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ON MEASURE SOLUTIONS TO THE ZERO-PRESSURE GAS MODEL AND THEIR UNIQUENESS

Jiequan Li, Beijing, Gerald Warnecke, Magdeburg

Abstract. The system of zero-pressure gas dynamics conservation laws describes the dynamics of free particles sticking under collision while mass and momentum are conserved. The existence of such solutions was established some time ago. Here we report a uniqueness result that uses the Oleinik entropy condition and a cohesion condition. Both of these conditions are automatically satisfied by solutions obtained in previous existence results. Important tools in the proof of uniqueness are regularizations, generalized characteristics and flow maps. The solutions may contain vacuum states as well as singular measures.

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1. Introduction

Let us consider a particle mass density ϱ and a velocity field u depending on one space variable x and time t. The hyperbolic system of conservation laws for zero-pressure gas dynamics is

$$\begin{aligned} \varrho_t + (\varrho u)_x &= 0, \\ (\varrho u)_t + (\varrho u^2)_x &= 0. \end{aligned} \tag{1}$$

We consider it together with the initial data

$$(\varrho, u)(0, x) = (\varrho_0, u_0)(x). \tag{2}$$

This system is related to the sticky particle model of Shandarin and Zeldovich [13] describing the motion of free particles sticking under collision while mass and momentum are conserved. For physical interpretations of this model in several space dimensions, we refer to Kofman et al. [8]. Formally, the model can be obtained from the compressible Euler equations by taking the pressure to zero or from the Boltzmann equation by letting the temperature go to zero, see Bouchut [2]. A rigorous proof of the reduction from the Euler equations to the system (1) taking account of the structure of solutions can be found in Li [9]. The system (1) was also related to scalar conservation laws and even to Hamilton-Jacobi equations in Brenier and Grenier [3]. In a class of fractional step methods for gas dynamics one splits evolution operators. For instance one may split the system of compressible Euler equations into the zero-pressure model, we are considering here, and the pressure gradient flow, see Agarwal and Halt [1] or Li and Cao [10].

The system (1) is non-strictly hyperbolic with the velocity u as a double eigenvalue. For smooth solutions it is easily seen that the system decouples in the following manner. Using the first equation, the second equation becomes the well understood Burgers equation $u_t + (\frac{u^2}{2})_x = 0$. Solving it first and then using the known velocity u, the first equation for the non-negative mass density ϱ is then solved as a scalar equation, too.

The density may develop into a singular measure, even when the initial data are regular. The weak solution in the sense of distributions is not a function, possibly discontinuous, anymore but must be considered in the sense of signed measures. These will be a measure m for the density ϱ and a signed measure I for the momentum ϱu, which is the other conserved dependent variable. We also allow the mass density to vanish. This is the vacuum state. The velocity makes physical sense only when mass is present. Therefore the momentum will be assumed to be absolutely continuous with respect to the mass measure.

Definition 1. We denote by $\mathcal{B}$ the σ-algebra of Borel measurable subsets of $\mathbb{R}$ and by $\mathcal{M}(\mathbb{R})$ the space of signed Borel measures on $\mathbb{R}$. Take $m, I \in L^\infty(\mathbb{R}^+, \mathcal{M}(\mathbb{R}))$, i.e. assume that $(m, I)(t, \cdot)$ are signed Borel measures on $\mathbb{R}$ for any $t \in [0, \infty[$. By $(m, I)(t, \Delta)$ we denote the signed measure of the Borel measurable set $\Delta \in \mathcal{B}$ at time $t \in \mathbb{R}^+ = [0, \infty[$.

Further, let the measure $I(t, \cdot)$ for the momentum be absolutely continuous with respect to the measure for the particle mass density $m(t, \cdot)$. Then a velocity $u(t, x)$ can be defined using the Radon-Nikodym Theorem as the density function given by

$$(3) \qquad u(t, x) = \frac{\mathrm{d}I}{\mathrm{d}m}, \qquad \text{i.e.} \int_\Delta \mathrm{d}I = \int_\Delta u \,\mathrm{d}m.$$

For clarity we also use the notation $m(t, \mathrm{d}x) := \mathrm{d}m$ and $I(t, \mathrm{d}x) := \mathrm{d}I$. We say that a pair of signed measures (m, I) is a *measure solution* of the system (1) for the initial data (2) if and only if the equations

$$
(4)\qquad
\begin{aligned}
&\int\int_{\mathbb{R}^+\times\mathbb{R}} [\varphi_t m(t,\,\mathrm{d}x) + \varphi_x I(t,\,\mathrm{d}x)]\,\mathrm{d}t + \int_{-\infty}^{\infty} \varphi(0,x) m(0,\,\mathrm{d}x) = 0,\\
&\int\int_{\mathbb{R}^+\times\mathbb{R}} [(\varphi_t + \varphi_x u(t,x)) I(t,\,\mathrm{d}x)]\,\mathrm{d}t + \int_{-\infty}^{\infty} \varphi(0,x) I(0,\,\mathrm{d}x) = 0
\end{aligned}
$$

are satisfied for all $\varphi \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$.

Based on this definition, the Cauchy problem was solved in Cheng et al. [4]. For an alternative proof see [6]. For the initial data one may either consider locally finite measures or one needs a boundedness condition in case the initial mass does not have a compact support, see [4].

2. Uniqueness

For weak solutions to conservation laws it is well-known that some additional entropy condition is needed in order to obtain the *uniqueness* of solutions. It selects the physically relevant solution for given initial data. From the existence proof in [4], [6] one may see: The solutions constructed there automatically satisfy the classical Oleinik Condition: There exists a constant $E > 0$ such that

$$
(5)\qquad \frac{u(t,x+a) - u(t,x)}{a} \leqslant \frac{E}{t}
$$

for all $x \in \mathbb{R},\ a > 0$ and $t \in]0,\infty[$.

This condition is sufficient to prove uniqueness for the case of the Burgers equation, see Smoller [12]. For the system (1) the following simple counterexample shows that an additional condition must be imposed. Take δ_0 to be the Dirac point measure with unit mass at zero and $\overline{u}$ to be any constant. Let (2) be the simple initial state $(m_0, I_0)(x) = (\delta_0, \overline{u}\delta_0)$. Basically, the initial data for the velocity are $\overline{u}$ at $x = 0$ and completely arbitrary elsewhere because the mass density vanishes. We can construct an infinite family of solutions in the follwing manner:

$$
(6)\qquad m(t,\cdot) := \varrho_1 \delta_{(x-a_1 t)} + \varrho_2 \delta_{(x-a_2 t)},\quad u(t,x) := \begin{cases} a_1, & x = a_1 t,\\ a_2, & x = a_2 t,\\ 0, & \text{everywhere else.}\end{cases}
$$

We choose constants $a_1 < a_2$ as well as $\varrho_1, \varrho_2 \geqslant 0$ in order to have conservation of mass by taking $\varrho_1 + \varrho_2 = 1$ and to have conservation of momentum by setting $\varrho_1 a_1 + \varrho_2 a_2 = \overline{u}$. These solutions all satisfy Definition 1 and the Oleinik condition.

The Oleinik condition is satisfied in the sense that the function $u(t,\cdot)$ in (6) may be replaced equivalently almost everywhere with respect to $m(t,\cdot)$ by a linear interpolant between the values $0, a_1, a_2, 0$ that has a maximal slope E/t for an appropriate constant E, since the values of u on the set where the mass density vanishes may be chosen arbitrarily. Note that this extension of the Oleinik condition to vacuum states prevents any further splitting of the mass from occuring at times $t > 0$. The slope of any interpolant must become infinite in that case. The factor $1/t$ in the Oleinik condition allows this to occur only at $t = 0$.

However, the Oleinik condition does not reflect the fact that massive particles represented by a singular measure should never separate, not even initially. The above example motivates the following condition suggested by the first author.

Cohesion Condition: For $x_0 \in \mathbb{R}$, if $m_0(\{x_0\}) > 0$, then writing $t \to 0^+$ for the limit from above, i.e. using $t > 0$, we require that

$$\lim_{t\to 0^+} m\left(t, \left\{x \in \mathbb{R}; \left|\frac{x - x_0}{t} - u_0(x_0)\right| \leqslant \varepsilon\right\}\right) = m_0(\{x_0\}) \tag{7}$$

for all $\varepsilon > 0$.

The Oleinik and the cohesion conditions together form an entropy condition for measure solutions of the zero-pressure gas dynamics initial value problem (1) and (2). We point out that the solutions constructed in [4] satisfy this condition as well. Note further that the cohesion condition prevents the splitting up of singular masses at time $t = 0$. But not even both the cohesion and Oleinik's condition together prevent singular masses from separating from pieces of regular mass at time $t = 0$. Similarly one may have also $u_0(x^-) < u_0(x^+)$ for some $x \in \mathbb{R}$ in the regular part. Here $u_0(x^{\pm})$ denotes the limits obtained by approaching x from above and below, respectively. In all allowed cases of initial separation of velocity a vacuum is created due to the conservation of mass. This is different from the creation of rarefaction waves for the Burgers equation using the same initial data u_0. At later times any separation of mass at all is prevented by the Oleinik condition.

In this paper we will highlight the main elements of the uniqueness proof of the solutions to the system (1) with the initial condition (2) under the restriction of the above entropy condition. The full details may be found in [11]. The uniqueness theorem is stated as follows.

Theorem 2 (Uniqueness)**.** *Measure solutions to the system* (1) *under the initial condition* (2) *satisfying the entropy condition are unique in the following sense. Assume that* (m_i, I_i) *for* $i = 1, 2$ *are two measure solutions of* (1) *and* (2) *satisfying Definition* 1 *and the Oleinik condition* (5) *as well as the cohesion condition* (7)*. Then* *they are equal, i.e. the equations*

$$\int\int_{\mathbb{R}^+\times\mathbb{R}} \varphi m_1(t,\,\mathrm{d}x)\,\mathrm{d}t = \int\int_{\mathbb{R}^+\times\mathbb{R}} \varphi m_2(t,\,\mathrm{d}x)\,\mathrm{d}t,$$
$$\int\int_{\mathbb{R}^+\times\mathbb{R}} \varphi I_1(t,\,\mathrm{d}x)\,\mathrm{d}t = \int\int_{\mathbb{R}^+\times\mathbb{R}} \varphi I_2(t,\,\mathrm{d}x)\,\mathrm{d}t \tag{8}$$

hold for all bounded test functions $\varphi \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$.

This result completes the program of giving an existence and uniqueness theory for the initial value problem (1) and (2).

We would like to point out that independently Huang and Wang [7] re-proved the existence result and gave a uniqueness result for solutions to (1) using a different entropy condition. They utilized the Lebesgue-Stieltjes integral to equivalently define weak solutions to (1) in terms of the distribution functions of the measures. Then they proved uniqueness of solutions using techniques quite different from those outlined here. They studied the adjoint problem and needed a new existence as well as uniqueness theorem for a linear equation with discontinuous cooefficients. Their entropy condition consists of the Oleinik condition and an energy condition that states that the energy should be weakly continuous initially. It must be pointed out that the sticky particles in this model lose energy immediately under collision as time evolves, see [2], and so one cannot guarantee that the energy is continuous in the solution. Therefore, their condition does not reflect a natural property of the model. Here we use instead the initial cohesion condition which reflects the physical fact that massive particles never split, not even initially.

3. Generalized characteristics and flow maps

The method of proof for Theorem 2 is basically a method of characteristics for the entropy solutions given by the existence theory. One has to overcome two main difficulties in extending the concept of generalized characteristics to the system (1). One is the possible presence of vacuum states in the solutions, where no velocity is specified. The other is the irregularity of solutions, i.e. the solutions are not necessarily functions of bounded variation but may contain singular measures. For this purpose, we first regularize the entropy solutions so that the generalized characteristics, see Dafermos [5], for the regularized solutions are well-defined in the usual sense. Then generalized characteristics for the irregular problem are obtained in the limit of the vanishing regularization parameter. An important tool in the analysis are the characteristic maps which are flow maps of the flow generated by the generalized characteristics. It is shown that they naturally satisfy the properties of conservation of mass and momentum, which is to be expected. Further, the dynamic behaviour of the center of mass of sets, subjected to the characteristic maps in time, is studied. The results thus obtained finally lead to our uniqueness proof. As a by-product, we also verify the Generalized Variational Principle used in [6].

The regularization procedure is rather technical. First note that the mass m is always a nonnegative measure. Using the Galilean invariance of the system, i.e. going to a reference frame moving with the largest negative velocity, one may assume all velocities to be nonnegative. This implies that the momentum I may also be assumed to be nonnegative. It means that some proofs concerning properties of I are identical to those for m. The distribution functions of the measures for mass m and momentum I are regularized by a suitable space-time averaging. It can be shown that the limit of taking the regularization parameter to zero gives a measure equivalent to the original one, for details see [11].

The generalized characteristics are Lipschitz curves that are generalized solutions to the ordinary differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = u(t,x)$ for the double eigenvalue u. If u is a smooth function, they are just the forward in time solution curves of this ordinary differential equation through the point (t_0, x_0) denoted by $x = x^0(s; t_0, x_0)$, $s \geqslant t_0$. Here these notions are not only extended to the well known case where u has jumps, see Dafermos [5], but also to the case of characteristics near a vacuum state or a singular measure. For this purpose the velocity function u has to be extended into the vacuum by linear interpolation or extension and then regularized. This has to be done in a manner compatible with the Oleinik condition. As a result one can determine the generalized characteristics in the regularized situation. The extended generalized characteristics needed here are then obtained by taking uniform limits for the vanishing regularization parameter, see [11]. For all $\tau > 0$ and $x_0 \in \mathbb{R}$, these extended generalized characteristic curves $x = x^0(t) = x^0(t; \tau, x_0)$ with $x^0(\tau) = x_0$ again satisfy the differential equation $\frac{\mathrm{d}x^0(t)}{\mathrm{d}t^+} = u(t, x^0(t))$ almost everywhere with respect to $m(t,\cdot)$. The derivative $\frac{\mathrm{d}}{\mathrm{d}t^+}$ denotes the one-sided derivative from above in time.

The generalized characteristic curves $x = x^0(s; t_0, x_0)$, $s \geqslant t_0$, once defined, allow the introduction of the family of *characteristic* or *flow maps*

$$x = h_{t_0,s}(x_0) = x^0(s; t_0, x_0), \quad s \geqslant t_0. \tag{9}$$

Also needed are the inverse maps $h^{-1}_{t_0,s}$ which are not always proper inverses as point mappings, because mass can accumulate. They may be set valued mappings, i.e. the image $h^{-1}_{t_0,s}(x)$ may be a set. Due to this property one may prefer to think of characteristic maps as maps between subsets of $\mathbb{R}$ from the outset. If the solution contains vacuum states the set $h^{-1}_{t_0,s}(x)$ may have gaps. In this case we add these gaps to $h^{-1}_{t_0,s}(x)$ so that the set will always be an interval. Important properties of the flow maps proved in [11] are given in the following lemma.

Lemma 3. *The maps* $h_{t_0,t}\colon \mathbb{R} \to \mathbb{R}$ *are non-decreasing and continuous. Furthermore, they satisfy*

$$h_{t_1,t_3}(x) = h_{t_2,t_3}\big(h_{t_1,t_2}(x)\big) \tag{10}$$

for all $\tau \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant T$.

Further, one can prove

Theorem 4. *For all bounded measurable functions* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *we have the conservation of mass*

$$\int_{-\infty}^{\infty} \varphi(x) m(T,\, \mathrm{d}x) = \int_{-\infty}^{\infty} \varphi(h_{\tau,T}(x)) m(\tau,\, \mathrm{d}x) \tag{11}$$

for all $0 \leqslant \tau < T < \infty$.

This theorem directly leads to the following conservation of mass and momentum property. For all Borel sets $A \in \mathcal{B}$ and $T > \tau$ we have

$$m(T, A) = m(\tau, h^{-1}_{\tau,T}(A)), \qquad I(T, A) = I(\tau, h^{-1}_{\tau,T}(A)). \tag{12}$$

An important concept for our analysis is the center of mass of a set and its dynamic behavior. We denote the first m o m e n t o f m a s s of a Borel set $a \in \mathcal{B}$ by

$$K(t, A) := \int_A \xi m(t,\, \mathrm{d}\xi), \qquad t \geqslant 0. \tag{13}$$

Then the c e n t e r o f m a s s of a set $A \in \mathcal{B}$ with $m(t, A) > 0$ is defined as

$$M(t, A) := \frac{K(t, A)}{m(t, A)}. \tag{14}$$

Consider a point x such that $m(t, \{x\}) = 0$ but the measure does not vanish in any open neigborhood of x, i.e. x does not lie in the vacuum. Then we can still define the center of mass for the one point set $\{x\} = A$. Note that by the Lebesgue Theorem one has $M(t, A) = x$ for a Lebesgue point x by taking the limit of the center of mass on open neighborhoods of x. This holds almost everywhere in the sense of the Lebesgue measure. We may assume that this holds everywhere in the complement of the vacuum set.

The following formula gives the shift of the center of mass of a set under the flow map due to the characteristic velocity. For all $A \in \mathcal{B}$, if $m(t, A) > 0$ and $t \geqslant \tau$, then

$$M(t, A) = \frac{\int_{h_{\tau,t}^{-1}(A)} (\xi + (t-\tau)u(\tau,\xi)) m(\tau, \mathrm{d}\xi)}{\int_{h_{\tau,t}^{-1}(A)} m(\tau, \mathrm{d}\xi)}. \tag{15}$$

For the shifted center of mass of any Borel set $B \in \mathcal{B}$ we introduce the notation

$$C(B; \tau, t) := \frac{\int_B [\xi + (t-\tau)u(\tau,\xi)] m(\tau, \mathrm{d}\xi)}{\int_B m(\tau, \mathrm{d}\xi)}. \tag{16}$$

This quantity tells us where the center of mass of the set B considered at time τ would be at a later time t, provided the generalized characteristics originating in B do not interact with others that start outside B at time τ. Note that the set B here is not necessarily an image under the map $h_{\tau,t}^{-1}$ as required for the validity of formula (15). In view of (15) we have $M(t, A) = C(h_{\tau,t}^{-1}(A); \tau, t)$.

Let $0 < \tau < T < \infty$ and $y \in \mathbb{R}$, and let us set $[a, b] := h_{\tau,T}^{-1}(y)$. Then we have $C([a, b]; \tau, T) = y$. One is mostly interested in the case $a < b$, which means that generalized characteristics are merging and mass is being accumulated in y. Taking any $x_0 \in [a, b]$, one considers the generalized characteristic $x^0(t) = x^0(t; x_0, \tau)$ emerging there. All generalized characteristics originating in the interval $[a, b]$ must pass through y at time T, i.e. satisfy $x^0(T) = y$. The fundamental inequalities

$$C([a, x_0]; \tau, T) \geqslant y \geqslant C([x_0, b]; \tau, T) \tag{17}$$

may be proven for all $x_0 \in [a, b]$. The possibility that the strict inequality holds comes from the fact that the sets $[a, x_0]$ and $[x_0, b]$ are not images under the map $h_{\tau,t}^{-1}$ unless $a = x_0 = b$.

Let $[a(t), b(t)] = h_{t,T}^{-1}(\{y\})$ for all $t \in [\tau, T]$ and $y \in \mathbb{R}$ not in the vacuum set at time T. Now the uniqueness proof works along the following lines. At time T one studies the initial intervals $[a(0), b(0)]$ for two solutions to the same initial data that differ at y. It is then proven via a number of technical results that the parts of the intervals that are not in their intersection must have mass zero. Assuming that this is not the case, the behaviour of the center of mass of the difference of the intervals is studied and a contradiction can be derived. The complications in the proof are essentially twofold. One is due to the fact that at $a(0)$ and $b(0)$ an initial separation of mass may occur. The other comes from the fact that the interval $[a(0), b(0)]$ may contain vacuum sets.

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Authors' addresses: *Jiequan Li*, Institute of Mathematics, Academia Sinica, Beijing, 100080, P. R. China, e-mail: `jiequan@math.sinica.edu.tw`; *Gerald Warnecke*, Institut für Analysis und Numerik, Otto-von-Guericke-Universität Magdeburg, Postfach 4120, 39016 Magdeburg,Germany, e-mail: `gerald.warnecke@mathematik.uni-magdeburg.de`.