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Large orbits of operators and operator semigroups

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LARGE ORBITS OF OPERATORS AND OPERATOR SEMIGROUPS

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Abstract. We give a survey of results concerning "large" orbits of operators and strongly continuous operator semigroups.

1. Introduction

Denote by $B(X)$ the set of all bounded linear operators acting on a Banach space X. For simplicity we assume that all Banach spaces are complex unless stated explicitly otherwise. However, all the notions make sense also in real Banach spaces and most of the results remain true (with slight modifications) in the real case as well.

Let X be a Banach space and let $T \in B(X)$. By an orbit of T we mean a sequence of the form $(T^n x)_{n=0}^{\infty}$, where $x \in X$ is a fixed vector.

By a weak orbit of T we mean a sequence of the form $(\langle T^n x, x^* \rangle)_{n=0}^{\infty}$, where $x \in X$ and x^* are fixed vectors.

Orbits of operators appear frequently in operator theory. They are closely connected with the famous invariant subspace/subset problem, they play a central role in the linear dynamics and appear also in other branches of functional analysis, for example in harmonic analysis, local spectral theory or in the theory of operator semigroups.

Typically, the behaviour of an orbit $(T^n x)$ depends essentially on the choice of the initial vector x. This can be illustrated by the following simple example:

Example 1.1. Let S be the backward shift on a Hilbert space H, i.e., $Se_0 = 0$ and $Se_i = e_{i-1} \quad (i \geq 1)$, where $\{e_i : i = 0, 1, 2, \dots\}$ is an orthonormal basis in H. Let $T = 2S$. Then:

(i) there is a dense subset of points $x \in H$ such that $\|T^n x\| \to 0$;
(ii) there is a dense subset of points $x \in H$ such that $\|T^n x\| \to \infty$;
(iii) there is a residual subset (= complement of a set of the first category) of points $x \in H$ such that the set $\{T^n x : n = 0, 1, \dots\}$ is dense in H.

Vectors of type (iii) (i.e., with dense orbits) are called hypercyclic. Hypercyclic vectors (and similar classes of vectors, like supercyclic, chaotic, frequently hypercyclic, mixing, weakly mixing vectors etc.) have been studied intensely by many authors. We refer to [BMa] or [GP] for a detailed information about these vectors.

The aim of this note is to give a survey of results concerning vectors with "large" orbits (for example orbits of type (ii) in the above example).

Apart from the orbits of operators we consider also orbits of operator semigroups. Let $\mathcal{T} = (T(t))_{t \geq 0}$ be a strongly continuous semigroup of operators on a Banach space X, i.e, a mapping

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$T : [0,\infty) \to B(X)$ continuous in the strong operator topology such that $T(0) = I_X$ and $T(t+s) = T(t)T(s)$ for all $t, s \in [0,\infty)$. By an orbit of $\mathcal{T}$ we mean a function $t \mapsto T(t)x \quad (t \geq 0)$ where $x \in X$ is a fixed vector. Similarly, a weak orbit of $\mathcal{T}$ is a mapping $t \mapsto \langle T(t)x, x^* \rangle$ where $x \in X$ and $x^* \in X^*$ are fixed.

2. Large orbits

There are various possibilities how an orbit $\{T^n x : n \in \mathbb{N}\}$ can be large. We consider the following possibilities:
A. $\|T^n x\|$ is large for infinitely many powers n;
B. $\|T^n x\|$ is large for all n;
C. $\|T^n x\|$ is large for many n (for example, for all n in a set of density 1, or of positive upper density);
D. various summability conditions (for example $\sum \|T^n x\|/\|T^n\| = \infty$).

A. Orbits of the first type are described by the Banach-Steinhaus theorem.

Theorem 2.1. *Let $T \in B(X)$. The following statements are equivalent:*
(i) $\sup_n \|T^n\| = \infty$;
(ii) there exists $x \in X$ such that $\sup_n \|T^n x\| = \infty$;
(iii) there exists a residual subset of vectors x with the property that $\sup_n \|T^n x\| = \infty$.

More precisely, it is possible to prove the following stronger result:

Theorem 2.2. *Let $T \in B(X)$, let $(a_n)_{n\geq 0}$ be a sequence of positive numbers such that $a_n \to 0$. Then the set of all $x \in X$ with the property that*

$$\|T^n x\| \geq a_n \|T^n\| \qquad \text{for infinitely many powers } n$$

is residual.

Indeed, for $k \in \mathbb{N}$ set

$$M_k = \big\{x \in X : \text{ there exists } n \geq k \text{ such that } \|T^n x\| > a_n \|T^n\|\big\}.$$

Clearly, M_k is an open set. We prove that M_k is dense. Let $x \in X$ and $\varepsilon > 0$. Choose $n \geq k$ such that $a_n \varepsilon^{-1} < 1$. There exists $z \in X$ of norm 1 such that $\|T^n z\| > a_n \varepsilon^{-1} \|T^n\|$. Then

$$2a_n \|T^n\| < \|T^n(2\varepsilon z)\| \leq \|T^n(x+\varepsilon z)\| + \|T^n(x - \varepsilon z)\|,$$

and so either $\|T^n(x+\varepsilon z)\| > a_n\|T^n\|$ or $\|T^n(x-\varepsilon z)\| > a_n \|T^n\|$. Thus either $x + \varepsilon z \in M_k$ or $x - \varepsilon z \in M_k$, and so $\operatorname{dist}\{x, M_k\} \leq \varepsilon$. Since x and ε were arbitrary, the set M_k is dense.

By the Baire category theorem, the intersection $\bigcap_{k=1}^\infty M_k$ is a dense G_δ-set, hence it is residual. Clearly, each $x \in \bigcap_{k=1}^\infty M_k$ satisfies $\|T^n x\| > a_n \|T^n\|$ for infinitely many powers n.

The previous simple result can be applied for example in the local spectral theory.

Definition 2.3. Let $T \in B(X)$ and $x \in X$. The local spectral radius of T at x is defined by $r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$.

Recall that the spectral radius $r(T)$ of T satisfies $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$.

It is easy to see that $r_T(x) \le r(T)$ for each $x \in X$ and $r(T^n) = r(T)^n$ for all n.

If we set $a_n = n^{-1}$ in Theorem 2.2, we obtain

$$r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{1/n} \ge \limsup_{n\to\infty} \left(\frac{\|T^n\|}{n}\right)^{1/n} = r(T)$$

for each x satisfying $\|T^n x\| \ge n^{-1}\|T^n\|$ for infinitely many n.

Corollary 2.4. *Let $T \in B(X)$. Then the set $\{x \in X : r_T(x) = r(T)\}$ is residual.*

B. It is much more difficult to construct orbits such that $\|T^n x\|$ is large for all n. The following result is based on the spectral theory.

Theorem 2.5. [M1], see also [Be] *Let $T \in B(X)$, let $(a_n)_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $\lim_{n\to\infty} a_n = 0$. Then:*

(i) for each $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| \le \sup\{a_n : j = 0, 1, \dots\} + \varepsilon$ and $\|T^n x\| \ge a_n r(T^n)$ for all $n \ge 0$;

(ii) there is a dense subset L of X with the following property: for each $y \in L$ we have $\|T^n y\| \ge a_n r(T^n)$ for all n sufficiently large.

As mentioned above, the previous result is based on the spectral theory, therefore it requires complex Banach spaces. However, considering the complexification of a real Banach space it is possible to prove statement (i) without changes and a modified version of (ii), see [M2].

Corollary 2.6. *The set $\bigl\{x \in X : \limsup_{n\to\infty} \|T^n x\|^{1/n} = r(T)\bigr\}$ is residual for each $T \in B(X)$. The set $\bigl\{x \in X : \liminf_{n\to\infty} \|T^n x\|^{1/n} = r(T)\bigr\}$ is dense.*

In particular, there is a dense subset of points $x \in X$ with the property that the limit $\lim_{n\to\infty} \|T^n x\|^{1/n}$ exists and is equal to $r(T)$.

For every operator T we have $r(T) = \inf_n \|T^n\|^{1/n} = \inf_n \sup_{\substack{x\in X\\ \|x\|=1}} \|T^n x\|^{1/n}$. The previous corollary implies that the infimum and supremum can be interchanged.

Corollary 2.7. *Let $T \in B(X)$. Then*

$$\sup_{\substack{x\in X\\ \|x\|=1}} \inf_{n\ge1} \|T^n x\|^{1/n} = \inf_{n\ge1} \sup_{\substack{x\in X\\ \|x\|=1}} \|T^n x\|^{1/n} = r(T).$$

In general it is not possible to replace the word "dense" in Corollary 2.6 by "residual".

Example 2.8. Let H be a separable Hilbert space with an orthonormal basis $\{e_j : j = 0, 1, \dots\}$ and let S be the backward shift, $Se_j = e_{j-1} \quad (j \ge 1)$, $Se_0 = 0$. Then $r(S) = 1$ and the set $\bigl\{x \in H : \liminf_{n\to\infty} \|S^n x\|^{1/n} = 0\bigr\}$ is residual.

In particular, the set $\bigl\{x \in H :$ the limit $\lim_{n\to\infty} \|S^n x\|^{1/n}$ exists$\bigr\}$ is of the first category (but it is always dense by Corollary 2.6).

Indeed, for $k \in \mathbb{N}$ let

$$M_k = \{x \in X : \text{ there exists } n \geq k \text{ such that } \|S^n x\| < k^{-n}\}.$$

Clearly, M_k is an open subset of X. Further, M_k is dense in X. To see this, let $x \in X$ and $\varepsilon > 0$. Let $x = \sum_{j=0}^{\infty} \alpha_j e_j$ and choose $n \geq k$ such that $\sum_{j=n}^{\infty} |\alpha_j|^2 < \varepsilon^2$. Set $y = \sum_{j=0}^{n-1} \alpha_j e_j$. Then $\|y - x\| < \varepsilon$ and $S^n y = 0$. Thus $y \in M_k$ and M_k is a dense open subset of X.

By the Baire category theorem, the intersection $M = \bigcap_{k=0}^{\infty} M_k$ is a dense G_δ-subset of X, hence it is residual.

Let $x \in M$. For each $k \in \mathbb{N}$ there is an $n_k \geq k$ such that $\|S^{n_k} x\| < k^{-n_k}$, and so $\liminf_{n\to\infty} \|S^n x\|^{1/n} = 0$.

Since the set $\{x \in H : \limsup_{n\to\infty} \|S^n x\|^{1/n} = r(S) = 1\}$ is also residual, we see that the set $\{x \in H : \text{ the limit } \lim_{n\to\infty} \|S^n x\|^{1/n} \text{ exists}\}$ is of the first category.

Remark 2.9. If $r(T) = 1$ and $a_n > 0$, $a_n \to 0$, then Theorem 2.5 says that there exists x such that $\|T^n x\| \geq a_n$ for all n. This is the best possible result since the previous example $S \in B(H)$ satisfies $S^n x \to 0$ for all $x \in H$. By Theorem 2.5, there are orbits converging to 0 arbitrarily slowly.

Second method of constructing orbits with $\|T^n x\|$ large for all n is based on the plank results of K. Ball:

Theorem 2.10. [B1] *Let X be a real or complex Banach space, $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n < 1$, $f_n \in X^*$, $\|f_n\| = 1 \quad (n \in \mathbb{N})$ and let $y \in X$. Then there exists $x \in X$ such that $\|x - y\| \leq 1$ and*

$$|\langle x, f_n \rangle| \geq a_n$$

for all n.

Theorem 2.11. [B2] *Let X be a complex Hilbert space, $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n^2 \leq 1$, $f_n \in X$, $\|f_n\| = 1 \quad (n \in \mathbb{N})$. Then there exists $x \in X$ such that $\|x\| = 1$ and*

$$|\langle x, f_n \rangle| \geq a_n$$

for all n.

It is interesting to note that there is a difference between real and complex Hilbert spaces. From this point of view real Hilbert spaces are not better than general Banach spaces.

The plank theorems imply easily the existence of large orbits of operators, see [M4], Sec. 37.

Theorem 2.12. *Let X, Y be Banach spaces, let $(T_n) \subset B(X, Y)$ be a sequence of operators. Let (a_n) be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Then there exists $x \in X$ such that $\|T_n x\| \geq a_n \|T_n\|$ for all $n \in \mathbb{N}$.*

Moreover, it is possible to choose such an x in each ball in X of radius greater than $\sum_{n=1}^{\infty} a_n$.

Corollary 2.13. [MV] *Let $T \in B(X)$ satisfy $\sum_{n=1}^{\infty} \|T^n\|^{-1} < \infty$. Then there exists a dense subset of points $x \in X$ such that $\|T^n x\| \to \infty$.*

Better results can be obtained for Hilbert space operators.

Theorem 2.14. [M4] *Let H, K be Hilbert spaces and let $(T_n) \subset B(H,K)$ be a sequence of operators. Let a_n be a sequence of positive numbers such that $\sum_{n=1}^\infty a_n^2 < \infty$. Let $\varepsilon > 0$. Then:*

(i) there exists $x \in H$ such that $\|x\| \le \left(\sum_{n=1}^\infty a_n^2\right)^{1/2} + \varepsilon$ and $\|T_n x\| \ge a_n \|T_n\|$ for all n;
(ii) there is a dense subset of vectors $x \in H$ such that $\|T_n x\| \ge a_n\|T_n\|$ for all n sufficiently large.

Corollary 2.15. [MV] *Let $T \in B(H)$ be a Hilbert space operator and $\sum_{n=1}^\infty \|T^n\|^{-2} < \infty$. Then there exists a dense subset of points $x \in H$ such that $\|T^n x\| \to \infty$.*

Corollary 2.16. *Let $T \in B(X)$ satisfy $\sum_{n=1}^\infty \|T^n\|^{-1} < \infty$. Then T has a non-trivial closed invariant subset.*

If X is a Hilbert space, then it is sufficient to assume that $\sum_{n=1}^\infty \|T^n\|^{-2} < \infty$.

Corollary 2.17. *Let $T \in B(X)$ satisfy $r(T) \ne 1$. Then T has a non-trivial closed invariant subset.*

Indeed, if $r(T) > 1$, then there exists an $x \in X$ with $\|T^n x\| \to \infty$. If $r(T) < 1$, then $\|T^n x\| \to 0$ for each $x \in X$. In both cases the orbit of x is a non-trivial closed subset invariant for T.

The previous results are in some sense the best possible.

Example 2.18. [MV] There exist a Banach space X and an operator $T \in B(X)$ such that $\|T^n\| = n+1$ for all n, but there is no vector $x \in X$ with $\|T^n x\| \to \infty$.

There exists a Hilbert space operator T such that $\|T^n\| = \sqrt{n+1}$ for all n and there in no vector x with $\|T^n x\| \to \infty$.

C. **Definition 2.19.** Let $A \subset \mathbb{N}$. The lower (upper) density of A is defined by

$$\underline{\text{dens}}\, A = \liminf_{n\to\infty} n^{-1} \operatorname{card}\{a \in A : 1 \le a \le n\}$$

and

$$\overline{\text{dens}}\, A = \limsup_{n\to\infty} n^{-1} \operatorname{card}\{a \in A : 1 \le a \le n\},$$

respectively. If $\underline{\text{dens}}\, A = \overline{\text{dens}}\, A$ then this common value is called the density of A and denoted by $\text{dens}\, A$.

Definition 2.20. Let $T \in B(X)$, $x \in X$. We say that the orbit of x is distributionally unbounded if there exists a subset $A \subset \mathbb{N}$ with $\overline{\text{dens}}\, A = 1$ such that $\lim_{n\in A} \|T^n x\| = \infty$.

Theorem 2.21. [BBMP] *Let $T \in B(X)$. Then the following statements are equivalent:*

(i) There exists $x \in X$ with distributionally unbounded orbit;
(ii) there exists a residual subset of points $x \in X$ with distributionally unbounded orbits;
(iii) there exist $x \in X$ and a subset $A \subset \mathbb{N}$ with $\overline{\text{dens}}\, A > 0$ such that $\lim_{n\in A} \|T^n x\| = \infty$.

The following example is a modification of Example 2.18.

Example 2.22. Let $\varepsilon > 0$. Then there exist a Banach space X and an operator $T \in B(X)$ such that $\|T^n\| = (n+1)^{1-\varepsilon}$ for all n and there is no vector x with distributionally unbounded orbit. By the previous theorem, if $x \in X$ and $A \subset \mathbb{N}$ such that $\lim_{n\in A} \|T^n x\| = \infty$ then $\operatorname{dens} A = 0$.

D. It is also possible to consider orbits that are large in the sense of $\sum \frac{\|T^n x\|}{\|T^n\|}$, see [M4], Sec. 37.

Theorem 2.23. *Let $T \in B(X)$, $r(T) \neq 0$ and let $0 < p < \infty$. Then the set*

$$\left\{ x \in X : \sum_{j=0}^{\infty} \left(\frac{\|T^j x\|}{r(T^j)} \right)^p = \infty \right\}$$

is residual.

Theorem 2.24. *Let $T \in B(X)$ be a non-nilpotent operator and $0 < p < 1$. Then the set*

$$\left\{ x \in X : \sum_{j=0}^{\infty} \left(\frac{\|T^j x\|}{\|T^j\|} \right)^p = \infty \right\}$$

is residual.

The previous result is not true for $p = 1$.

Example 2.25. There are a Banach space X and a non-nilpotent operator $T \in B(X)$ such that $\sum_{n=0}^{\infty} \frac{\|T^n x\|}{\|T^n\|} < \infty$ for all $x \in X$.

For complex Hilbert spaces the last two statements are modified in the following way:

Theorem 2.26. *Let H be a complex Hilbert space, let $T \in B(H)$ be a non-nilpotent operator and $0 < p < 2$. Then the set*

$$\left\{ x \in H : \sum_{j=0}^{\infty} \left(\frac{\|T^j x\|}{\|T^j\|} \right)^p = \infty \right\}$$

is residual.

Morover, there exists a non-nilpotent operator $T \in B(H)$ on a separable Hilbert space H such that $\sum_{n=0}^{\infty} \left(\frac{\|T^n x\|}{\|T^n\|} \right)^2 < \infty$ for all $x \in H$.

The results 2.23 - 2.26 can be found in [M4]. The last example - a Hilbert space operator satisfying $\sum \left(\frac{\|T^n x\|}{\|T^n\|} \right)^2 < \infty$ for all x - is an easy modification of Example 2.25. Hint: consider the backward weighted shift with weights $\frac{k+1}{k}$ $\quad (k = 1, 2, \dots)$.

3. Orbits of operator semigroups

Many results from the previous section can be reformulated also for strongly continuous semigroups of operators.

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup of operators on a Banach space X. Let ω_0 be the growth bound of $\mathcal{T}$. Recall that $r(T(t)) = e^{\omega_0 t}$ for all $t \geq 0$. Recall also that $\mathcal{T}$ is locally bounded by the Banach-Steinhaus theorem. Let $K = \sup_{0\leq t\leq 1} \|T(t)\|$.

The local boundedness of operator semigroups enables to reformulate all results for orbits of operators to the semigroup case. Indeed, if $n-1 < t \leq n$ for some positive integer n and $x \in X$ then

$$\|T(n)x\| \leq \|T(n-t)\| \cdot \|T(t)x\| \leq K\|T(t)x\|,$$

so $\|T(t)x\| \geq K^{-1}\|T(n)x\| = K^{-1}\|T(1)^n x\|$. Thus if an orbit of the operator $T(1)$ is large (in any sense), the corresponding orbit of the semigroup $\mathcal{T}$ is also large.

We mention only two sample results:

Theorem 3.1. *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X, let $f : [0,\infty) \to (0,\infty)$ be a bounded function such that $\lim_{t\to\infty} f(t) = 0$.*

Then there exists $x \in X$ such that

$$\|T(t)x\| \geq f(t)e^{\omega_0 t}$$

for all $t \geq 0$.

Moreover, for any $\varepsilon > 0$ is is possible to find such an $x \in X$ with $\|x\| < \sup\{f(t) : t \geq 0\} + \varepsilon$, cf. [N], p. 78.

Theorem 3.2. *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X, let $f : [0,\infty) \to [0,\infty)$ be a non-increasing function, $f \in L^1$.*

Then there exists $x \in X$ such that

$$\|T(t)x\| \geq f(t) \cdot \|T(t)\| \qquad (t \geq 0).$$

If X is a Hilbert space then it is possible to take $f \in L^2$.

4. Weak orbits of operators

Some results for orbits can be generalized also for weak orbits.

The following result can be obtained by applying the plank theorem twice.

Theorem 4.1. [M4], Sec. 39. *Let $T \in B(X)$, $a_n \geq 0$, $\sum_{n=1}^{\infty} \sqrt{a_n} < \infty$. Then there exist $x \in X$ and $x^* \in X^*$ such that*

$$|\langle T^n x, x^*\rangle| \geq a_n \|T^n\| \qquad (n \in \mathbb{N}).$$

Moreover, given any balls $B \subset X$ and $B' \subset X^*$ of radii greater than $\sum a_n^{1/2}$, then it is possible to find such $x \in B$ and $x^* \in B'$.

If X is a Hilbert space, then it is sufficient to require that $\sum a_n < \infty$.

Generalizations of results based on the spectral theory (for example Theorem 2.5) are more complicated. Usually it is necessary to assume that T is power bounded or that $T^n \to 0$ in some sense.

Theorem 4.2. [BM] *Let H be a Hilbert space, $T \in B(H)$, $T^n \to 0$ in the weak operator topology, let $r(T) = 1$. Let (a_n) be any sequence of nonnegative numbers such that $a_n \to 0$. Then there exists $x \in H$ such that*

$$|\langle T^n x, x\rangle| \geq a_n$$

for all $n \in \mathbb{N}$.

In particular, the statement is true for all completely non-unitary contractions T satisfying $r(T) = 1$.

A better result can be obtained if we assume that $1 \in \sigma(T)$.

Theorem 4.3. [BM] *Let H be a Hilbert space, $T \in B(H)$, $T^n \to 0$ in the weak operator topology, $1 \in \sigma(T)$, let (a_n) be a sequence of nonnegative numbers, $a_n \to 0$. Then there exists $x \in H$ such that*

$$\mathrm{Re}\,\langle T^n x, x\rangle \geq a_n \qquad (n \in \mathbb{N}).$$

Let X be a Banach space. A subset $C \subset X$ is called a cone if $C + C \subset C$ and $tC \subset C$ for each $t \geq 0$.

Corollary 4.4. [M3] *Let H be a Hilbert space, let $T \in B(H)$ be power bounded and $1 \in \sigma(T)$. Then T has a nontrivial closed invariant cone.*

Indeed, a standard reduction technique gives that without loss of generality we may assume that $T^n \to 0 \quad (WOT)$. Let x be the vector constructed in the previous theorem. Let C be the closed cone generated by the vectors $T^n x \quad (n \geq 0)$. Then $\mathrm{Re}\,\langle y, x\rangle \geq 0$ for all $y \in C$, and so C is non-trivial since $-x \notin C$. Clearly C is a closed cone invariant for T.

Similar results can be proved also for Banach space operators, see [M3].

Theorem 4.5. *Let X be a Banach space, $c_0 \not\subset X$, $T \in B(X)$, $T^n \to 0$ in the strong operator topology, $1 \in \sigma(T)$. Then there exist nonzero vectors $x \in X, x^* \in X^*$ such that*

$$\mathrm{Re}\,\langle T^n x, x^*\rangle \geq 0$$

for all $n \in \mathbb{N}$.

Corollary 4.6. *Let X be a reflexive Banach space, $T \in B(X)$ power bounded, $1 \in \sigma(T)$. Then T has a nontrivial closed invariant cone.*

We apply now the previous results to unitary operators. This gives statements about Fourier coefficients of L^1 functions.

Let μ be a non-negative finite Borel measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Consider the Hilbert space $H := L^2(\mu)$ and the operator $U_\mu \in B(H)$ deefined by $(U_\mu f)(z) = zf(z) \quad (f \in H, z \in \mathbb{T})$. Clearly U_μ is a unitary operator.

Let $f, g \in H$. Then

$$\langle U_\mu^n f, g\rangle = \int z^n f\bar{g}\,\mathrm{d}\mu,$$

so the weak orbit of U_μ is formed by the Fourier coefficients of the function $\bar{f}g \in L^1(\mu)$.

Recall that μ is called Rajchman if its Fourier transform $\hat{\mu}(n) = \int e^{-2\pi int} d\mu(t)$ vanishes at infinity, i.e., $\lim_{|n|\to\infty} \hat{\mu}(n) = 0$. In particular, each absolutely continuous measure is Rajchman (the converse is not true). It is easy to see that μ is Rajchman if and only if $U_\mu^n \to 0$ in the weak operator topology.

Theorem 4.7. [BM] *Let μ be a Rajchman measure. Let $(a_n)_{n=0}^\infty$ be a sequence of positive numbers satisfying $a_n \to 0$ and $\sup a_n < 1$. Then there exists $f \in L^1(\mu)$ of norm 1 such that $f \geq 0$ a.e. and $|\hat{f}(n)| > a_{|n|}$ for all integers n.*

If $1 \in \operatorname{supp} \mu$ then it is possible to find $f \geq 0$ such that $\operatorname{Re} \hat{f}(n) > a_{|n|}$ for all n.

The previous theorem is not true in general for non-Rajchman measures. For an example see [BM].

Example 4.8. There exists a unitary operator $U \in B(H)$ and a sequence (a_n) of positive numbers with $a_n \to 0$ such that there are no $x, y \in H$ satisfying

$$|\langle U^n x, y \rangle| \geq a_n$$

for all $n \in \mathbb{N}$.

The following result is an example of weak orbits that are large in the sense of density.

Theorem 4.9. [MT] *Let T be a power bounded operator on a Hilbert space H. Let $r(T) = 1$, let (a_n) be a sequence of non-negative numbers such that $\lim_{n\to\infty} a_n = 0$. Then there exist $x \in H$, $\|x\| = 1$ and a subset $A \subset \mathbb{N}$ with $\operatorname{dens} A = 1$ such that*

$$|\langle T^n x, x \rangle| \geq a_n$$

for all $n \in A$.

Theorem 4.10. *Let $T \in B(X)$, $r(T) \geq 1$, let (a_n) be a sequence of non-negative numbers such that $\lim_{n\to\infty} a_n = 0$. Then there exist $x \in X$, $x^* \in X^*$, $\|x\| = \|x^*\| = 1$ and a subset $A \subset \mathbb{N}$ with $\overline{\operatorname{dens}}\, A = 1$ such that*

$$|\langle T^n x, x^* \rangle| \geq a_n$$

for all $n \in A$.

The following results are an analogy to Theorems 2.23 – 2.26, see [M4], Sec. 39.

Theorem 4.11. *Let $T \in B(X)$, $r(T) \neq 0$ and let $0 < p < \infty$. Then the set*

$$\left\{ (x, x^*) \in X \times X^* : \sum_{j=0}^{\infty} \left(\frac{|\langle T^j x, x^* \rangle|}{r(T^j)} \right)^p = \infty \right\}$$

is residual in $X \times X^*$.

Theorem 4.12. *Let $T \in B(X)$ be a non-nilpotent operator and $0 < p < 1/2$. Then the set*

$$\left\{ (x, x^*) \in X \times X^* : \sum_{j=0}^{\infty} \left(\frac{|\langle T^j x, x^* \rangle|}{\|T^j\|} \right)^p = \infty \right\}$$

is residual in $X \times X^*$.

If X is a complex Hilbert space then the result is true for all p, $0<p<1$.

Again the conditions on p cannot be improved.

Example 4.13. There are a Banach space X and a non-nilpotent operator $T\in B(X)$ such that $\sum_{n=0}^\infty\left(\frac{|\langle T^nx,x^*\rangle|}{\|T^n\|}\right)^{1/2}<\infty$ for all $x\in X$ and $x^*\in X^*$.

There exists a non-nilpotent Hilbert space operator $T\in B(H)$ and a such that $\sum_{n=0}^\infty\frac{|\langle T^nx,y\rangle|}{\|T^n\|}<\infty$ for all $x,y\in X$.

We finish this section with a results concerning large weak orbits of the type $\langle T^nx,x\rangle$ for a Hilbert space operator T. This type of orbits is connected with questions concerning the joint numerical ranges.

Theorem 4.14. [DM] *Let T be an operator on a Hilbert space H. Let $a_n\ge 0$, $\sum_{n=1}^\infty a_n^{1/2}<1$. Then there exists a unit vector $x\in H$ such that*

$$|\langle T^nx,x\rangle|\ge\frac{a_n}{4}\|T^n\|$$

for all $n\in\mathbb{N}$.

We do not know if the condition $\sum a_n^{1/2}<1$ can be improved.

5. Weak orbits of operator semigroups

It is also possible to prove some results concerning large weak orbits for semigroups of operators. We refer to [MT] for the proofs and more details.

We start with an analogy to Theorem 4.2.

Theorem 5.1. *Let H be a Hilbert space, $\mathcal{T}=(T(t))_{t\ge0}$ a strongly continuous semigroup on H, $\mathcal{T}$ weakly stable (i.e., $T(t)\to0$ in the weak operator topology as $t\to\infty$), let $\omega_0=0$. Let $f:[0,\infty)\to[0,\infty)$ be a bounded function, $\lim_{t\to\infty}f(t)=0$. Then there exists $x\in H$ such that*

$$|\langle T(t)x,x\rangle|\ge f(t)\qquad(t\ge0).$$

Moreover, for any $\varepsilon>0$ it is possible to find such an $x\in H$ with $\|x\|\le\sup\{f(t):t\ge0\}+\varepsilon$.

The only known application of the plank theorem for weak orbits of operator semigroups is the following result.

Theorem 5.2. *Let H be a Hilbert space, $\mathcal{T}=(T(t))_{t\ge0}$ norm continuous semigroup and $\varepsilon>0$ Then there exist $x,y\in H$ such that*

$$|\langle T(t)x,y\rangle|\ge\frac{1}{(t+1)^{2+\varepsilon}}\|T(t)\|$$

for all $t\ge0$.

The upper density of a subset $A \subset [0, \infty)$ is defined by

$$\overline{\text{dens}}\, A = \limsup_{t\to\infty} t^{-1} m\big(A \cap [0,t]\big),$$

where m is the Lebesgue measure. Similarly, the density of A is defined by

$$\text{dens}\, A = \lim_{t\to\infty} t^{-1} m\big(A \cap [0,t]\big),$$

if the limit exists.

Theorem 5.3. *Let H be a Hilbert space, $\mathcal{T} = (T(t))_{t\ge 0}$ a bounded strongly continuous semigroup on H, $\omega_0 = 0$. Let $f : [0,\infty) \to [0,\infty)$ be a bounded function, $\lim_{t\to\infty} f(t) = 0$. Then there exists $x \in H$, $\|x\| = 1$ and $B \subset [0,\infty)$ with $\text{dens}\, B = 1$ such that*

$$|\langle T(t)x, x\rangle| \ge f(t)$$

for all $t \in B$.

The next result is an application of the previous theorem to Fourier transforms.

Theorem 5.4. *Let μ be a finite positive Borel measure on $\mathbb{R}$, let $f : [0,\infty) \to [0,\infty)$ be a function satisfying $\lim_{t\to\infty} f(t) = 0$. Then there exists a positive function $g \in L^1(\mu)$ and a subset $B \subset [0,\infty)$ with $\text{dens}\, B = 1$ such that the Fourier transform $\hat g$ satisfies*

$$|\hat g(t)| \ge f(|t|)$$

for all $t \in B \cup (-B)$.

Theorem 5.5. *Let $\mathcal{T} = (T(t))_{t\ge 0}$ be a strongly continuous semigroup on a Banach space X with generator A, let $\sigma(A) \cap \{z : \text{Re}\, z \ge 0\} \ne \emptyset$. Let $f : [0,\infty) \to [0,\infty)$ be a function such that $\lim_{t\to\infty} f(t) = 0$. Then there exist $x \in X$, $x^* \in X^*$ with $\|x\| \le 1, \|x^*\| \le 1$ and a set $B \subset [0,\infty)$ with $\overline{\text{dens}}\, B = 1$ such that*

$$|\langle T(t)x, x^*\rangle| \ge f(t)$$

for all $t \in B$.

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