous flow. To appear in Mathematical Models and Methods in Applied Sciences.

[4] *D. Colton, R. Kress*: Integral Equation Methods in Scattering Theory. John Wiley & Sons, New York, 1983.

[5] *M. Feistauer*: Mathematical Methods in Fluid Dynamics. Pitman Monographs and Surveys in Pure and Applied Math., Longman Sci. & Technical, Harlow, 1993.

[6] *M. Feistauer, J. Felcman, M. Lukáčová-Medvidová*: Combined finite element-finite volume solution of compressible flow. J. Comp. Appl. Math. *63* (1995), 179–199.

[7] *M. Feistauer, J. Nečas*: On the solvability of transonic potential flow problems. Z. Anal. Anw. *4* (1985), 305–329.

[8] *L. Ferm*: Open boundary conditions for external flow problems. J. Comput. Phys. *91* (1990), 55–70.

[9] *F. Gastaldi, A. Quarteroni*: On the coupling of hyperbolic and parabolic systems: analytical and numerical approach. Appl. Num. Math. *6* (1990), 3–31.

[10] *C. Hirsch*: Numerical Computation of Internal and External Flows. Wiley, Chichester, 1988.

[11] *A. Quarteroni, L. Stolcis*: Homogeneous and heterogeneous domain decomposition for compressible fluid flows at high Reynolds numbers. Numerical Methods for Fluid Dynamics V (K. W. Morton, M. J. Baines, eds.). Oxford Univ. Press, 1995, pp. 113–128.

[12] *V. S. Ryaben'kii, S. V. Tsynkov*: Artificial boundary conditions for the numerical solution of external viscous flow problems. SIAM J. Numer. Anal. *32* (1995), 1355–1389.

[13] *I. L. Sofronov*: Coupling of potential and Euler equations to describe far field in 3D transonic flows. Book of Abstracts of the 2nd Seminar on Euler and Navier-Stokes Equations held in Prague, 1996, pp. 69–70.

[14] *I. L. Sofronov, W. L. Wendland*: Exact linear far-field conditions for three-dimensional aerodynamic stationary transonic flows. Preprint 99–16, Math. Inst. A, University of Stuttgart (1999).

[15] *A. Verhoff, D. Stookesberry, S. Agrawal*: Far-field computational boundary conditions for two-dimensional external flow problems. AIAA Journal *30* (1992), 2585–2594.

Authors' addresses: *C. A. Coclici*, Mathematisches Institut A, Universität Stuttgart, 70569 Stuttgart, Germany, e-mail: `cristi@mathematik.uni-stuttgart.de`, *I. L. Sofronov*, Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya sq. 4, 125047 Moscow, Russia, e-mail: `sofronov@spp.keldysh.ru`, *W. L. Wendland*, Mathematisches Institut A, Universität Stuttgart, 70569 Stuttgart, Germany, e-mail: `wendland@mathematik.uni-stuttgart.de`.