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**Fraïssé sequences: category-theoretic
approach to universal homogeneous
structures**

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Abstract

We develop a category-theoretic framework for universal homogeneous objects, with some applications in the theory of Banach spaces, linear orderings, and in the topology of compact Hausdorff spaces.

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1 Introduction

When studying a given class of structures, it is an interesting and important issue to find a “special” object that is universal for the class, namely, every other object “embeds” into this special one. A better property of this special object would be some sort of homogeneity, namely, that every isomorphism between two “small” substructures extends to an automorphism of the special object (as we shall see later, the exact meaning of “small” will depend on the class under consideration). Probably one of the earliest results in this spirit was Cantor’s theorem on the uniqueness of the set of rational numbers among all countable linear orders. More precisely, \mathbb{Q} is the only (up to isomorphism) countable linear order which contains all other

countable linear orders and every isomorphism between two finite subsets extends to an automorphism of \mathbb{Q} . A typical statement of Cantor’s theorem is different: the extension property is equivalent to saying that \mathbb{Q} is order dense and has no endpoints. The argument used in the proof is now called the “back-and-forth” method: an automorphism extending a given finite isomorphism is constructed inductively, interchanging the domain and the co-domain at each step.

It was Urysohn who found an analogue of Cantor’s back-and-forth argument¹ for metric spaces. Namely, he found a complete separable metric space \mathbb{U} which has very similar properties to the rationals, now considering isometries between finite subsets. Urysohn’s work [40] was actually forgotten for many years; it received some significant attention in the end of the 20th century due to problems in topological dynamics; perhaps the most representative work is [22].

One of the most important works on this subject, independent of Urysohn, was done by Roland Fraïssé [12] in 1954. This is a model-theoretic approach to the back-and-forth argument, which can be partially applied also to the case of Urysohn’s metric space. Roughly speaking, Fraïssé considers a class \mathfrak{K} of finite (or, at least, finitely generated) models of some first order language. The class should have the amalgamation property, that is, each two embeddings of the same model $Z \in \mathfrak{K}$ can be extended to a further embedding into a bigger model in \mathfrak{K} . In other words, given two embeddings $f: Z \rightarrow X$, $g: Z \rightarrow Y$, there should exist a model $W \in \mathfrak{K}$ and embeddings $f': X \rightarrow W$ and $g': Y \rightarrow W$ satisfying $f' \circ f = g' \circ g$. If such a class has only countably many isomorphic types and each two models embed into a common one (the joint embedding property), then there exists a countable model \mathbb{U} that can be represented as the union of a chain of models from the class \mathfrak{K} , contains isomorphic copies of all models in \mathfrak{K} and has the following strong homogeneity property: every isomorphism between submodels of \mathbb{U} which are in \mathfrak{K} extends to an automorphism of \mathbb{U} . Furthermore, the model \mathbb{U} is unique up to isomorphism. It is often called the *Fraïssé limit* of the class \mathfrak{K} .

This line of investigation was further continued by Jónsson [17] and Morley & Vaught [30] (see also Yasuhara [41]), where uncountable classes of models were studied. Besides the amalgamation property, typical cardinal-arithmetic assumption $\kappa = 2^{<\kappa}$ is needed for the existence of the universal homogeneous structure of cardinality κ . One has to mention a curious independent work of Trnková [39] with metamathematical results on universal categories, in the setting of Bernays-Gödel set theory. One of the main tools is the amalgamation property for certain classes of categories, treated just as first-order structures.

All the authors cited above assume that the class of structures is closed under unions of chains of length less than the size of the universal homogeneous structure (in the last case, the union of any chain of small categories is a category). One of the objectives of this work is to relax this assumption and to make the theory general enough for capturing new cases and obtaining new examples of universal

¹ It seems that the back-and-forth argument was actually developed by Huntington (1904) and Hausdorff (1914), see http://en.wikipedia.org/wiki/Cantor_back-and-forth_method.

homogeneous objects.

We believe that category theory is the proper language for Fraïssé-Jónsson limits. In fact, this has already been confirmed by the works of Droste & Göbel [10, 11], where the authors consider some categories of first-order structures with special types of embeddings, obtaining new applications in algebra and theoretical computer science. In this context, one has to mention a recent work of Pech & Pech [34] where the authors, based on the results of Droste & Göbel, develop the theory of Fraïssé limits in comma categories, leading to universal homomorphisms and universal retractions. Finally, Irwin & Solecki [14] presented a variant of Fraïssé theory with reversed arrows, i.e., epimorphisms of finite structures instead of embeddings. By this way, they obtained an interesting new characterization of the *pseudo-arc*—a certain connected compact metric space, never associated to Fraïssé limits before.

There is no doubt that universal structures with strong homogeneity properties can be discovered or identified in various areas of pure mathematics, theoretical computer science and even mathematical physics (see [8]). Model-theoretic Fraïssé limits are nowadays important objects of study in combinatorics, permutation group theory and topological dynamics, see Macpherson’s survey [29] for more information and further references. Category theory brings much more freedom for dealing with Fraïssé limits, offering the possibility of constructing new objects from old and eliminating superfluous assumptions. It has been demonstrated in [10, 11, 34] that category-theoretic approach brings new important examples of universal homogeneous objects and the work [14] shows that one of the simplest constructions in category theory, namely, passing to the opposite category, leads to new and somewhat surprising examples. Actually, one of the author’s construction of a universal pre-image for a certain class of compact linearly ordered spaces [24] turns out to be the Fraïssé limit of a category whose arrows are increasing quotient maps.

Summarizing, category-theoretic approach offers powerful tools for constructing new universal homogeneous objects as well as identifying existing objects, discovering their homogeneity properties.

Here a notational issue has to be pointed out. Namely, in category theory the notion of a “universal object” is totally different from the notion of a “universal structure” in model theory and related areas. In order to avoid this confusion, we shall replace the adjective “universal” by “cofinal” in the latter case. Namely, an object U is defined to be *cofinal* for a class \mathfrak{K} if every object from \mathfrak{K} embeds into U , where “embedding” will be just an arrow of a category under consideration.



As mentioned above, the aim of this work is extending the theory of Fraïssé-Jónsson limits in the framework of pure category theory. The key point of our approach is dealing with sequences instead of (co-)limits, where a *sequence* is nothing but a functor from an ordinal into the category. The crucial notion is that of a *Fraïssé sequence*, a sequence which is supposed to “converge” to a universal (cofinal, in our terminology) homogeneous object in a bigger category, where homogeneity is meant

with respect to the original category. In other words, we deal with the base category of “small” objects and we use sequences for encoding the category of “large” objects. This approach is similar in spirit to the idea of forcing in set theory, where one deals with approximations of a generic object, often working in the base model instead of its forcing extension.

The paper is organized as follows. Section 2 contains basic notation, including a very short treatment of categories of sequences. Section 3 presents the main ideas of this work: Fraïssé sequences, their existence and properties. Section 4 deals with discontinuity: We describe a way of dealing with Fraïssé sequences that are continuous with respect to some quasi-limiting operator. We also introduce the notion of an amalgamation structure, specializing the result on cofinality of a Fraïssé sequence in the category of sequences. Section 5 contains selected examples of classes of categories with Fraïssé sequences, demonstrating the ideas of previous sections. In particular, we discuss categories of functors (diagrams), monoids, and certain categories of trees. We also present an example of a category with many incomparable Fraïssé sequences. Section 6 deals with categories of embedding-projection pairs, aiming at applications to Banach space theory and topology of compact spaces.

Finally, Section 7 contains applications, the first four of them are equivalent to the Continuum Hypothesis. One of the results there, which actually could fit into the model-theoretic framework, is the existence of a unique Banach space of density \aleph_1 that is isometrically universal for this class of spaces and isometrically homogeneous for separable Banach spaces². It turns out that such a Banach space has not been discovered before. Another result is a Banach space \mathbf{P} , complementably universal for the class of spaces with monotone Schauder bases of length $\leq \omega_1$. More precisely, every Banach space from this class is linearly isometric to a 1-complemented subspace of \mathbf{P} . Yet another result shows the existence of a Banach space of density \aleph_1 that is complementably universal for the class of Banach spaces with the so-called *projectional resolution of the identity*. This class of Banach spaces had been studied extensively by several authors, in particular, in connections with renorming theory, see the monographs [7] and [9] for further references. We also present a dual result in topology, for the class of *Valdivia compact spaces* that can be described in the language of retractions onto metrizable compacta. We show that there exists a Valdivia compact K of weight \aleph_1 such that all other Valdivia compacta of the same weight embed as retracts into K . Both results are straightforward applications of Section 6, by considering two natural categories of embedding-projection pairs. Another result, now without any cardinal-arithmetic assumptions, is the existence of a sequence of continuous functions on the Cantor set which is universal both for quotient maps and for topological embeddings. This is obtained by adapting the ideas of embedding-projection pairs to the category of finite nonempty sets. We finally describe, again without any cardinal-arithmetic assumptions, a linearly ordered set

²There is a recent work [3] elaborating the method of constructing Gurariï-like Banach spaces using amalgamations. The article [3] was however partially inspired by our result, with a reference to an early draft of this work.

of size \aleph_1 which could serve as a natural generalization of the rationals, with homogeneity property involving increasing embeddings and projections at the same time. Its topological counterpart identifies as a Fraïssé limit the author's example [24] of a universal pre-image for the class of linearly ordered Valdivia compacta, thus showing its homogeneity and uniqueness with respect to these properties.

2 Preliminaries

Categories will usually be denoted by letters \mathfrak{K} , \mathfrak{L} , \mathfrak{M} , etc. All functors considered here are assumed to be covariant, unless otherwise specified. Let \mathfrak{K} be a category. We shall write “ $a \in \mathfrak{K}$ ” for “ a is an object of \mathfrak{K} ”. Given $a, b \in \mathfrak{K}$, we shall denote by $\mathfrak{K}(a, b)$ the set of all \mathfrak{K} -morphisms from a to b . The composition of two compatible arrows f and g will be denoted by $g \circ f$. A *subcategory* of \mathfrak{K} is a category \mathfrak{L} such that each object of \mathfrak{L} is an object of \mathfrak{K} and each arrow of \mathfrak{L} is an arrow of \mathfrak{K} (with the same domain and co-domain). We write $\mathfrak{L} \subseteq \mathfrak{K}$. Recall that a subcategory \mathfrak{L} of \mathfrak{K} is *full* if $\mathfrak{L}(a, b) = \mathfrak{K}(a, b)$ for every objects $a, b \in \mathfrak{L}$. We say that \mathfrak{L} is *cofinal* in \mathfrak{K} if for every object $x \in \mathfrak{K}$ there exists an object $y \in \mathfrak{L}$ such that $\mathfrak{K}(x, y) \neq \emptyset$. The opposite category to \mathfrak{K} will be denoted by \mathfrak{K}^{op} . That is, the objects of \mathfrak{K}^{op} are the objects of \mathfrak{K} and all arrows are reversed, i.e. $\mathfrak{K}^{\text{op}}(a, b) = \mathfrak{K}(b, a)$ for every $a, b \in \mathfrak{K}$. Recall that a category \mathfrak{K} is *ordered* if $|\mathfrak{K}(x, y)| \leq 1$ and $\mathfrak{K}(x, y) \neq \emptyset \neq \mathfrak{K}(y, x)$ implies $x = y$ for every \mathfrak{K} -objects x, y . Removing the last condition we get the notion of a *quasi-ordered category*. Every (not necessarily ordered) category induces a partial order \leq on the objects of \mathfrak{K} defined by the formula $\mathfrak{K}(x, y) \neq \emptyset$ iff $x \leq y$. Every partially ordered set $\langle P, \leq \rangle$ can be viewed as an ordered category \mathfrak{K}_P with P the class of objects and the class of arrows defined by $\mathfrak{K}_P(x, y) = \{\langle x, y \rangle\}$ whenever $x \leq y$ and $\mathfrak{K}_P(x, y) = \emptyset$ otherwise. In particular, ordinals treated as well ordered sets are important examples of ordered categories.

Let \mathfrak{K} be a category. We say that \mathfrak{K} has the *amalgamation property* if for every $a, b, c \in \mathfrak{K}$ and for every morphisms $f \in \mathfrak{K}(a, b)$, $g \in \mathfrak{K}(a, c)$ there exist $d \in \mathfrak{K}$ and morphisms $f' \in \mathfrak{K}(b, d)$ and $g' \in \mathfrak{K}(c, d)$ such that $f' \circ f = g' \circ g$. If, additionally, for every arrows f'', g'' such that $f'' \circ f = g'' \circ g$ there exists a unique arrow h satisfying $h \circ f' = f''$ and $h \circ g' = g''$ then the pair $\langle f', g' \rangle$ is a *pushout* of $\langle f, g \rangle$. Reversing the arrows, we define the *reversed amalgamation* and the *pullback*. We say that \mathfrak{K} is *directed* if for every $a, b \in \mathfrak{K}$ there exists $g \in \mathfrak{K}$ such that both sets $\mathfrak{K}(a, g)$, $\mathfrak{K}(b, g)$ are nonempty. In model theory, where the arrows are embeddings, this is usually called the *joint embedding property*.

Fix a category \mathfrak{K} and fix an ordinal $\delta > 0$. An *inductive δ -sequence* in \mathfrak{K} is formally a covariant functor from δ (treated as a poset category) into \mathfrak{K} . In other words, it could be described as a pair of the form $\langle \{a_\xi\}_{\xi < \delta}, \{a_\xi^\eta\}_{\xi < \eta < \delta} \rangle$, where δ is an ordinal, $\{a_\xi : \xi < \delta\} \subseteq \mathfrak{K}$ and $a_\xi^\eta \in \mathfrak{K}(a_\xi, a_\eta)$ are such that $a_\eta^\rho \circ a_\xi^\eta = a_\xi^\rho$ for every $\xi < \eta < \rho < \delta$. We shall denote such a sequence shortly by \vec{a} . The ordinal δ is the *length* of \vec{a} .

Let κ be an infinite cardinal. A category \mathfrak{K} is κ -complete if all inductive sequences of length $< \kappa$ have co-limits in \mathfrak{K} . Every category is \aleph_0 -complete, since the co-limit of a finite sequence is its last object. A category \mathfrak{K} is κ -bounded if for every inductive sequence \vec{x} in \mathfrak{K} of length $\lambda < \kappa$ there exist $y \in \mathfrak{K}$ and a co-cone of arrows $\{y_\alpha\}_{\alpha < \lambda}$ such that $y_\alpha: x_\alpha \rightarrow y$ and $y_\beta \circ x_\alpha^\beta = y_\alpha$ for every $\alpha < \beta < \lambda$. Obviously, every κ -complete category is κ -bounded. We shall write “ σ -complete” and “ σ -bounded” for “ \aleph_1 -complete” and “ \aleph_1 -bounded” respectively.

We shall need the following notion concerning families of arrows. Fix a family of arrows \mathcal{F} in a given category \mathfrak{K} . We shall write $\text{Dom}(\mathcal{F})$ for the set $\{\text{dom}(f) : f \in \mathcal{F}\}$. We say that \mathcal{F} is *dominating* in \mathfrak{K} if the family of objects $\text{Dom}(\mathcal{F})$ is cofinal in \mathfrak{K} and moreover for every $a \in \text{Dom}(\mathcal{F})$ and for every arrow $f: a \rightarrow x$ in \mathfrak{K} there exists an arrow g in \mathfrak{K} such that $g \circ f \in \mathcal{F}$.

For all undefined category-theoretic notions we refer to Mac Lane [28] or Johnstone [16].

2.1 Categories of sequences

Fix a category \mathfrak{K} and denote by $\text{Seq}_{<\kappa}(\mathfrak{K})$ the class of all sequences in \mathfrak{K} which have length $< \kappa$. We shall write $\text{Seq}_{\leq\kappa}(\mathfrak{K})$ instead of $\text{Seq}_{<\kappa+}(\mathfrak{K})$ and $\sigma\mathfrak{K}$ instead of $\text{Seq}_{\leq\aleph_0}(\mathfrak{K})$. We would like to turn $\text{Seq}_{<\kappa}(\mathfrak{K})$ into a category in such a way that an arrow from a sequence \vec{a} into a sequence \vec{b} induces an arrow from $\lim \vec{a}$ into $\lim \vec{b}$, whenever \mathfrak{K} is embedded into a category in which sequences \vec{a}, \vec{b} have co-limits.

Fix two sequences \vec{a} and \vec{b} in a given category \mathfrak{K} . Let $\lambda = \text{dom}(\vec{a})$, $\varrho = \text{dom}(\vec{b})$. A *transformation* from \vec{a} to \vec{b} is, by definition, a natural transformation from \vec{a} into $\vec{b} \circ \varphi$, where $\varphi: \lambda \rightarrow \varrho$ is an order preserving map (i.e. a covariant functor from λ into ϱ , treated as ordered categories).

In order to define an arrow from \vec{a} to \vec{b} we need to identify some transformations. Fix two natural transformations $F: \vec{a} \rightarrow \vec{b} \circ \varphi$ and $G: \vec{a} \rightarrow \vec{b} \circ \psi$. We shall say that F and G are *equivalent* if the following conditions hold:

- (1) For every α there exists $\beta \geq \alpha$ such that $\varphi(\alpha) \leq \psi(\beta)$ and $b_{\varphi(\alpha)}^{\psi(\beta)} \circ F(\alpha) = G(\beta) \circ a_\alpha^\beta$.
- (2) For every α there is $\beta \geq \alpha$ such that $\psi(\alpha) \leq \varphi(\beta)$ and $b_{\psi(\alpha)}^{\varphi(\beta)} \circ G(\alpha) = F(\beta) \circ a_\alpha^\beta$.

It is rather clear that this defines an equivalence relation, which is actually a congruence on the category of transformations. Every equivalence class of this relation will be called an *arrow* (or *morphism*) from \vec{a} to \vec{b} . It is easy to check that this indeed defines a category structure on all sequences in \mathfrak{K} . Formally, this is the quotient category with respect to the equivalence relation described above. The identity arrow of \vec{a} is the equivalence class of the identity natural transformation $\text{id}_{\vec{a}}: \vec{a} \rightarrow \vec{a}$.

Categories of sequences are special cases (or rather “parts”) of more general categories called *Ind-completions*, see Chapter VI of Johnstone’s monograph [16].

We shall later need the following two facts.

Lemma 2.1. *Let \mathfrak{K} be a category, $\kappa \geq \aleph_0$ a regular cardinal, and let \vec{x} be a κ -sequence in $\text{Seq}_{\leq \kappa}(\mathfrak{K})$ whose elements are sequences of length κ . Then \vec{x} has co-limit in $\text{Seq}_{\leq \kappa}(\mathfrak{K})$.*

Proof. Refining inductively each \vec{x}_α ($\alpha < \kappa$) to a cofinal subsequence, we may assume that all bonding maps are natural transformations. Now look at \vec{x} as a functor from $\kappa \times \kappa$ into \mathfrak{K} and let \vec{y} be the diagonal sequence. Then \vec{y} is easily seen to be the co-limit of \vec{x} . \square

Proposition 2.2. *Assume \mathfrak{K} is a κ -complete category, where $\kappa = \text{cf } \kappa \geq \aleph_0$. Then $\text{Seq}_{\leq \kappa}(\mathfrak{K})$ is κ^+ -complete.*

Proof. The fact that \mathfrak{K} is κ -complete implies that so is $\text{Seq}_{\leq \kappa}(\mathfrak{K})$. Finally, κ^+ -completeness follows from Lemma 2.1. \square

2.2 Partial orders and trees

By a *tree* we mean a partially ordered set $\langle T, \leq \rangle$ which is a meet semilattice, i.e. every two elements of T have the greatest lower bound, and for every $t \in T$ the interval $\{x \in T : x < t\}$ is well ordered. Every tree T has a single minimal element 0_T , called the *root* of T . An *immediate successor* of $t \in T$ is an element $s > t$ such that no $x \in T$ satisfies $t < x < s$. The set of all immediate successors of t will be denoted by t^+ . A *subtree* of a tree T is a subset $S \subseteq T$ which is again a semilattice, possibly with a different infimum. A tree T is *binary* if every $t \in T$ has at most two immediate successors. We shall denote by $\max T$ the set of all maximal elements of T . A tree T is *bounded* if for every $x \in T$ there is $t \in \max T$ such that $x \leq t$. Recall that an *initial segment* of a poset $\langle T, \leq \rangle$ is a subset A of T satisfying $\{x \in T : x \leq t\} \subseteq A$ for every $t \in A$. Given a tree T and $t \in T$, we define the *level* of t in T as the order type of the interval $[0, t) = \{s \in T : s < t\}$. We denote it by $\text{Lev}_T(t)$. The *height* of T , $\text{ht}(T)$ is the minimal ordinal α such that no $t \in T$ satisfies $\text{Lev}_T(t) = \alpha$.

For all undefined set-theoretic notions we refer to Jech [15] or Kunen [27].

3 Fraïssé sequences

Below we introduce the key notion of this work.

Let \mathfrak{K} be a category and let κ be a cardinal. A *Fraïssé sequence* of length κ in \mathfrak{K} (briefly: a κ -*Fraïssé sequence*) is an inductive sequence \vec{u} satisfying the following conditions:

- (U) For every $x \in \mathfrak{K}$ there exists $\xi < \kappa$ such that $\mathfrak{K}(x, u_\xi) \neq \emptyset$.
- (A) For every $\xi < \kappa$ and for every arrow $f \in \mathfrak{K}(u_\xi, y)$, where $y \in \mathfrak{K}$, there exist $\eta \geq \xi$ and $g \in \mathfrak{K}(y, u_\eta)$ such that $u_\xi^\eta = g \circ f$.

An inductive sequence satisfying (U) will be called \mathfrak{K} -*cofinal*. More generally, a collection \mathcal{U} of objects of \mathfrak{K} is \mathfrak{K} -*cofinal* if for every $x \in \mathfrak{K}$ there is $u \in \mathcal{U}$ such that $\mathfrak{K}(x, u) \neq \emptyset$. Condition (A) will be called *amalgamation property*.

3.1 Basic properties

Let \vec{v} be a κ -sequence in a category \mathfrak{K} . We say that \vec{v} has the *extension property* if the following holds:

- (E) For every arrows $f: a \rightarrow b$, $g: a \rightarrow v_\alpha$ in \mathfrak{K} , where $\alpha < \kappa$, there exist $\beta \geq \alpha$ and an arrow $h: b \rightarrow v_\beta$ such that $v_\alpha^\beta \circ g = h \circ f$.

Clearly, this condition implies (A).

Proposition 3.1. *Let \vec{u} be a κ -Fraïssé sequence in a category \mathfrak{K} . Then \mathfrak{K} is directed. Moreover, the following conditions are equivalent:*

- (a) \vec{u} has the extension property.
(b) \mathfrak{K} has the amalgamation property.

Proof. The first statement is trivial. Assume (a) and fix arrows $f: z \rightarrow x$, $g: z \rightarrow y$. Using (U), find $h: x \rightarrow u_\alpha$, $\alpha < \kappa$. Using (E), find $\beta \geq \alpha$ and $k: y \rightarrow u_\beta$ such that $k \circ g = u_\alpha^\beta \circ h \circ f$. Thus (b) holds.

Finally, assume (b) and fix arrows $f: a \rightarrow b$ and $g: a \rightarrow u_\alpha$, $\alpha < \kappa$. Using (b), find arrows $f_1: b \rightarrow w$ and $g_1: u_\alpha \rightarrow w$ so that $f_1 \circ f = g_1 \circ g$. Using (A) for the sequence \vec{u} , we find $\beta \geq \alpha$ and $h: w \rightarrow u_\beta$ so that $h \circ g_1 = u_\alpha^\beta$. Thus $(h \circ f_1) \circ f = h \circ g_1 \circ g = u_\alpha^\beta \circ g$, which shows that (a) holds. \square

Proposition 3.2. *Assume \mathfrak{K} is a directed category. Then every sequence in \mathfrak{K} satisfying condition (A) is Fraïssé.*

Proof. Let \vec{u} be a sequence in \mathfrak{K} satisfying (A). Fix $x \in \mathfrak{K}$. Since \mathfrak{K} is directed, there exist $w \in \mathfrak{K}$ and arrows $f: u_0 \rightarrow w$, $g: x \rightarrow w$. Using (A), we find an arrow $h: w \rightarrow u_\xi$ such that $h \circ f = u_0^\xi$. Thus $\mathfrak{K}(x, u_\xi) \neq \emptyset$, which shows (U). \square

Proposition 3.3. *Let \mathfrak{K} be a category, let \vec{u} be an inductive sequence of length κ in \mathfrak{K} and let $S \subseteq \kappa$ be unbounded in κ .*

- (a) *If \vec{u} is a Fraïssé sequence in \mathfrak{K} then $\vec{u} \upharpoonright S$ is Fraïssé in \mathfrak{K} .*
(b) *If \mathfrak{K} has the amalgamation property and $\vec{u} \upharpoonright S$ is a Fraïssé sequence in \mathfrak{K} then so is \vec{u} .*

Proof. Assume \vec{u} is a Fraïssé sequence. Then $\vec{u} \upharpoonright S$ clearly satisfies (U). In order to check (A), fix $f: u_\xi \rightarrow y$ with $\xi \in S$. Then $u_\xi^\eta = g \circ f$ for some arrow g and for some $\eta \geq \xi$. Since S is unbounded in κ , there is $\alpha \in S$ such that $\alpha \geq \eta$. Then $u_\xi^\alpha = u_\eta^\alpha \circ g \circ f$, which shows that $\vec{u} \upharpoonright S$ satisfies (A).

Now assume $\vec{u} \upharpoonright S$ is a Fraïssé sequence. Clearly, \vec{u} satisfies (U). Fix $f: u_\xi \rightarrow y$, $\xi < \kappa$. Find $\alpha \in S$ with $\alpha \geq \xi$. Using the amalgamation property of \mathfrak{K} , find $f': u_\alpha \rightarrow z$ such that the diagram

$$\begin{array}{ccc} u_\alpha & \xrightarrow{f'} & z \\ u_\xi^\alpha \uparrow & & \uparrow g \\ u_\xi & \xrightarrow{f} & y \end{array}$$

commutes for some arrow g in \mathfrak{K} . Now, using (A) for $\vec{u} \upharpoonright S$, we can find $\beta \in S$ such that $\beta \geq \alpha$ and $h \circ f' = u_\alpha^\beta$ holds for some $h: z \rightarrow u_\beta$. This shows that \vec{u} satisfies (A). \square

Recall that we consider sequences up to equivalence with respect to the relation defined in Section 2. Thus, we need to show that being a Fraïssé sequence does not depend on the representation. This is indeed true, assuming amalgamations:

Proposition 3.4. *Let \mathfrak{K} be a category with the amalgamation property and let $\vec{g}: \vec{u} \rightarrow \vec{v}$, $\vec{h}: \vec{v} \rightarrow \vec{u}$ be transformations of sequences such that $\vec{g} \circ \vec{h}$ is equivalent to the identity of \vec{v} . If \vec{u} is a Fraïssé sequence then so is \vec{v} .*

Proof. Condition (U) is obvious. Fix a \mathfrak{K} -arrow $e: v_\alpha \rightarrow y$ and assume $h_\alpha: v_\alpha \rightarrow u_{\alpha'}$. Using the amalgamation property, find $k: u_{\alpha'} \rightarrow w$ and $\ell: y \rightarrow w$ such that $\ell \circ e = k \circ h_\alpha$. Using the fact that \vec{u} is Fraïssé, find $\beta \geq \alpha'$ and a \mathfrak{K} -arrow f such that $f \circ k = u_{\alpha'}^\beta$. Finally, if $\beta' \geq \alpha$ is such that $g_\beta: u_\beta \rightarrow v_{\beta'}$, then $v_{\alpha'}^{\beta'} = g_\beta \circ u_{\alpha'}^\beta \circ h_\alpha = g_\beta \circ f \circ \ell \circ e$. This shows (A). \square

A Fraïssé sequence can possibly have finite length. In that case, by Proposition 3.3(a), there is also a Fraïssé sequence of length one—it is an object u which is cofinal in \mathfrak{K} and which satisfies the following version of (A): given $f \in \mathfrak{K}(u, x)$, where $x \in \mathfrak{K}$, there exists $g \in \mathfrak{K}(x, u)$ such that $g \circ f = \text{id}_u$. We shall call u a *Fraïssé object* in \mathfrak{K} . Given a Fraïssé object u , the sequence $u \rightarrow u \rightarrow \dots$, where each arrow is identity, is a Fraïssé sequence of length ω . Thus, it follows from Theorem 3.15 below that a possible Fraïssé object is unique, up to isomorphism. Below we give a direct proof of this fact.

Proposition 3.5. *Assume u, v are Fraïssé objects in a category \mathfrak{K} . Then $u \approx v$. If moreover all arrows in \mathfrak{K} are monomorphisms then every arrow $f: u \rightarrow x$ is an isomorphism.*

Proof. Applying (U) for v , we find a morphism $f_0: u \rightarrow v$ which, using (A) for u , has a left inverse $g_0: v \rightarrow u$, i.e. $g_0 \circ f_0 = \text{id}_u$. Now, using (A) for v , we obtain an arrow $f_1: u \rightarrow v$ such that $f_1 \circ g_0 = \text{id}_v$. Observe that

$$f_1 = f_1 \circ \text{id}_u = f_1 \circ (g_0 \circ f_0) = (f_1 \circ g_0) \circ f_0 = \text{id}_v \circ f_0 = f_0.$$

Hence $f_0 \circ g_0 = \text{id}_v$, which shows that f_0 is an isomorphism.

Finally, let $f: u \rightarrow x$ be a morphism in \mathfrak{K} . Again by (A), f has a left inverse $g: x \rightarrow u$. Assuming g is a monomorphism, we deduce that $f \circ g = \text{id}_x$, because $g \circ (f \circ g) = g \circ \text{id}_x$. Thus f is an isomorphism. \square

We finish this subsection with the following stability result which is well-known for model-theoretic Fraïssé limits.

Theorem 3.6. *Let \mathfrak{K} be a category, $\kappa \geq \aleph_0$ a regular cardinal, and let $\vec{u}: \kappa \rightarrow \text{Seq}_{\leq \kappa}(\mathfrak{K})$ be a sequence in $\text{Seq}_{\leq \kappa}(\mathfrak{K})$ such that each \vec{u}_α is a κ -Fraïssé sequence in \mathfrak{K} . Then the co-limit of \vec{u} in $\text{Seq}_{\leq \kappa}(\mathfrak{K})$ is a Fraïssé sequence in \mathfrak{K} .*

Recall that by Lemma 2.1, the co-limit of \vec{u} indeed exists.

Proof. Since every cofinal subsequence of a Fraïssé sequence is again Fraïssé, we can inductively refine each \vec{u}_α so that the bonding maps become natural transformations. We look at \vec{u} as a “two-dimensional sequence” in \mathfrak{K} and we denote by $u_{\alpha,\beta}$ the β th object of the sequence \vec{u}_α . Now the co-limit of \vec{u} is the diagonal sequence \vec{v} , where $v_\alpha = u_{\alpha,\alpha}$ for $\alpha < \kappa$. It is clear that \vec{v} satisfies (U). Fix a \mathfrak{K} -arrow $f: v_\xi \rightarrow y$. Since \vec{u}_ξ is Fraïssé, there are $\eta > \xi$ and a \mathfrak{K} -arrow $g: y \rightarrow u_{\xi,\eta}$ such that $g \circ f = u_{\xi,\eta}^{\xi,\eta}$. Finally, let $h = u_{\xi,\eta}^{\eta,\eta} \circ g$, where $u_{\xi,\eta}^{\eta,\eta}$ is the suitable component of the natural transformation from \vec{u}_ξ to \vec{u}_η . Then $h \circ f = v_\eta^\eta$, which shows that \vec{v} satisfies (A). \square

3.2 The existence

We present below a simple yet useful criterion for the existence of a Fraïssé sequence. In case of sequences of length $\leq \aleph_1$, this criterion becomes a characterization.

Recall that a family of arrows \mathcal{F} is *dominating* in \mathfrak{K} if it satisfies the following two conditions.

- (D1) The family $\text{Dom}(\mathcal{F})$ is cofinal in \mathfrak{K} , i.e. for every $x \in \mathfrak{K}$ there is $a \in \text{Dom}(\mathcal{F})$ such that $\mathfrak{K}(x, a) \neq \emptyset$.
- (D2) Given $a \in \text{Dom}(\mathcal{F})$ and $f: a \rightarrow y$ in \mathfrak{K} , there exist $g: y \rightarrow b$ in \mathfrak{K} such that $g \circ f \in \mathcal{F}$.

Theorem 3.7 (Existence). *Let κ be an infinite regular cardinal and let \mathfrak{K} be a κ -bounded directed category with the amalgamation property. Assume further that $\mathcal{F} \subseteq \text{Arr}(\mathfrak{K})$ is dominating in \mathfrak{K} and $|\mathcal{F}| \leq \kappa$. Then there exists a Fraïssé sequence of length κ in \mathfrak{K} .*

Proof. Let $\text{Dom}(\mathcal{F}) = \{a_\alpha\}_{\alpha < \kappa}$ and enumerate $\mathcal{F} \times \kappa$ as $\{\langle f_\alpha, i_\alpha \rangle\}_{\alpha < \kappa}$ so that for each $p \in \mathcal{F} \times \kappa$ the set $\{\alpha < \kappa: p = \langle f_\alpha, i_\alpha \rangle\}$ is unbounded in κ . We shall construct inductively a κ -Fraïssé sequence \vec{u} , so that the following conditions are satisfied:

- (i) $u_\eta^\alpha \circ u_\xi^\eta = u_\xi^\alpha$ for every $\xi < \eta < \alpha$ (i.e. \vec{u} is indeed a sequence).

- (ii) $u_\alpha \in \text{Dom}(\mathcal{F})$ and $\mathfrak{K}(a_\alpha, u_\alpha) \neq \emptyset$.
- (iii) Given $\xi < \alpha$, if $\xi = i_\alpha$ and $\text{dom}(f_\alpha) = u_\xi$ then there exists an arrow h in \mathfrak{K} such that $h \circ f_\alpha = u_\xi^\alpha$.

We start with $u_0 = a_0$. Assume that $\beta < \kappa$ is such that u_ξ and u_ξ^η have been constructed for all $\xi < \eta < \beta$. Using the fact that \mathfrak{K} is κ -bounded, find $v \in \mathfrak{K}$ and $j_\alpha: u_\alpha \rightarrow v$ such that $j_\xi = j_\eta \circ u_\xi^\eta$ holds for every $\xi < \eta < \beta$. Using the fact that \mathfrak{K} is directed, we may ensure that $\mathfrak{K}(a_\beta, v) \neq \emptyset$. Now, if $f_\beta: u_\beta \rightarrow y$ and $\xi = i_\beta < \beta$ then, using amalgamation, we may find arrows $h: v \rightarrow w$ and $g: y \rightarrow w$ so that $g \circ f_\beta = h \circ j_\beta$ holds. Using (D1), we may further assume that $w \in \text{Dom}(\mathcal{F})$. Finally, set $u_\beta := w$ and $u_\eta^\beta := h \circ j_\eta$ for $\eta < \beta$. It is clear that conditions (i) – (iii) are satisfied.

It follows that the construction can be carried out. It remains to check that $\vec{u}: \kappa \rightarrow \mathfrak{K}$ is a Fraïssé sequence. Condition (i) says that \vec{u} is indeed an inductive sequence. Conditions (D1) and (ii) imply (U). In order to justify (A), fix $\xi < \kappa$ and $f \in \mathfrak{K}(u_\xi, x)$, where $x \in \mathfrak{K}$. We need to find $\alpha > \xi$ and an arrow g so that $g \circ f = u_\xi^\alpha$. Since $u_\xi \in \text{Dom}(\mathcal{F})$, using (D2), we can find $g \in \mathcal{F}$ such that $g = k \circ f$ for some arrow k . Now find $\alpha > \xi$ such that $f_\alpha = g$ and $i_\alpha = \xi$. By (iii), $h \circ g = u_\xi^{\alpha+1}$ for some arrow h . Hence $(h \circ k) \circ f = u_\xi^{\alpha+1}$, which completes the proof. \square

The existence of a Fraïssé sequence can also be proved by using a Baire category argument. We shall demonstrate it below for the countable case, where the classical Baire Category Theorem is invoked.

Corollary 3.8. *Let \mathfrak{K} be a directed category with the amalgamation property. Assume that \mathfrak{K} is dominated by a countable family of arrows. Then \mathfrak{K} has an ω -Fraïssé sequence.*

Proof. Without loss of generality, we may assume that a countable dominating family \mathcal{F} is a subcategory of \mathfrak{K} . An ω -sequence \vec{x} in \mathcal{F} may be regarded as a function from $\Delta = \{\langle m, n \rangle : m \leq n\}$ into \mathcal{F} satisfying the obvious conditions. Thus, the set S of all ω -sequences in \mathcal{F} is a closed subspace of the Polish space \mathcal{F}^Δ , endowed with the product topology. Given an object x in \mathcal{F} , let U_x be the set of all $\vec{x} \in S$ for which there exists an arrow $x \rightarrow \vec{x}$. Clearly, U_x is open and dense in S . Given an arrow $f: a \rightarrow b$ in \mathcal{F} and $n \in \omega$, let

$$V_{f,n} = \{\vec{x} \in S : x_n = a \implies (\exists m > n)(\exists g) g \circ f = x_n^m\}.$$

Again, $V_{f,n}$ is open and dense in S (for the density one needs to use amalgamations). Using the Baire Category Theorem, we can find a sequence $\vec{u} \in S$ that belongs to all the sets defined above. It is easy to check, using the fact that \mathcal{F} is dominating in \mathfrak{K} , that \vec{u} is a Fraïssé sequence in \mathfrak{K} . \square

The argument above can be repeated in the general case, however one has to use the topological space S^κ , where κ is a regular cardinal and the topology is generated

by open sets of the form $V_s = \{x: s \subseteq x\}$, where $s \in S^{<\kappa}$. It is a standard and easy fact that in this topology, the intersection of κ many open dense sets is dense in S^κ .

We shall sometimes need the following variant of Theorem 3.7, involving a functor:

Proposition 3.9. *Let \mathfrak{K} be a directed category with the amalgamation property. Let $\kappa \geq \aleph_0$ be a regular cardinal and let $\Phi: \mathfrak{K} \rightarrow \mathfrak{L}$ be a functor satisfying the following condition:*

- (C) *For every $\vec{x} \in \text{Seq}_{<\kappa}(\mathfrak{K})$ there exist $a \in \mathfrak{K}$ and a co-cone of arrows \mathcal{F} in \mathfrak{K} such that $\langle \Phi(a), \Phi[\mathcal{F}] \rangle$ is the co-limit of $\Phi[\vec{x}]$ in \mathfrak{L} .*

If \mathfrak{K} has a Fraïssé sequence of length κ then there exists a κ -Fraïssé sequence \vec{u} in \mathfrak{K} such that $\Phi[\vec{u}]$ is continuous in \mathfrak{L} .

Proof. Note that the assumptions on the existence of a κ -Fraïssé sequence imply that \mathfrak{K} is dominated by at most κ many arrows. Repeat the construction from the proof of Theorem 3.7, taking care that $\Phi(u_\delta)$ be the co-limit of $\Phi[\vec{u} \upharpoonright \delta]$ at every limit stage δ . \square

Selected examples of Fraïssé sequences will be described in Sections 5 and 7. Below we illustrate the main concepts in the case of ordered categories.

Example 3.10. Let $\langle P, \leq \rangle$ be a partially ordered (or, more generally, quasi-ordered) set, treated as a category. Observe that the amalgamation property is equivalent to directedness. A sequence in P is nothing but a well ordered chain. Note that a subset of P is dominating iff it is cofinal. In particular, a sequence in P is Fraïssé iff it is cofinal in P . Finally, it is well known and easy to show by a simple transfinite induction that a directed poset with no maximal element has a well ordered cofinal chain of type κ (a κ -Fraïssé sequence in our terminology) if and only if it is κ -bounded (i.e. every subset of cardinality $< \kappa$ has an upper bound in P) and has any cofinal set of cardinality κ . Such a cardinal κ is necessarily regular.

3.3 Cofinality

Below we discuss the crucial property of a Fraïssé sequence: cofinality in the category of sequences.

Theorem 3.11 (Cofinality for ω -sequences). *Assume \vec{u} is a Fraïssé sequence in a category \mathfrak{K} with the amalgamation property. Then for every countable inductive sequence \vec{x} in \mathfrak{K} there exists a morphism of sequences $F: \vec{x} \rightarrow \vec{u}$.*

Proof. We use the extension property (property (E)) of the sequence \vec{u} , which is equivalent to the amalgamation property of \mathfrak{K} (Proposition 3.1). Let \vec{x} be an ω -sequence in \mathfrak{K} . Using (U), find an arrow $f_0: x_0 \rightarrow u_{\alpha_0}$. Now assume that arrows

f_0, \dots, f_{n-1} have been defined so that $f_m: x_m \rightarrow u_{\alpha_m}$ and the diagram

$$\begin{array}{ccc} x_\ell & \xrightarrow{f_\ell} & u_{\alpha_\ell} \\ x_k^\ell \uparrow & & \uparrow u_{\alpha_k}^{\alpha_\ell} \\ x_k & \xrightarrow{f_k} & u_{\alpha_k} \end{array}$$

commutes for every $k < \ell < n$ (in particular $\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_{n-1}$). Using (E), find $\alpha_n \geq \alpha_{n-1}$ and an arrow $f: x_n \rightarrow u_{\alpha_n}$ so that $f \circ x_{n-1}^n = u_{\alpha_{n-1}}^{\alpha_n} \circ f_{n-1}$ and define $f_n := f$. Given $m < n - 1$, by the induction hypothesis, we get

$$f_n \circ x_m^n = f_n \circ x_{n-1}^n \circ x_m^{n-1} = u_{\alpha_{n-1}}^{\alpha_n} \circ f_{n-1} \circ x_m^{n-1} = u_{\alpha_{n-1}}^{\alpha_n} \circ u_{\alpha_m}^{\alpha_{n-1}} \circ f_m = u_{\alpha_m}^{\alpha_n} \circ f_m.$$

Finally, setting $F = \{f_n\}_{n \in \omega}$, we obtain the required morphism $F: \vec{x} \rightarrow \vec{u}$. \square

The proof above can be easily extended to uncountable sequences, assuming continuity:

Theorem 3.12. *Let \mathfrak{K} be a category with the amalgamation property and let \vec{u} be a Fraïssé sequence of regular length κ in \mathfrak{K} . Then for every continuous sequence $\vec{x} \in \text{Seq}_{\leq \kappa}(\mathfrak{K})$ there exists an arrow of sequences $F: \vec{x} \rightarrow \vec{u}$.*

Proof. We repeat the construction from the proof of Theorem 3.11. In the case of a limit ordinal δ , we let α_δ to be the supremum of $\{\alpha_\xi: \xi < \delta\}$ and we define f_δ to be the unique arrow satisfying $f_\delta \circ x_\alpha^\delta = f_\alpha$ for every $\alpha < \delta$. This is possible, because x_δ together with the co-cone of arrows $\{x_\xi^\delta\}_{\xi < \delta}$ is, by assumption, the co-limit of $\vec{x} \upharpoonright \delta$. Thus, the construction from the proof of Theorem 3.11 can be carried out, obtaining the desired arrow $F: \vec{x} \rightarrow \vec{u}$. \square

We shall see later that an uncountable Fraïssé sequence may not be cofinal for ω_1 -sequences. From Theorem 3.11 we immediately get the following characterization of the existence of a Fraïssé sequence of length ω_1 .

Corollary 3.13. *Let \mathfrak{K} be a category with the amalgamation property. There exists a Fraïssé sequence of length ω_1 in \mathfrak{K} if and only if \mathfrak{K} is σ -bounded and dominated by a family consisting of at most \aleph_1 arrows.*

Proof. The “if” part is a special case of Theorem 3.7. Let \vec{u} be an ω_1 -Fraïssé sequence in \mathfrak{K} . Then \mathfrak{K} is directed and the family $\{u_\alpha^\beta: \alpha \leq \beta < \omega_1\}$ is dominating in \mathfrak{K} . Fix $\vec{x} \in \sigma\mathfrak{K}$. Theorem 3.11 says that there exists an arrow of sequences $\vec{f}: \vec{x} \rightarrow \vec{u}$, so some u_α provides a bound for \vec{x} . Thus, every countable sequence is bounded in \mathfrak{K} . \square

3.4 The back-and-forth principle

Fix a category \mathfrak{K} . Let \vec{u}, \vec{v} be Fraïssé sequences in \mathfrak{K} . We shall say that $\langle \vec{u}, \vec{v} \rangle$ satisfies the *back-and-forth principle* if for every α below the length of \vec{u} , for every arrow $f: u_\alpha \rightarrow \vec{v}$ there exists an isomorphism of sequences $h: \vec{u} \rightarrow \vec{v}$ such that $h \circ u_\alpha^\infty = f$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \vec{u} & \xrightarrow{h} & \vec{v} \\ u_\alpha^\infty \uparrow & & \nearrow f \\ u_\alpha & & \end{array}$$

Since there exists at least one arrow $f: u_0 \rightarrow \vec{v}$, this implies that $\vec{u} \approx \vec{v}$. We shall say that \vec{u} satisfies the *back-and-forth principle* if $\langle \vec{u}, \vec{u} \rangle$ does. The following simple statement shows the importance of this property.

Proposition 3.14 (Homogeneity). *Assume \mathfrak{K} is a category with the amalgamation property and \vec{u} is a Fraïssé sequence in \mathfrak{K} which satisfies the back-and-forth principle. Then for every arrows $f: a \rightarrow b$, $i: a \rightarrow \vec{u}$ and $j: b \rightarrow \vec{u}$ there exists an isomorphism of sequences $h: \vec{u} \rightarrow \vec{u}$ such that $h \circ i = j \circ f$, i.e. the following diagram commutes.*

$$\begin{array}{ccc} \vec{u} & \xrightarrow{h} & \vec{u} \\ i \uparrow & & \uparrow j \\ a & \xrightarrow{f} & b \end{array}$$

Proof. Let α be such that $i = u_\alpha^\infty \circ i_\alpha$ for some $i_\alpha: a \rightarrow u_\alpha$. Let \vec{v} be the sequence obtained from \vec{u} by cutting all objects with indices below α and adding the arrow $i_\alpha: a \rightarrow u_\alpha$ at the beginning. That is,

$$a \xrightarrow{i_\alpha} u_\alpha \xrightarrow{u_\alpha^{\alpha+1}} u_{\alpha+1} \longrightarrow \dots$$

Since \mathfrak{K} has the amalgamation property, by Proposition 3.4, \vec{v} is Fraïssé. The back-and-forth principle applied to $j \circ f$ yields the required isomorphism h . \square

It turns out that countable Fraïssé sequences always satisfy the back-and-forth principle. We shall see in Section 5.5 that this not true for sequences of length ω_1 .

Theorem 3.15 (Uniqueness). *Assume that \vec{u}, \vec{v} are ω -Fraïssé sequences in a given category \mathfrak{K} . Assume further that $k, \ell < \omega$ and $f: u_k \rightarrow v_\ell$ is an arrow in \mathfrak{K} . Then there exists an isomorphism $F: \vec{u} \rightarrow \vec{v}$ in $\sigma\mathfrak{K}$ such that the diagram*

$$\begin{array}{ccc} \vec{u} & \xrightarrow{F} & \vec{v} \\ u_k^\infty \uparrow & & \uparrow v_\ell^\infty \\ u_k & \xrightarrow{f} & v_\ell \end{array}$$

commutes. In particular $\vec{u} \approx \vec{v}$.

Notice that in the statement above we do not need to assume that the given category has the amalgamation property.

Proof. We construct inductively arrows $f_n: u_{k_n} \rightarrow v_{\ell_n}$, $g_n: v_{\ell_n} \rightarrow u_{k_{n+1}}$, where $k_0 \leq \ell_0 < k_1 \leq \ell_1 < \dots$ and for each $n \in \omega$ the diagram

$$\begin{array}{ccccc}
 u_{k_n} & \xrightarrow{u_{k_n}^{k_{n+1}}} & u_{k_{n+1}} & \xrightarrow{u_{k_{n+1}}^{k_{n+2}}} & u_{k_{n+2}} \dashrightarrow \\
 & \searrow f_n & & \searrow f_{n+1} & \\
 & & v_{\ell_n} & \xrightarrow{v_{\ell_n}^{\ell_{n+1}}} & v_{\ell_{n+1}} \dashrightarrow \\
 & & & \nearrow g_n & \nearrow g_{n+1}
 \end{array}$$

commutes.

We start with $f_0 := f$, $k_0 := k$, $\ell_0 := \ell$, possibly replacing f by some arrow of the form $j_\ell^m \circ f$ to ensure that $k_0 \leq \ell_0$. Using property (A) of the sequence \vec{u} , find $k_1 > k_0$ and $g_1: v_0 \rightarrow u_{k_1}$ such that $g_1 \circ f = u_0^{k_1}$. Assume that f_m, g_m have already been constructed for $m \leq n$. Using the amalgamation of \vec{v} , find $\ell_{n+1} \geq k_{n+1}$ and an arrow $f_{n+1}: u_{k_{n+1}} \rightarrow v_{\ell_{n+1}}$ such that $f_{n+1} \circ g_n = v_{\ell_n}^{\ell_{n+1}}$. Now, using the amalgamation of \vec{u} , we find $k_{n+2} > \ell_{n+1}$ and an arrow $g_{n+1}: v_{\ell_{n+1}} \rightarrow u_{k_{n+2}}$ such that $g_{n+1} \circ f_{n+1} = u_{k_{n+1}}^{k_{n+2}}$. By the induction hypothesis, $g_n \circ f_n = u_{k_n}^{k_{n+1}}$, therefore the above diagram commutes. This finishes the construction.

Finally, set $F = \{f_n\}_{n \in \omega}$ and $G = \{g_n\}_{n \in \omega}$. Then $F: \vec{u} \rightarrow \vec{v}$, $G: \vec{v} \rightarrow \vec{u}$ are morphisms of sequences and by a simple induction we show that

$$(*) \quad g_n \circ v_{\ell_m}^{\ell_n} \circ f_m = u_{k_m}^{k_{n+1}} \quad \text{and} \quad f_n \circ u_{k_{m+1}}^{k_n} \circ g_m = v_{\ell_m}^{\ell_n}$$

holds for every $m < n < \omega$. This shows that $F \circ G = \text{id}_{\vec{v}}$ and $G \circ F = \text{id}_{\vec{u}}$, therefore F is an isomorphism. The equality $v_0^\infty \circ f = F \circ u_0^\infty$ means that $v_0^{\ell_n} \circ f = f_n \circ u_0^{k_n}$ should hold for every $n \in \omega$. Fix $n > 0$. Applying (*) twice (with $m = 0$ and $m = n - 1$ respectively), we get

$$f_n \circ u_0^{k_n} = f_n \circ g_{n-1} \circ v_0^{\ell_{n-1}} \circ f = v_{\ell_{n-1}}^{\ell_n} \circ v_0^{\ell_{n-1}} \circ f = v_0^{\ell_n} \circ f.$$

Thus $v_0^\infty \circ f = F \circ u_0^\infty$.

Finally, notice that, by property (U) of the sequence \vec{v} , for some $\ell < \omega$ there exists an arrow $f: u_0 \rightarrow v_\ell$, so applying the first part we see that $\vec{u} \approx \vec{v}$. \square

3.5 Fraïssé sequences and functors

We now discuss a possible extension of the notion of a Fraïssé sequence, involving a functor. The motivation is two-fold: First, we may want to have a sequence that is not Fraïssé in a given category \mathfrak{K} , yet resembles the Fraïssé property when moving to a different, richer category \mathfrak{L} . Second, given a Fraïssé sequence, we may be unable to show its cofinality, however we may succeed after transferring the sequence to

a different category. Of course, this makes sense only if the functor we are using preserves relevant information.

Fix a functor $\Phi: \mathfrak{K} \rightarrow \mathfrak{L}$. We say that a sequence $\vec{u}: \kappa \rightarrow \mathfrak{L}$ is *Fraïssé over Φ* if the following conditions are satisfied.

(U_Φ) For every $x \in \mathfrak{L}$ there is $\alpha < \kappa$ such that $\mathfrak{L}(x, \Phi(u_\alpha)) \neq \emptyset$.

(A_Φ) Given $\alpha < \kappa$ and $f: u_\alpha \rightarrow y$ in \mathfrak{K} , there are $\beta \geq \alpha$ and an arrow $g: y \rightarrow \Phi(u_\beta)$ such that $\Phi(u_\alpha^\beta) = g \circ \Phi(f)$.

In case of the identity functor (where $\mathfrak{K} = \mathfrak{L}$) this is just the usual notion of a Fraïssé sequence.

Proposition 3.16. *Let $\Phi: \mathfrak{K} \rightarrow \mathfrak{L}$ be a covariant functor and let \vec{u} be a Fraïssé sequence in \mathfrak{K} . If $\Phi[\mathfrak{K}]$ is cofinal in \mathfrak{L} then \vec{u} is Fraïssé over Φ .*

Aiming at the problem of existence and cofinality, we now introduce the following natural concept. Let $\Phi: \mathfrak{K} \rightarrow \mathfrak{L}$ be a covariant functor. We say that Φ has the *mixed amalgamation property* if for every objects a, b, c in \mathfrak{K} , for every arrow $f: a \rightarrow b$ in \mathfrak{K} and for every arrow $g: \Phi(a) \rightarrow \Phi(c)$ in \mathfrak{L} , there exist $d \in \mathfrak{K}$ and arrows $f': c \rightarrow d$, $g': \Phi(b) \rightarrow \Phi(d)$ in \mathfrak{K} and \mathfrak{L} respectively, such that the following diagram

$$\begin{array}{ccc} \Phi(b) & \xrightarrow{g'} & \Phi(d) \\ \Phi(f) \uparrow & & \uparrow \Phi(f') \\ \Phi(a) & \xrightarrow{g} & \Phi(c) \end{array}$$

is commutative. The usual amalgamation can be rephrased as the amalgamation property of the identity functor. Given a category \mathfrak{L} and its subcategory \mathfrak{K} , we say that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the *mixed amalgamation property* if the inclusion functor $\mathfrak{K} \subseteq \mathfrak{L}$ has this property.

In order to formulate the existence result, we need to adapt the notion of a dominating family. Let $\Phi: \mathfrak{K} \rightarrow \mathfrak{L}$ be a functor. A family $\mathcal{F} \subseteq \mathfrak{K}$ will be called *Φ -dominating* if

(D1 $_\Phi$) For every \mathfrak{K} -object x , there is $a \in \text{Dom}(\mathcal{F})$ such that $\mathfrak{L}(\Phi(x), \Phi(a)) \neq \emptyset$.

(D2 $_\Phi$) Given $a \in \text{Dom}(\mathcal{F})$ and a \mathfrak{K} -arrow $f: a \rightarrow y$, there exist $h: a \rightarrow b$ in \mathcal{F} and an \mathfrak{L} -arrow $g: \Phi(y) \rightarrow \Phi(b)$ such that $\Phi(h) = g \circ \Phi(f)$.

Theorem 3.17. *Let κ be an infinite regular cardinal and let $\Phi: \mathfrak{K} \rightarrow \mathfrak{L}$ be a functor satisfying the following conditions:*

(F_1) *For every \mathfrak{K} -objects x, y there exists a \mathfrak{K} -object z such that $\mathfrak{K}(x, z) \neq \emptyset \neq \mathfrak{L}(\Phi(y), \Phi(z))$.*

(F_2) *Φ has the mixed amalgamation property.*

(F₃) \mathfrak{K} is κ -bounded and has a Φ -dominating family consisting of $\leq \kappa$ arrows.

Then there exists a κ -Fraïssé sequence over Φ . If, additionally, Φ satisfies condition (C) of Proposition 3.9 then there exists a κ -Fraïssé sequence over Φ such that $\Phi[\vec{u}]$ is continuous.

Proof. This is actually a repetition of the arguments from the proof of Theorem 3.7 combined with Proposition 3.9. The only difference is that we need to use mixed amalgamations. More precisely, having defined u_ξ and u_ξ^η for $\xi < \eta \leq \beta$, as in the proof of Theorem 3.7, given a \mathfrak{K} -arrow $f: u_\alpha \rightarrow y$ that belongs to a fixed Φ -dominating family \mathcal{F} , we use the mixed amalgamation property followed by condition (D2 $_\Phi$) in order to get a \mathfrak{K} -object $u_{\beta+1}$ and a \mathfrak{K} -arrow $u_\beta^{\beta+1}: u_\beta \rightarrow u_{\beta+1}$ such that $u_\beta^{\beta+1} \in \mathcal{F}$ and $\Phi(u_\beta^{\beta+1}) = g \circ \Phi(f)$ for some $g \in \mathcal{L}$.

If Φ satisfies (C) then we use this condition at limit steps in order to make the sequence Φ -continuous. \square

The mixed amalgamation property is also the main tool for showing cofinality of a Fraïssé sequence “transferred” to a different, possibly much bigger, category. As we shall see in Subsection 5.6, there exist quite natural pairs of categories $\mathfrak{K} \subseteq \mathcal{L}$ such that \mathfrak{K} has neither the amalgamation property nor a Fraïssé sequence, yet there exists a Fraïssé sequence over the inclusion $\mathfrak{K} \subseteq \mathcal{L}$.

Theorem 3.18. *Let κ be an infinite regular cardinal and let $\Phi: \mathfrak{K} \rightarrow \mathcal{L}$ be a functor with the mixed amalgamation property. Assume further that $\vec{u}: \kappa \rightarrow \mathfrak{K}$ is a Fraïssé sequence over Φ . Then for every sequence $\vec{x} \in \text{Seq}_{\leq \kappa}(\mathfrak{K})$ such that $\Phi[\vec{x}]$ is continuous in \mathcal{L} , there exists an arrow $F: \Phi[\vec{x}] \rightarrow \Phi[\vec{u}]$ in $\text{Seq}_{\leq \kappa}(\mathcal{L})$.*

Proof. Fix a sequence \vec{x} in \mathfrak{K} , of length $\lambda = \text{cf } \lambda \leq \kappa$, and assume that $\Phi[\vec{x}]$ is continuous in \mathcal{L} . We define inductively arrows $f_\alpha: \Phi[x_\alpha] \rightarrow \Phi[u_{\theta(\alpha)}]$ so that $\vec{f} = \{f_\alpha\}_{\alpha < \lambda}$ will be an arrow of sequences in \mathcal{L} . We start with an arbitrary arrow $f_0: \Phi[x_0] \rightarrow \Phi[u_{\theta(0)}]$, using the fact that \vec{u} is Fraïssé over Φ . Fix $\alpha > 0$ and suppose f_ξ have already been defined for all $\xi < \alpha$ so that

$$f_\eta \circ \Phi[x_\xi^\eta] = \Phi[u_{\theta(\xi)}^{\theta(\eta)}] \circ f_\xi$$

holds for every $\xi < \eta < \alpha$.

If α is a limit ordinal, define $\theta(\alpha) = \sup_{\xi < \alpha} \theta(\xi)$ and, using the continuity of $\Phi[\vec{x}]$, let $f_\alpha: \Phi[x_\alpha] \rightarrow \Phi[u_{\theta(\alpha)}]$ be the unique arrow satisfying $f_\alpha \circ \Phi[x_\xi^\alpha] = \Phi[u_{\theta(\xi)}^{\theta(\alpha)}] \circ f_\xi$ for every $\xi < \alpha$.

Now assume that $\alpha = \beta + 1$. Using the mixed amalgamation property of Φ , we find an arrow $g: u_{\theta(\beta)} \rightarrow w$ in \mathfrak{K} and an arrow $h: \Phi[x_\alpha] \rightarrow \Phi[w]$ in \mathcal{L} such that $h \circ \Phi[j_\beta^\alpha] = \Phi[g] \circ f_\beta$. Since \vec{u} is Fraïssé over Φ , there are $\gamma \geq \theta(\beta)$ and $k: \Phi[w] \rightarrow \Phi[u_\gamma]$ such that $\Phi[u_{\theta(\beta)}^\gamma] = k \circ \Phi[g]$. Finally, set $\theta(\alpha) := \gamma$ and $f_\alpha := k \circ h$.

This finishes the construction and completes the proof. \square

4 Fraïssé-Jónsson categories

In this section we describe a way of dealing with discontinuous Fraïssé sequences. Namely, it may happen that “short” sequences do not have co-limits, however there is a (sometimes quite natural) operator of quasi-limit (see the definition below). If this operator is good enough, the back-and-forth principle holds. If moreover the quasi-limit commutes with certain amalgamations, it is possible to prove cofinality of a quasi-continuous Fraïssé sequence. The results of this section are motivated by applications, described later.

4.1 Quasi-limiting operators

What is really needed for the back-and-forth principle? It is rather clear that the existence of co-limits for “short” sequences is sufficient, because then one can restrict attention to continuous Fraïssé sequences. Looking for specific applications, we are forced to relax this condition. So, let us analyze co-limits for a moment.

Given a category \mathfrak{K} , let $\Delta(\mathfrak{K})$ be the category of all diagrams of the form $\{f_\xi\}_{\xi < \lambda}$ where λ is an ordinal and all f_ξ s have the same co-domain. In other words, $\Delta(\mathfrak{K})$ is the category of all functors from posets of the form $\lambda \cup \{\infty\}$ where λ has the discrete order (different elements are incomparable) and $\infty > \xi$ for every $\xi \in \lambda$. The arrows are natural transformations. Assume all sequences in \mathfrak{K} of length $< \kappa$ have co-limits in \mathfrak{K} . Then we have an operator $\lim: \text{Seq}_{< \kappa}(\mathfrak{K}) \rightarrow \Delta(\mathfrak{K})$ which assigns to each sequence $\vec{x}: \lambda \rightarrow \mathfrak{K}$ its co-limiting co-cone $\lim \vec{x}$. This is indeed a functor, because of the universality of the co-limit. In the back-and-forth argument, we actually deal with isomorphisms at each limit step, so it turns out that only some properties of \lim are needed.

Denote by $\text{Seq}_{< \kappa}^{\text{iso}}(\mathfrak{K})$ the category $\text{Seq}_{< \kappa}(\mathfrak{K})$ restricted to isomorphisms of sequences only. A *quasi-limiting operator* for $\text{Seq}_{< \kappa}(\mathfrak{K})$ is a functor $\mathcal{L}: \text{Seq}_{< \kappa}^{\text{iso}}(\mathfrak{K}) \rightarrow \Delta(\mathfrak{K})$ assigning to each sequence \vec{x} a co-cone $\mathcal{L}\vec{x}$ for \vec{x} . More precisely, given an isomorphism of sequences $\vec{h}: \vec{x} \rightarrow \vec{y}$ there exists an isomorphism $h_\infty: \mathcal{L}\vec{x} \rightarrow \mathcal{L}\vec{y}$ commuting with \vec{h} in the usual sense. We say that \mathcal{L} is a *partial quasi-limiting operator* if it is defined on a subcategory of $\text{Seq}_{< \kappa}^{\text{iso}}(\mathfrak{K})$ with the property that if $\mathcal{L}\vec{x}$ is defined then so is $\mathcal{L}\vec{y}$ whenever $\vec{y} \approx \vec{x}$. An example is the co-limiting operator, defined only for sequences that have co-limits.

Typical examples of quasi-limiting operators come from co-limits, “computed” in a bigger category containing \mathfrak{K} .

Proposition 4.1. *Let $F: \mathfrak{K} \rightarrow \mathfrak{L}$ be a faithful functor, $\kappa = \text{cf } \kappa > \aleph_0$, and assume the following conditions are satisfied:*

(\otimes_1) *For every sequence \vec{x} in \mathfrak{K} of length $\lambda < \kappa$ there exists a co-cone $\mathcal{L}\vec{x} := \{x_\xi^\infty\}_{\xi < \lambda} \subseteq \mathfrak{K}$ such that $\{F(x_\xi^\infty)\}_{\xi < \lambda}$ is the co-limiting co-cone of $F(\vec{x})$ in \mathfrak{L} .*

(\otimes_2) *For every \mathfrak{K} -objects a, b , for every \mathfrak{L} -isomorphism $h: F(a) \rightarrow F(b)$ there exists a \mathfrak{K} -isomorphism $g: a \rightarrow b$ such that $h = F(g)$.*

Then \mathcal{L} is a quasi-limiting operator on $\text{Seq}_{<\kappa}(\mathfrak{K})$ satisfying $F(\mathcal{L}\vec{x}) = \lim F(\vec{x})$ for every $\vec{x} \in \text{Seq}_{<\kappa}(\mathfrak{K})$.

Proof. Fix an isomorphism of sequences $\vec{h}: \vec{x} \rightarrow \vec{y}$ in \mathfrak{K} . Let the co-cones $\mathcal{L}\vec{x}$, $\mathcal{L}\vec{y}$ have co-domains \bar{x} and \bar{y} , respectively. There is an isomorphism $k: \bar{x} \rightarrow \bar{y}$ in \mathfrak{L} , commuting with $F(\vec{h})$. That is, for every α, β such that $h_\alpha: x_\alpha \rightarrow x_\beta$, we have a commuting diagram

$$\begin{array}{ccc} F(x_\alpha) & \xrightarrow{F(x_\alpha^\infty)} & F(\bar{x}) \\ F(h_\alpha) \downarrow & & \downarrow k \\ F(y_\beta) & \xrightarrow{F(y_\beta^\infty)} & F(\bar{y}) \end{array}$$

commutes. Let $g: \bar{x} \rightarrow \bar{y}$ be an isomorphism such that $F(g) = k$. Then the diagram

$$\begin{array}{ccc} x_\alpha & \xrightarrow{x_\alpha^\infty} & \bar{x} \\ h_\alpha \downarrow & & \downarrow g \\ y_\beta & \xrightarrow{y_\beta^\infty} & \bar{y} \end{array}$$

is commutative, because F is faithful. This shows that \mathcal{L} is indeed a functor from $\text{Seq}_{<\kappa}^{\text{iso}}(\mathfrak{K})$ into the co-cones of \mathfrak{K} . The faithfulness of F guarantees that \mathcal{L} is well defined, i.e. the co-cone in condition (\otimes_1) is unique, up to isomorphism. \square

The next example shows that quasi-limiting operators may have nothing to do with co-limits.

Example 4.2. Consider ω_1 as a poset category. For each limit ordinal $\delta < \omega_1$ add its copy δ^+ , incomparable with δ and such that $\xi < \delta^+ < \eta$ whenever $\xi < \delta < \eta$. Let $\mathfrak{K} = \omega_1 \cup \{\delta^+ : \delta \in \text{lim}(\omega_1)\}$. It is clear that no strictly increasing sequence has a co-limit in \mathfrak{K} . On the other hand, there exists a quasi-limiting operator $\mathcal{L}: \text{Seq}_{<\omega_1}(\mathfrak{K}) \rightarrow \Delta^{\text{iso}}(\mathfrak{K})$ defined by

$$\mathcal{L}\vec{x} = \sup_{n \in \omega} P(x_n),$$

where P is the canonical projection of \mathfrak{K} onto ω_1 .

This example also shows that a quasi-limiting operator may not be unique, since the formula $\mathcal{L}^+\vec{x} = (\mathcal{L}\vec{x})^+$ defines another one.

Note that in the example above, as in every poset category, the back-and-forth principle holds for every two Fraïssé sequences. A Fraïssé sequence in a poset is a cofinal chain whose length is a regular cardinal. If such a chain exists, then it is equivalent to any other cofinal chain. Summarizing: an infinite poset \mathbb{P} has a Fraïssé sequence if and only if it is κ -directed (i.e. every subset of cardinality $< \kappa$ has an upper bound) for some cardinal κ , and dominated by κ many elements, i.e. \mathbb{P} has a cofinal subset of cardinality κ . Of course, such a cardinal κ must be regular.

4.2 The back-and-forth principle revisited

Let \mathfrak{K} be a category and let \mathcal{L} be a partial quasi-limiting operator on $\text{Seq}_{<\kappa}(\mathfrak{K})$. Let $\varrho < \kappa$ be an ordinal. We say that a sequence $\vec{x}: \varrho \rightarrow \mathfrak{K}$ is \mathcal{L} -continuous if for every limit ordinal $\delta < \varrho$ it holds that

$$x_\delta = \mathcal{L}(\vec{x} \upharpoonright \delta)$$

with the co-cone $\{x_\xi^\delta\}_{\xi < \delta}$. This means in particular that \mathcal{L} is defined on all subsequences of \vec{x} .

The following fact is rather trivial, knowing the proof of Theorem 3.7.

Proposition 4.3. *Let \mathfrak{K} be a directed category with the amalgamation property. Let $\kappa \geq \aleph_0$ be a regular cardinal and let \mathcal{L} be a quasi-limiting operator on $\text{Seq}_{<\kappa}(\mathfrak{K})$. If \mathfrak{K} has a Fraïssé sequence of length κ then \mathfrak{K} also has an \mathcal{L} -continuous one.*

Theorem 4.4. *Let $\kappa > \aleph_0$ be a regular cardinal and let \mathfrak{K} be a category with a partial quasi-limiting operator \mathcal{L} on $\text{Seq}_{<\kappa}(\mathfrak{K})$. Then every two \mathcal{L} -continuous κ -Fraïssé sequences in \mathfrak{K} satisfy the back-and-forth principle.*

Proof. Let \vec{u}, \vec{v} be \mathcal{L} -continuous Fraïssé sequences of length κ in \mathfrak{K} and fix an arrow $f: u_0 \rightarrow v_0$. We construct inductively arrows $f_\alpha: u_{\psi(\alpha)} \rightarrow v_{\varphi(\alpha)}$ and $g_\alpha: v_{\varphi(\alpha)} \rightarrow u_{\psi(\alpha+1)}$, where $\varphi: \kappa \rightarrow \kappa$, $\psi: \kappa \rightarrow \kappa$ are increasing functions, so that the following conditions are satisfied.

- (i) $\alpha \leq \psi(\alpha) \leq \varphi(\alpha) < \psi(\alpha + 1)$.
- (ii) $g_\alpha \circ f_\alpha = u_{\psi(\alpha)}^{\psi(\alpha+1)}$ and $f_{\alpha+1} \circ g_\alpha = v_{\varphi(\alpha)}^{\varphi(\alpha+1)}$.
- (iii) $\xi < \eta \implies v_{\varphi(\xi)}^{\varphi(\eta)} \circ f_\xi = f_\eta \circ u_{\psi(\xi)}^{\psi(\eta)}$.

We start with $\psi(0) = 0$, $\varphi(0) = \alpha$ and $f_0 = f$, where $0 < \alpha < \kappa$ is such that f is equivalent to $f_0: u_0 \rightarrow v_\alpha$.

Fix $\beta > 0$ and assume that $\{f_\alpha\}_{\alpha < \beta}$ and $\{g_\alpha\}_{\alpha < \beta}$ have already been defined, so in particular $\varphi(\alpha)$, $\psi(\alpha)$, $\psi(\alpha + 1)$ have been defined for every $\alpha < \beta$.

Suppose first that β is a successor ordinal, say $\beta = \alpha + 1$. Using the fact that \vec{v} is Fraïssé, find $\varphi(\beta) \geq \psi(\alpha)$ and $f_\beta: u_{\psi(\beta)} \rightarrow v_{\varphi(\beta)}$ so that the second part of (ii) holds. Using the fact that \vec{u} is Fraïssé, find $\psi(\beta + 1) > \varphi(\beta)$ and $g_\beta: v_{\varphi(\beta)} \rightarrow u_{\psi(\beta+1)}$ so that the first part of (ii) holds. Clearly, (i) is satisfied and (iii) follows easily from the inductive hypothesis and (ii).

Assume now that β is a limit ordinal. Let $\varrho := \sup_{\alpha < \beta} \varphi(\alpha) = \sup_{\alpha < \beta} \psi(\alpha)$ and define $\varphi(\beta) := \psi(\beta) := \varrho$, $\psi(\beta + 1) := \varrho + 1$. By the induction hypothesis $\varrho \geq \beta$, therefore (i) holds. Observe that ϱ is a limit ordinal.

First, observe that $F = \{f_\xi\}_{\xi < \varrho}$ and $G = \{g_\xi\}_{\xi < \varrho}$ witness an isomorphism between $\vec{u} \upharpoonright \varrho$ and $\vec{v} \upharpoonright \varrho$, which extends to an isomorphism of their \mathcal{L} -limits $h: u_\varrho \rightarrow v_\varrho$. More specifically,

$$(\times) \quad h \circ u_{\psi(\xi)}^\varrho = v_{\varphi(\xi)}^\varrho \circ f_\xi \quad \text{and} \quad h^{-1} \circ v_{\varphi(\xi)}^\varrho = u_{\psi(\xi+1)}^\varrho \circ g_\xi$$

hold for every $\xi < \varrho$.

Define $f_\beta := h$ and $g_\beta := u_\varrho^{\varrho+1} \circ h^{-1}$. Then, $g_\beta \circ f_\beta = u_\varrho^{\varrho+1}$, i.e. the first part of (ii) holds for β in place of α (the second part is taken care of in the next, successor, inductive step). The first equation in (✕) shows (iii). This finishes the inductive construction.

We claim that $\vec{f} = \{f_\alpha\}_{\alpha < \kappa}$ is an isomorphism from \vec{u} to \vec{v} which witnesses the back-and-forth principle. For this aim, we first observe that \vec{f} is an arrow of sequences, because of (iii). Define $\vec{g} = \{g_\alpha\}_{\alpha < \kappa}$. We claim that \vec{g} is an arrow of sequences, inverse to \vec{f} .

Using (ii) and (iii), we see that

$$\begin{aligned} g_\eta \circ v_{\varphi(\xi)}^{\varphi(\eta)} &= g_\eta \circ v_{\varphi(\xi+1)}^{\varphi(\eta)} \circ v_{\varphi(\xi)}^{\varphi(\xi+1)} = g_\eta \circ v_{\varphi(\xi+1)}^{\varphi(\eta)} \circ f_{\xi+1} \circ g_\xi \\ &= g_\eta \circ f_\eta \circ u_{\psi(\xi+1)}^{\psi(\eta)} \circ g_\xi = u_{\psi(\eta)}^{\psi(\eta+1)} \circ u_{\psi(\xi+1)}^{\psi(\eta)} \circ g_\xi \\ &= u_{\psi(\xi+1)}^{\psi(\eta+1)} \circ g_\xi \end{aligned}$$

for every $\xi < \eta$, which shows that $\vec{g} = \{g_\alpha\}_{\alpha < \kappa}$ is indeed an arrow of sequences. Furthermore, again by (ii), we deduce that \vec{g} is the inverse of \vec{f} , which shows that \vec{f} is an isomorphism.

Finally, $\vec{f} \circ u_0^\infty = f$, by the first stage of the construction. \square

As a corollary, we obtain the result proved in [11], stated there in a different language.

Corollary 4.5. *Let \mathfrak{K} be a category. Every two continuous Fraïssé sequences in \mathfrak{K} of the same regular length satisfy the back-and-forth principle.*

Theorem 4.6. *Let $\kappa > \aleph_0$ be a regular cardinal and let \mathfrak{K} be a full, cofinal and relatively κ -complete subcategory of a category \mathfrak{L} with the amalgamation property. Then*

- (a) *There exists at most one, up to isomorphism, κ -Fraïssé sequence in \mathfrak{K} .*
- (b) *A possible κ -Fraïssé sequence in \mathfrak{K} is cofinal for $\text{Seq}_{\leq \kappa}(\mathfrak{K})$ and satisfies the back-and-forth principle.*

Proof. Let \vec{u} be a κ -Fraïssé sequence in \mathfrak{K} . Since \mathfrak{K} is full and cofinal in \mathfrak{L} , \vec{u} is Fraïssé in \mathfrak{L} . Let us make this sequence continuous in \mathfrak{L} , by adding co-limits at each limit ordinal below κ . Denote the extended sequence by \tilde{u} . Since \mathfrak{L} has the amalgamation property, the sequence \tilde{u} is Fraïssé in \mathfrak{L} (by Proposition 3.3(b)). Given another κ -Fraïssé sequence \vec{v} , let us extend it in the same way, obtaining a Fraïssé sequence \tilde{v} in \mathfrak{L} . By Corollary 4.5, the two extended sequences \tilde{u}, \tilde{v} satisfy the back-and-forth principle. Since \mathfrak{K} is a full subcategory of \mathfrak{L} , we conclude that \vec{u}, \vec{v} satisfy the back-and-forth principle in \mathfrak{K} . This shows (a) and the first part of (b). The second part of (b) follows from Theorem 3.12 and the fact that each κ -sequence in \mathfrak{K} can be completed to a continuous κ -sequence in \mathfrak{L} . \square

Corollary 4.7. *Let κ be a regular cardinal and let \mathfrak{K} be a directed category with the amalgamation property and with a dominating family consisting of at most κ arrows. Assume further that either \mathfrak{K} is κ -complete or \mathfrak{K} is κ -bounded and $\text{Seq}_{<\kappa}(\mathfrak{K})$ has the amalgamation property. Then*

- (a) *There exists a unique, up to isomorphism, κ -Fraïssé sequence \vec{u} in \mathfrak{K} .*
- (b) *The sequence \vec{u} is cofinal for $\text{Seq}_{\leq\kappa}(\mathfrak{K})$ and satisfies the back-and-forth principle.*

4.3 Amalgamation structures

We now describe a general way of obtaining cofinality, using a quasi-limiting operator and amalgamations. As we shall see later, it may happen that sequences of amalgamations do not converge and any inductive procedure of mapping a given sequence into a Fraïssé sequence fails at the first limit step. It is possible, however, that some special amalgamations survive limit steps. For this aim, we consider the following concepts.

Fix a category \mathfrak{K} with the amalgamation property. An *amalgamation structure* on \mathfrak{K} is a class \mathcal{A} of commuting squares of \mathfrak{K} (formally: functors from the 4-element Boolean algebra into \mathfrak{K}) satisfying the following conditions:

- (A1) Every square with two parallel identities is in \mathcal{A} .
- (A2) Given squares

$$\begin{array}{ccc} b & \xrightarrow{g'} & d \\ g \uparrow & & \uparrow f' \\ c & \xrightarrow{f} & a \end{array} \quad \begin{array}{ccc} d & \xrightarrow{f''} & v \\ f' \uparrow & & \uparrow h' \\ a & \xrightarrow{h} & u \end{array}$$

in \mathcal{A} , the composition square

$$\begin{array}{ccc} b & \xrightarrow{f'' \circ g'} & v \\ g \uparrow & & \uparrow h' \\ c & \xrightarrow{h \circ f} & u \end{array}$$

is again in \mathcal{A} .

- (A3) Given \mathfrak{K} -arrows $f: c \rightarrow a$, $g: c \rightarrow b$, there exist \mathfrak{K} -arrows $f': a \rightarrow v$ and $g': b \rightarrow v$ such that the square

$$\begin{array}{ccc} b & \xrightarrow{g'} & v \\ g \uparrow & & \uparrow f' \\ c & \xrightarrow{f} & a \end{array}$$

is in \mathcal{A} .

Clearly, the class \mathcal{A} of all commuting squares in \mathfrak{K} is an amalgamation structure as long as \mathfrak{K} has the amalgamation property. In Section 6 we shall consider an important type of amalgamation structures in categories of embedding-projection pairs.

Now assume \mathcal{L} is a quasi-limiting operator on $\text{Seq}_{<\kappa}(\mathfrak{K})$. We say that an amalgamation structure \mathcal{A} is \mathcal{L} -continuous if for every limit ordinal $\varrho < \kappa$, for every \mathcal{L} -continuous sequences $\vec{x}: \varrho + 1 \rightarrow \mathfrak{K}$ and $\vec{y}: \varrho + 1 \rightarrow \mathfrak{K}$, for every sequence of \mathfrak{K} -arrows $\{f_\xi: x_\xi \rightarrow y_\xi\}_{\xi < \varrho}$ such that for each $\xi < \eta < \varrho$ the square

$$\begin{array}{ccc} y_\xi & \xrightarrow{y_\xi^\eta} & y_\eta \\ f_\xi \uparrow & & \uparrow f_\eta \\ x_\xi & \xrightarrow{x_\xi^\eta} & x_\eta \end{array}$$

is in \mathcal{A} , there exists $f_\varrho: x_\varrho \rightarrow y_\varrho$ such that for each $\xi < \varrho$ the square

$$\begin{array}{ccc} y_\xi & \xrightarrow{y_\xi^\varrho} & y_\varrho \\ f_\xi \uparrow & & \uparrow f_\varrho \\ x_\xi & \xrightarrow{x_\xi^\varrho} & x_\varrho \end{array}$$

is in \mathcal{A} .

The existence of an \mathcal{L} -continuous amalgamation structure is sufficient for proving cofinality, as we shall see in a moment.

We now introduce the main notion of this section. For defining one of the conditions, we need to consider posets $[\lambda]^{\leq 2}$ of at most two-element subsets of an ordinal λ . This poset can be regarded as the set of all ordered pairs $\langle \xi, \eta \rangle$ in $\lambda \times \lambda$ with $\xi \leq \eta$. The ordering is coordinate-wise. Given a functor $X: [\lambda]^{\leq 2} \rightarrow \mathfrak{K}$, we shall denote by X_Δ its *diagonal sequence* $X \circ \delta$, where $\delta: \lambda \rightarrow [\lambda]^{\leq 2}$ is defined by $\delta(\xi) = \{\xi, \xi\}$. Furthermore, we shall denote by X^α the α th *vertical sequence* $X \circ v_\alpha$, where $v_\alpha: \lambda \rightarrow [\lambda]^{\leq 2}$ is defined by $v_\alpha(\xi) = \{\alpha, \alpha + \xi\}$.

A *Fraïssé-Jónsson category* is a triple of the form $\langle \mathfrak{K}, \mathcal{L}, \mathcal{A} \rangle$, where \mathcal{L} is a quasi-limiting operator on $\text{Seq}_{<\kappa}(\mathfrak{K})$ for some infinite regular cardinal κ , \mathcal{A} is an \mathcal{L} -continuous amalgamation structure on \mathfrak{K} and the following condition is satisfied.

(Δ) Given an infinite cardinal $\lambda < \kappa$, given a functor $X: [\lambda]^{\leq 2} \rightarrow \mathfrak{K}$ such that for every $\alpha_0 < \alpha_1 \leq \beta_0 < \beta_1$ the square

$$\begin{array}{ccc} X_{\{\alpha_0, \beta_1\}} & \xrightarrow{X_{\{\alpha_0, \beta_1\}}^{\{\alpha_1, \beta_1\}}} & X_{\{\alpha_1, \beta_1\}} \\ X_{\{\alpha_0, \beta_0\}}^{\{\alpha_0, \beta_1\}} \uparrow & & \uparrow X_{\{\alpha_1, \beta_0\}}^{\{\alpha_1, \beta_1\}} \\ X_{\{\alpha_0, \beta_0\}} & \xrightarrow{X_{\{\alpha_0, \beta_0\}}^{\{\alpha_1, \beta_0\}}} & X_{\{\alpha_1, \beta_0\}} \end{array}$$

is in \mathcal{A} , it holds that

$$\mathcal{L}(X_\Delta) = \mathcal{L}(\mathcal{L}X),$$

where $\mathcal{L}X$ is the sequence obtained from the \mathcal{L} -continuity of \mathcal{A} :

$$\mathcal{L}X^0 \longrightarrow \mathcal{L}X^1 \longrightarrow \cdots \longrightarrow \mathcal{L}X^\alpha \longrightarrow \cdots$$

and the corresponding arrows and co-cones are natural.

We shall call condition (Δ) the *diagonalization property*.

Note that the diagonalization property is obvious when one deals with co-limits, even after moving by a faithful functor to a different category. Namely:

Proposition 4.8. *Let $F: \mathfrak{K} \rightarrow \mathfrak{L}$ be a faithful functor satisfying conditions (\otimes_1) , (\otimes_2) of Proposition 4.1 for some $\kappa = \text{cf } \kappa > \aleph_0$ and let \mathcal{L} be the quasi-limiting operator such that $F(\mathcal{L}\vec{x}) = \lim F(\vec{x})$ for every $\vec{x} \in \text{Seq}_{<\kappa}(\mathfrak{K})$. Then for every \mathcal{L} -continuous amalgamation structure \mathcal{A} , $\langle \mathfrak{K}, \mathcal{L}, \mathcal{A} \rangle$ has the diagonalization property, i.e., $\langle \mathfrak{K}, \mathcal{L}, \mathcal{A} \rangle$ is a Fraïssé-Jónsson category.*

Proof. Assume \mathcal{A} is an \mathcal{L} -continuous amalgamation structure on \mathfrak{K} . Fix a suitable functor $X: [\lambda]^2 \rightarrow \mathfrak{K}$ in which all squares are in \mathcal{A} . Note that the usual co-limit has the diagonalization property, therefore we have $\lim F(X_\Delta) = \lim(\lim F(X))$, with the notation above. Since F is faithful and satisfies (\otimes_2) , we conclude that $\mathcal{L}(X_\Delta) = \mathcal{L}(\mathcal{L}X)$. \square

We now give a simple example showing that (Δ) does not follow from the other assumptions.

Example 4.9. Let $P_0 = \{\langle m, n \rangle \in \omega \times (\omega + 1) : m \leq n\}$ treated as a poset category with the coordinate-wise ordering. Let $P = P_0 \cup \{a, b\}$ where a, b are incomparable and $\langle m, n \rangle < a$ precisely for $m \leq n < \omega$, while $\langle m, n \rangle < b$ for every $m \leq n \leq \omega$. We define a quasi-limiting operator \mathcal{L} on P as follows. Given a strictly increasing sequence \vec{x} we define $\mathcal{L}\vec{x} = \sup_{P_0} \vec{x}$ provided that $\vec{x} \subseteq k \times (\omega + 1)$ for some k . Otherwise, we have two possibilities: either $\vec{x} \subseteq \omega \times \omega$ and then we set $\mathcal{L}\vec{x} = a$, or else $\vec{x} = \{x_n\}_{n \in \omega}$, where $x_n \in \omega \times \{\omega\}$ for all but finitely many $n \in \omega$. In this case we set $\mathcal{L}\vec{x} = b$. It is clear that this defines a quasi-limiting operator and $P_0 \subseteq P$ witnesses the failure of (Δ) .

4.4 Cofinality revisited

Theorem 4.10. *Let $\langle \mathfrak{K}, \mathcal{L}, \mathcal{A} \rangle$ be a κ -Fraïssé-Jónsson category, where κ is an infinite regular cardinal. Assume \vec{u} is an \mathcal{L} -continuous κ -Fraïssé sequence in \mathfrak{K} . Then for every \mathcal{L} -continuous sequence $\vec{x} \in \text{Seq}_{\leq \kappa}(\mathfrak{K})$ there exists an arrow of sequences*

$$\vec{1}: \vec{x} \rightarrow \vec{u}.$$

Proof. Fix an \mathcal{L} -continuous sequence $\vec{x}: \lambda \rightarrow \mathfrak{K}$, where $\lambda \leq \kappa$. If $\lambda < \kappa$ then we may use \mathcal{L} extending the sequence to $\lambda + 1$ and we use condition (U) of the definition of a Fraïssé sequence for obtaining \vec{v} . Thus, we may assume that $\lambda = \kappa$.

We shall define inductively a functor $I: [\kappa]^{\leq 2} \rightarrow \mathfrak{K}$ such that $I^0 = \vec{x}$ and I_Δ is a cofinal subsequence $\vec{u} \circ \varphi$ of \vec{u} (so $\varphi: \kappa \rightarrow \kappa$ will be an increasing map). Using property (U) of the Fraïssé sequence, we may assume that $x_0 = I_{\{0,0\}} = u_0$ and we set $\varphi(0) = 0$. Suppose I has been constructed for all pairs $\{\xi, \eta\}$ with $\xi \leq \eta < \beta$.

Assume first that $\beta = \alpha + 1$. Using \mathcal{A} -amalgamations, we construct $I_{\{\xi, \beta\}}$ and the corresponding arrows by induction on $\xi < \alpha$. We start by using an \mathcal{A} -amalgamation, knowing that $I_{\{0, \beta\}}$ must be equal to x_β . At limit steps we use the fact that \mathcal{A} is \mathcal{L} -continuous. We finish this procedure at step α . Finally, using condition (A) of the Fraïssé sequence, we find $\varphi(\beta) > \varphi(\alpha)$ and an arrow $g: I_{\{\alpha, \beta\}} \rightarrow u_{\varphi(\beta)}$ such that

$$g \circ I_{\{\alpha, \alpha\}}^{\{\alpha, \beta\}} = u_{\varphi(\alpha)}^{\varphi(\beta)}.$$

Note that, by the inductive hypothesis, $I_{\{\alpha, \alpha\}} = u_{\varphi(\alpha)}$. We set

$$I_{\{\beta, \beta\}} = u_{\varphi(\beta)} \quad \text{and} \quad I_{\{\alpha, \beta\}}^{\{\beta, \beta\}} = g.$$

Assume now that β is a limit ordinal. We set $\varphi(\beta) = \sup_{\xi < \beta} \varphi(\xi)$. For each $\xi < \beta$ let $I_{\{\xi, \beta\}}$ be the \mathcal{L} -limit of the sequence that looks as follows.

$$(\spadesuit) \quad I_{\{\xi, \xi\}} \longrightarrow I_{\{\xi, \xi+1\}} \longrightarrow \cdots \longrightarrow I_{\{\xi, \alpha\}} \longrightarrow I_{\{\xi, \alpha+1\}} \longrightarrow \cdots$$

Since \mathcal{A} is \mathcal{L} -continuous, for each $\alpha < \beta$ there is an arrow from $I_{\{\alpha, \beta\}}$ to $I_{\{\alpha+1, \beta\}}$ compatible with all arrows below. Applying the diagonalization property, we obtain $I_{\{\beta, \beta\}} = u_{\varphi(\beta)}$ as the \mathcal{L} -limit of the sequence (\spadesuit) .

This finishes the inductive construction. Finally, since $I_{\{\alpha, \alpha\}} = u_{\varphi(\alpha)}$, the collection of arrows $\vec{v} = \{I_{\{0, \alpha\}}^{\{\alpha, \alpha\}}\}_{\alpha < \kappa}$ is the required arrow from \vec{x} to \vec{u} . \square

We shall now discuss a class of amalgamation structures for which the proof above can be simplified, avoiding the diagonalization.

An amalgamation structure \mathcal{A} will be called *stable* if it contains all squares of the form

$$\begin{array}{ccc} b & \xrightarrow{g} & c \\ f \uparrow & & \uparrow h \\ a & \xrightarrow{\text{id}_a} & a \end{array}$$

It turns out that amalgamation structures considered in Section 6 have this property.

Lemma 4.11. *Let \mathcal{L} be a quasi-limiting operator on $\text{Seq}_{< \kappa}(\mathfrak{K})$. Then every stable \mathcal{L} -continuous amalgamation structure in \mathfrak{K} has the diagonalization property.*

Proof. Fix a functor $X: [\lambda]^{\leq 2} \rightarrow \mathfrak{K}$ as in condition (Δ) . Given $\beta_0 < \beta < \lambda$, the natural composition of the following two diagrams is in \mathcal{A} :

$$\begin{array}{ccc}
 X_{\{0,\beta_1\}} & \xrightarrow{X_{\{0,\beta_1\}}^{\{\beta_0,\beta_1\}}} & X_{\{\beta_0,\beta_1\}} \\
 \uparrow X_{\{0,\beta_0\}}^{\{0,\beta_1\}} & & \uparrow X_{\{\beta_0,\beta_0\}}^{\{\beta_0,\beta_1\}} \\
 X_{\{0,\beta_0\}} & \xrightarrow{X_{\{0,\beta_0\}}^{\{\beta_0,\beta_0\}}} & X_{\{\beta_0,\beta_0\}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 X_{\{\beta_0,\beta_1\}} & \xrightarrow{X_{\{\beta_0,\beta_1\}}^{\{\beta_1,\beta_1\}}} & X_{\{\beta_1,\beta_1\}} \\
 \uparrow X_{\{\beta_0,\beta_0\}}^{\{\beta_0,\beta_1\}} & \nearrow X_{\{\beta_0,\beta_0\}}^{\{\beta_1,\beta_1\}} & \\
 X_{\{\beta_0,\beta_0\}} & &
 \end{array}$$

This is because \mathcal{A} is stable and the triangle above can be replaced by a square with an identity arrow at the bottom. Now the functor X can be reduced to a sequence of squares in \mathcal{A} which, by \mathcal{L} -continuity, converges to an arrow from $F: \rightarrow 0\mathcal{L}X^0\mathcal{L}(X_\Delta)$. The same holds when replacing 0 by a fixed positive ordinal $< \lambda$, obtaining a co-cone of arrows $\{F_\xi\}_{\xi < \lambda}$ for the sequence $\{\mathcal{L}X^\xi\}_{\xi < \lambda}$. In particular, $\mathcal{L}(\mathcal{L}X) = \mathcal{L}(X_\Delta)$. \square

It is usually much easier to check that a given amalgamation structure is stable, rather than checking the diagonalization property directly (unless $\mathcal{L} = \text{lim}$).

Notice that, in the case of stable amalgamation structures, the proof of Theorem 4.10 can be simplified by constructing an arrow of sequences directly. More important here is the property of this arrow. Namely, given an amalgamation structure \mathcal{A} , an arrow of sequences $F: \vec{x} \rightarrow \vec{y}$ will be called \mathcal{A} -admissible if it is equivalent to a natural transformation from \vec{x} to $\vec{y} \circ \varphi$ (where, as usual, φ is an order preserving map between the corresponding ordinals) in which all squares belong to \mathcal{A} . More precisely, all squares of the form

$$\begin{array}{ccc}
 x_\alpha & \xrightarrow{x_\alpha^\beta} & x_\beta \\
 f_\alpha \downarrow & & \downarrow f_\beta \\
 y_{\varphi(\alpha)} & \xrightarrow{y_{\varphi(\alpha)}^{\varphi(\beta)}} & y_{\varphi(\beta)}
 \end{array}$$

are required to be in \mathcal{A} . Summarizing, an easy adaptation of the proof of Theorem 6 gives:

Theorem 4.12. *Assume $\langle \mathfrak{K}, \mathcal{L}, \mathcal{A} \rangle$ is a κ -Fraïssé-Jónsson category, where $\kappa = \text{cf } \kappa \geq \aleph_0$ and \mathcal{A} is a stable amalgamation structure. Assume \vec{u} is an \mathcal{L} -continuous κ -Fraïssé sequence in \mathfrak{K} . Then for every \mathcal{L} -continuous sequence \vec{x} of length $\leq \kappa$ there exists an \mathcal{A} -admissible arrow $\vec{f}: \vec{x} \rightarrow \vec{u}$.*

Clearly, all \mathcal{A} -admissible arrows provide a subcategory of the category of sequences and the statement above brings more information than Theorem 6, even in the case of countable sequences. We shall use it in Section 6, aiming at specific applications.

5 Some examples

This section contains the first series of examples concerning Fraïssé sequences, illustrating our ideas and results.

Perhaps one of the simplest is the category of sets \mathbf{Set} . If we allow all possible functions as morphisms, then the singleton is trivially a Fraïssé object. If we deal with monics only, at the same time restricting to sets of cardinality $< \kappa$, we see that any κ -chain whose union has cardinality κ is Fraïssé. This, however, is trivial and fits into the classical Fraïssé-Jónsson theory. On the other hand, starting from the category of finite sets, one can get non-trivial Fraïssé sequences. This will be demonstrated in Section 6. Below we present some examples that do not fit completely into the model-theoretic framework.

5.1 Reversed Fraïssé limits

Here we describe briefly the theory of inverse limits of models, the so-called *projective Fraïssé* theory, developed by Irwin & Solecki [14]. Fix a first-order language L and consider some class \mathfrak{M} of L -models. The classical Fraïssé-Jónsson theory deals with the category \mathfrak{M}^e of all embeddings of L -models. Recall that $f: M \rightarrow N$ is an *embedding* if it is a one-to-one homomorphism such that for every n -ary relation symbol $R \in L$, the following equivalence

$$R^M(f(x_1), \dots, f(x_n)) \iff R^N(x_1, \dots, x_n)$$

holds for every $x_1, \dots, x_n \in M$. In the reversed Fraïssé-Jónsson theory one deals with *quotient maps* $f: M \rightarrow N$ that are, by definition, surjective homomorphisms satisfying the following formula

$$R^N(y_0, \dots, y_{n-1}) \implies (\exists x_0, \dots, x_{n-1} \in M) R^M(x_0, \dots, x_{n-1}) \wedge (\forall i < n) f(x_i) = y_i$$

for each n -ary relation symbol $R \in L$ and for each $y_0, \dots, y_{n-1} \in N$. The two notions are dual in some sense, due to the following easy fact.

Proposition 5.1. *Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be homomorphisms of models of a fixed first-order language. Then*

(a) $g \circ f$ is an embedding $\implies f$ is an embedding.

(b) $g \circ f$ is a quotient map $\implies g$ is a quotient map.

The following result from [14] is a direct application of the results from Section 3. It can also be derived from the results of [11], however our approach is more direct and explains why the topology is needed.

Theorem 5.2. *Let \mathfrak{M} be a countable class of finite models of a fixed first-order language L . Suppose \mathfrak{M} satisfies the following conditions:*

- (J) For every $a, b \in \mathfrak{M}$ there exist $c \in \mathfrak{M}$ and quotient maps $f: c \rightarrow a$ and $g: c \rightarrow b$.
- (A) Given quotient maps $f: c \rightarrow a$ and $g: c \rightarrow b$ with $a, b, c \in \mathfrak{M}$, there exist $w \in \mathfrak{M}$ and quotient maps f', g' for which the diagram

$$\begin{array}{ccc} b & \xleftarrow{g'} & w \\ g \downarrow & & \downarrow f' \\ c & \xleftarrow{f} & a \end{array}$$

commutes.

Then there exists a unique (up to a topological isomorphism) topological L -model \mathbb{M} satisfying the following conditions:

- (1) \mathbb{M} is the inverse limit of a sequence of models from \mathfrak{M} with quotient maps.
- (2) Every model from \mathfrak{M} is a continuous quotient of \mathbb{M} .
- (3) Given continuous quotients $p: \mathbb{M} \rightarrow a$ and $q: \mathbb{M} \rightarrow b$ with $a, b \in \mathfrak{M}$, given a quotient map $f: b \rightarrow a$, there exists a topological isomorphism $h: \mathbb{M} \rightarrow \mathbb{M}$ for which the diagram

$$\begin{array}{ccc} \mathbb{M} & \xleftarrow{h} & \mathbb{M} \\ p \downarrow & & \downarrow q \\ a & \xleftarrow{f} & b \end{array}$$

is commutative.

Furthermore, every L -model which is the inverse limit of a sequence of models from \mathfrak{M} with quotient maps is a continuous quotient of \mathbb{M} .

Proof. Let \mathfrak{K} be the category whose objects are elements of \mathfrak{M} and an arrow from $a \in \mathfrak{M}$ into $b \in \mathfrak{M}$ is a quotient map $f: b \rightarrow a$. It is obvious that conditions (J) and (A) translate to directedness and the amalgamation property. Since \mathfrak{M} is countable, the category \mathfrak{K} is countable. Now observe that the category $\sigma\mathfrak{K}$ can be naturally identified with the category of compact L -models that are inverse limits of sequences of models from \mathfrak{M} . In fact, the topology becomes natural here, because given two sequences \vec{x} and \vec{y} in \mathfrak{K} and taking X and Y to be their inverse limits in the category of sets, one can easily check that precisely the continuous quotient maps $f: Y \rightarrow X$ correspond to $\sigma\mathfrak{K}$ -arrows from \vec{x} to \vec{y} . Summarizing: the existence and properties of \mathbb{M} follow directly from Theorems 3.7, 3.11 and 3.15. \square

The main problem to get the reversed Fraïssé limit is, as usual, showing that the class of models in question has the reversed amalgamation property, i.e., satisfies condition (A) above. In fact, in [14] this property for the class of linear graphs is crucial and non-trivial. It turns out however, that for many classes of models the reversed amalgamation can be proved easily.

Lemma 5.3. *Let \mathfrak{M} be a class of models of a first-order language L . Assume \mathfrak{M} is closed under finite products and substructures. Then for every quotient maps $f: c \rightarrow a$ and $g: c \rightarrow b$ such that $a, b, c \in \mathfrak{M}$ there exist quotient maps $f': a \rightarrow w$ and $g': b \rightarrow w$ for which the diagram*

$$\begin{array}{ccc} b & \xleftarrow{g'} & w \\ g \downarrow & & \downarrow f' \\ c & \xleftarrow{f} & a \end{array}$$

is commutative.

Proof. Define

$$w = \{\langle x, y \rangle \in a \times b : f(x) = g(y)\}.$$

It is straight to check that w is a submodel of the product $a \times b \in \mathfrak{M}$. We let f' and g' to be the canonical projections. It is obvious that they are homomorphisms. It remains to check that f', g' are quotient maps. Fix an n -ary relation symbol R in L and fix $x_0, \dots, x_{n-1} \in a$ such that $R(x_0, \dots, x_{n-1})$ holds. Since $R(f(x_0), \dots, f(x_{n-1}))$ holds and g is a quotient map, there exist $b_0, \dots, b_{n-1} \in b$ such that $R(b_0, \dots, b_{n-1})$ holds and $g(b_i) = f(x_i)$ for $i < n$. Let $p_i = \langle x_i, b_i \rangle$ for $i < n$. Note that $\{p_i\}_{i < n} \subseteq w$ and $f'(p_i) = x_i$ for $i < n$. Finally, $R(p_0, \dots, p_{n-1})$ holds. This shows that f' is a quotient map. The same argument shows that g' is a quotient map. \square

The statement above is particularly useful for some classes of algebras, like groups, semilattices, rings, etc. Recall that a topological algebra is *pro-finite* if it is the inverse limit of a system of finite algebras with epimorphisms. Lemma 5.3 and the results above give the following:

Corollary 5.4. *Let \mathfrak{M} be a class of pro-finite algebras, stable under subalgebras and finite products. Assume that \mathfrak{M} contains countably many isomorphic types of finite algebras. Then there exists a unique, up to a topological isomorphism, second countable pro-finite algebra \mathbb{A} satisfying conditions (2) and (3) from Theorem 5.2. Furthermore, every second countable pro-finite algebra from \mathfrak{M} is a continuous quotient of \mathbb{A} .*

5.2 Monoids

Recall that a *semigroup* is a structure of the form $\langle S, \circ, 1 \rangle$, where \circ is an associative binary operation on S , and $1 \in S$ is such that $1 \circ s = s = s \circ 1$ holds for every $s \in S$. A semigroup $\mathfrak{S} = \langle S, \circ, 1 \rangle$ can be viewed as a category with a single object S , whose arrows are all elements of S and 1 is the identity arrow. Such a category is called a *monoid*. A sequence in a monoid \mathfrak{S} is naturally represented as $\{a_n\}_{n \in \omega}$, where $a_n \in S$ for every $n \in \omega$. This corresponds to the formal sequence

$$S \xrightarrow{a_0} S \xrightarrow{a_1} S \longrightarrow \dots \longrightarrow S \xrightarrow{a_n} S \longrightarrow \dots$$

In particular, the bonding arrow from the n th to the m th element of the sequence is the composition $a_{m-1} \circ \dots \circ a_n$.

Note that every monoid is automatically directed. The amalgamation property can be rephrased as

$$(\forall f, g)(\exists f', g') f' \circ f = g' \circ g.$$

This is automatically fulfilled when S is commutative. In particular, every countable commutative monoid has a Fraïssé sequence of length ω .

It turns out that, even the set of natural numbers provides a monoid with many non-isomorphic ω -sequences. Namely, consider \mathbb{N}^+ , the set of all positive integers, as a multiplicative monoid. Formally, a sequence in \mathbb{N}^+ is a function $\varphi: \Delta \rightarrow \mathbb{N}^+$, where $\Delta = \{(m, n) \in \omega \times \omega : m \leq n\}$, satisfying $\varphi(n, n) = 1$ and $\varphi(n, k) \cdot \varphi(m, n) = \varphi(m, k)$ for every $m < n < k$. Clearly, the relevant values are $\varphi(n, n+1)$, therefore $\sigma\mathbb{N}^+$ can be naturally identified with sequences of positive integers. Now, it is an easy exercise that a sequence $\{u_n\}_{n \in \omega}$ is Fraïssé if and only if for every prime $p \in \mathbb{N}^+$, for every $n \in \omega$, there is $m > n$ such that p divides $u_n u_{n+1} \cdots u_m$. Since there are infinitely many prime numbers, it is easy to see that $\sigma\mathbb{N}^+$ has continuum many pairwise incomparable sequences. This is not the case for the additive monoid \mathbb{N} , where there are only two isomorphic types in $\sigma\mathbb{N}$, namely, the equivalence class of the constant 0 sequence and the equivalence class of the constant 1 sequence. The latter one is Fraïssé in \mathbb{N} .

Another class of monoids are semilattices. Namely, fix a join semilattice $\langle L, \vee, 0 \rangle$ and let \leq be the partial order induced by \vee , i.e. $x \leq y$ iff $x \vee y = y$. Since this is a commutative monoid, it has the amalgamation property. Let $\vec{u}: \kappa \rightarrow L$ be a sequence in L . Given $\alpha < \beta < \kappa$ we have that $u_0^\beta = u_\alpha^\beta \vee u_0^\alpha$, therefore $u_0^\alpha \leq u_0^\beta$. Assuming that \vec{u} is Fraïssé and using condition (A), we see that for every $x \in L$ there are $\alpha < \kappa$ and $y \in L$ such that $u_0^\alpha = y \vee x$. In other words, if \vec{u} is Fraïssé, then $\{u_0^\alpha\}_{\alpha < \kappa}$ is an increasing cofinal well ordered chain in $\langle L, \leq \rangle$. Conversely, given an increasing cofinal sequence $\{a_\alpha\}_{\alpha < \kappa}$ in $\langle L, \leq \rangle$, the functor $\vec{u}: \kappa \rightarrow L$, defined by $u_\alpha^\alpha = 0$ and $u_\alpha^\beta = a_\beta$ for $\alpha < \beta < \kappa$, is a Fraïssé sequence in L .

An important class of monoids are transformation semigroups. We briefly discuss the particular case of such monoids induced by Fraïssé sequences. Fix a category \mathfrak{K} such that $\sigma\mathfrak{K}$ has the amalgamation property and assume \vec{u} is a countable Fraïssé sequence in \mathfrak{K} . Consider $M_{\vec{u}} = \sigma\mathfrak{K}(\vec{u}, \vec{u})$. By Theorem 3.6, $M_{\vec{u}}$ is a σ -complete monoid. It turns out that $M_{\vec{u}}$ “encodes” all information about the category $\sigma\mathfrak{K}$ relevant for the existence of an ω_1 -Fraïssé sequence:

Theorem 5.5. *Let \mathfrak{K} be a category with an ω -Fraïssé sequence \vec{u} and such that $\sigma\mathfrak{K}$ has the amalgamation property. Then $M_{\vec{u}}$ has the amalgamation property and is cofinal in $\sigma\mathfrak{K}$. Furthermore, $\sigma\mathfrak{K}$ has an ω_1 -Fraïssé sequence if and only if the monoid $M_{\vec{u}}$ has an ω_1 -Fraïssé sequence.*

Note that, by the first part, a Fraïssé sequence in $M_{\vec{u}}$ is Fraïssé in $\sigma\mathfrak{K}$.

Proof. The fact that $M_{\vec{u}}$ has the amalgamation property and is cofinal in $\sigma\mathfrak{K}$ follows directly from Theorem 3.11. It remains to prove that if $\sigma\mathfrak{K}$ has an ω_1 -Fraïssé sequence then so does $M_{\vec{u}}$. For this aim, it suffices to show that $M_{\vec{u}}$ is dominated by $\leq \aleph_1$ many arrows. Fix a Fraïssé sequence $\omega_1: \vec{v} \rightarrow \sigma\mathfrak{K}$. Cutting the sequence if necessary, we may assume that there is a $\sigma\mathfrak{K}$ -arrow $H: \vec{u} \rightarrow \vec{v}_0$. Using Theorem 3.11, for each $\alpha < \omega_1$ choose an arrow $F_\alpha: \vec{v}_\alpha \rightarrow \vec{u}$. Let G_α denote the bonding arrow from \vec{v}_0 to \vec{v}_α . Define

$$\mathcal{F} = \{F_\alpha \circ G_\alpha \circ H: \alpha < \omega_1\}.$$

We claim that \mathcal{F} is dominating in $M_{\vec{u}}$. Fix a $\sigma\mathfrak{K}$ -arrow $f: \vec{u} \rightarrow \vec{u}$. Using the amalgamation property, we find $\sigma\mathfrak{K}$ -arrows H', f' such that $f' \circ f = H' \circ H$. Using the fact that \vec{v} is Fraïssé, we find $\alpha > 0$ and a $\sigma\mathfrak{K}$ -arrow g such that $g \circ H' = G_\alpha$. Finally, $(F_\alpha \circ g \circ f') \circ f = F_\alpha \circ G_\alpha \circ H \in \mathcal{F}$. \square

It is clear that the result above can be generalized to uncountable sequences, adding suitable assumptions concerning completeness.

5.3 Diagrams: the role of pushouts

We shall describe a quite general procedure of building a category with the amalgamation property. First of all, let us note that typical categories with monics do not admit pushouts. For instance, this is the case with the category of finite sets with one-to-one maps. In order to capture these situations, we define the following natural notion.

Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two categories. We say that \mathfrak{K} has pushouts in \mathfrak{L} (or, that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has pushouts) if for every \mathfrak{K} -arrows $f: c \rightarrow a$, $g: c \rightarrow b$ there exist \mathfrak{K} -arrows $f': a \rightarrow w$ and $g': b \rightarrow w$ such that

$$\begin{array}{ccc} b & \xrightarrow{g'} & w \\ g \uparrow & & \uparrow f' \\ c & \xrightarrow{f} & a \end{array}$$

is a pushout in \mathfrak{L} .

Now, fix a small category \mathbb{S} and let $\mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$ be the category whose objects are covariant functors from \mathbb{S} to \mathfrak{L} taking objects from \mathfrak{K} and the arrows are natural transformations into \mathfrak{K} . More precisely, x is an object of iff x is a functor from \mathbb{S} into \mathfrak{L} such that $x(s)$ is an object of \mathfrak{K} for every object s of \mathbb{S} . Furthermore, $f: x \rightarrow y$ is an $\mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$ -arrow iff it is a natural transformation from x to y whose all components (arrows) are in \mathfrak{K} . In other words, for every object s of \mathbb{S} , the arrow $f(s)$ belongs to \mathfrak{K} .

Lemma 5.6. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ be a pair of categories such that \mathfrak{K} has pushouts in \mathfrak{L} and let \mathbb{S} be a small category. Then $\mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$ has pushouts in $\mathfrak{f}(\mathbb{S}, \mathfrak{L}, \mathfrak{L})$.*

Proof. Fix $a, b, c \in \mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$ and fix natural transformations $f: c \rightarrow a$, $g: c \rightarrow b$. We are going to define $w \in \mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$. Fix $p \in \mathbb{S}$. Let $\langle f'(p), g'(p) \rangle$ be the pushout of $\langle f(p), g(p) \rangle$ in \mathfrak{L} , and let $w(p)$ be the common co-domain of $f'(p)$ and $g'(p)$. Now fix an arrow $j: p \rightarrow q$ in \mathbb{S} . Using the property of a pushout, there is a unique arrow $w(j)$ making the following diagram commutative.

$$\begin{array}{ccccc}
 c(q) & \xrightarrow{g(q)} & b(q) & & \\
 \uparrow & \searrow f(q) & \uparrow & \searrow g'(q) & \\
 c(j) & & a(q) & \xrightarrow{f'(q)} & w(q) \\
 & & \uparrow a(j) & & \uparrow b(j) \\
 c(p) & \xrightarrow{g(p)} & b(p) & & \\
 \uparrow & \searrow f(p) & \uparrow & \searrow g'(p) & \\
 & & a(p) & \xrightarrow{f'(p)} & w(p) \\
 & & & & \uparrow w(j)
 \end{array}$$

Indeed, $\langle f'(q) \circ a(j), g'(q) \circ b(j) \rangle$ is an amalgamation of $\langle f(p), g(p) \rangle$. By uniqueness, we have that $w(j \circ k) = w(j) \circ w(k)$, whenever j, k are compatible arrows in \mathbb{S} .

Thus we have defined a functor $w: \mathbb{S} \rightarrow \mathfrak{K}$. Further, $p \mapsto f'(p)$ and $p \mapsto g'(p)$ are natural transformations from a to w and from b to w respectively.

It is clear that $f' \circ f = g' \circ g$.

It remains to check that $\langle f', g' \rangle$ is a pushout of $\langle f, g \rangle$ in $\mathfrak{f}(\mathbb{S}, \mathfrak{L}, \mathfrak{L})$. For this aim, fix $v \in \mathfrak{f}(\mathbb{S}, \mathfrak{L}, \mathfrak{L})$ and natural transformations $f'': a \rightarrow v$, $g'': b \rightarrow v$ such that $f'' \circ f = g'' \circ g$. By the fact that $\langle f', g' \rangle$ is a pushout of $\langle f, g \rangle$ in \mathfrak{L} , for each $p \in \mathbb{S}$ there is a unique \mathfrak{L} -arrow $h(p): w(p) \rightarrow v(p)$ satisfying $h(p) \circ f'(p) = f''(p)$ and $h(p) \circ g'(p) = g''(p)$. This defines uniquely a map $h: \mathbb{S} \rightarrow \text{Arr } \mathfrak{L}$ that satisfies $h \circ f' = f''$ and $h \circ g' = g''$. It remains to check that h is indeed a natural transformation.

Fix an arrow $j: p \rightarrow q$ in \mathbb{S} and let $k = h(q) \circ w(j)$, $\ell = v(j) \circ h(p)$. We need to show that $k = \ell$. Notice that $k \circ f'(p) = f''(q) \circ a(j)$ and $k \circ g'(p) = g''(q) \circ b(j)$. Also, $\ell \circ f'(p) = f''(q) \circ a(j)$ and $\ell \circ g'(p) = g''(q) \circ b(j)$. It follows that $k = \ell$, because $\langle f'(p), g'(p) \rangle$ is a pushout of $\langle f(p), g(p) \rangle$ in \mathfrak{L} . \square

The interesting part of the lemma above is that $\mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$ has the amalgamation property, whenever \mathfrak{K} has pushouts in \mathfrak{L} .

We now demonstrate a possible use of Lemma 5.6 in the context of Fraïssé model-theoretic structures.

Given categories $\mathfrak{K} \subseteq \mathfrak{L}$, we say that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the *mixed pushout property* if for every \mathfrak{K} -arrow $i: c \rightarrow a$, for every \mathfrak{L} -arrow $f: c \rightarrow b$, there exist $j \in \mathfrak{K}$ and $g \in \mathfrak{L}$

such that

$$\begin{array}{ccc} b & \xrightarrow{j} & w \\ f \uparrow & & \uparrow g \\ c & \xrightarrow{i} & a \end{array}$$

is a pushout in \mathfrak{L} . In case \mathfrak{M} is a class of models of some first-order language, it is natural to consider $\mathfrak{K}(\mathfrak{M})$ to be the category of all embeddings between models of \mathfrak{M} and $\mathfrak{L}(\mathfrak{M})$ to be the category of all homomorphisms between these models. We then say that \mathfrak{M} has the *mixed pushout property* if so does $\langle \mathfrak{K}(\mathfrak{M}), \mathfrak{L}(\mathfrak{M}) \rangle$. We denote by $\overline{\mathfrak{M}}$ the class of all (countable) models that are unions of ω -chains of models from \mathfrak{M} .

Theorem 5.7. *Let \mathfrak{M} be a countable Fraïssé class of finitely generated models, with the mixed pushout property. Let W denote the Fraïssé limit of \mathfrak{M} . Then there exists a unique (up to isomorphism) homomorphism $L: W \rightarrow W$ satisfying the following conditions.*

- (a) *For every $X, Y \in \overline{\mathfrak{M}}$, for every homomorphism $F: X \rightarrow Y$ there exist embeddings $I_X: X \rightarrow W$ and $I_Y: Y \rightarrow W$ such that the square*

$$\begin{array}{ccc} W & \xrightarrow{L} & W \\ I_X \uparrow & & \uparrow I_Y \\ X & \xrightarrow{F} & Y \end{array}$$

is commutative.

- (b) *Given finitely generated substructures x_0, x_1, y_0, y_1 of W such that $L[x_i] \subseteq y_i$ for $i < 2$, given isomorphisms $h_i: x_i \rightarrow y_i$ for $i < 2$ such that $L \circ h_0 = h_1 \circ L$, there exist automorphisms $H_i: W \rightarrow W$ extending h_i for $i < 2$, and such that $L \circ H_0 = H_1 \circ L$.*

Proof. Consider the category $\mathfrak{C} = \mathfrak{f}(2, \mathfrak{K}(\mathfrak{M}), \mathfrak{L}(\mathfrak{M}))$, where 2 denotes the two-element poset category. The assumptions above, combined with Lemma 5.6, give a Fraïssé sequence $\{\ell_n\}_{n \in \omega}$ which translated back to $\mathfrak{L}(\mathfrak{M})$ looks as follows.

$$\begin{array}{ccccccc} u_0 & \longrightarrow & u_1 & \longrightarrow & \dots & \longrightarrow & u_n & \longrightarrow & \dots \\ \ell_0 \downarrow & & \ell_1 \downarrow & & & & \ell_n \downarrow & & \\ v_0 & \longrightarrow & v_1 & \longrightarrow & \dots & \longrightarrow & v_n & \longrightarrow & \dots \end{array}$$

The horizontal arrows are embeddings and the vertical arrows are homomorphisms of models. Without loss of generality, we may assume that the horizontal arrows are inclusions. Now let $U = \bigcup_{n \in \omega} u_n$, $V = \bigcup_{n \in \omega} v_n$ and define $L = \bigcup_{n \in \omega} \ell_n$. It is clear that L satisfies conditions (a) and (b) above. It remains to check that both U and V are isomorphic to the \mathfrak{M} -Fraïssé limit W . For this aim, it suffices to show that both $\{u_n\}_{n \in \omega}$ and $\{v_n\}_{n \in \omega}$ are Fraïssé sequences in $\mathfrak{K}(\mathfrak{M})$.

It is clear that both sequences satisfy (U), because identity arrows can be viewed as objects of \mathfrak{C} .

Fix $n \in \omega$ and fix an embedding $j: u_n \rightarrow x$, where $x \in \mathfrak{M}$. Using the mixed pushout property, we can find an embedding $k: v_n \rightarrow y$ and a homomorphism $f: x \rightarrow y$ such that $k \circ \ell_n = f \circ j$. Now f can be regarded as an object of \mathfrak{C} and the pair $\langle j, k \rangle$ is an arrow of \mathfrak{C} . Using the Fraïssé property of the sequence $\{\ell_n\}_{n \in \omega}$, we find $m \geq n$ and embeddings $j': x \rightarrow u_m$, $k': y \rightarrow v_m$ such that $\ell_m \circ j' = k' \circ f$ and $j' \circ j$, $k' \circ k$ are inclusions. In particular, this shows that $\{u_n\}_{n \in \omega}$ is Fraïssé and therefore $U = W$.

Now fix an embedding $i: v_n \rightarrow z$. Let $g = i \circ \ell_n$. The pair $\langle \text{id}_{u_n}, i \rangle$ can be viewed as a \mathfrak{C} -arrow into g , therefore by the Fraïssé property of $\{\ell_n\}_{n \in \omega}$ we again find $m \geq n$ and embeddings $e: u_n \rightarrow u_m$, $i': z \rightarrow u_m$ such that, among other conditions, the composition $i' \circ i$ is the inclusion $v_n \subseteq v_m$. This shows that $\{v_n\}_{n \in \omega}$ is Fraïssé and consequently $V = W$. \square

Note that the *mixed* pushout property is needed for concluding that the domain of the universal homogeneous homomorphism is the Fraïssé limit of the class in question. Relaxing this by assuming only that embeddings have pushouts among all homomorphisms, one still gets a homomorphism with properties (a) and (b) above, whose co-domain is the Fraïssé limit of the considered class of models. It is clear how to formulate a similar statement for reversed Fraïssé limits, where pushouts are replaced by pullbacks.

It might be interesting to compare Theorem 5.7 with the recent work of Pech & Pech [34], where the authors consider comma categories, in particular obtaining universal homomorphisms as Fraïssé limits, constructed from categories with mixed pushouts.

5.4 Universal homogeneous categories

It is tempting to ask for the existence of a Fraïssé sequence in “categories of all categories”. Of course, one needs to be consistent with the axioms of set theory, therefore we should only talk about classes of small categories. Actually, a small category is nothing but a directed graph endowed with an additional “composition” operation, satisfying the obvious axioms. Arrows between small categories are just covariant functors. In order to avoid trivialities, we should consider injective functors (possibly onto full subcategories). Summarizing, we can ask for a Fraïssé sequence in a directed class of categories that has the amalgamation property with respect to injective functors. Recall that a functor is *injective* if it is one-to-one on the class of arrows (in particular, such a functor must be one-to-one on the class of objects).

Actually, this question had already been addressed by Trnková [39] in 1966. Her work was set up within the Bernays-Gödel axiom system for set theory. In this framework, every category is the union of a chain of small categories and the amalgamation property (which appears to be quite non-trivial—proved by Trnková in her

earlier work [38]) allows concluding the existence of a “Fraïssé chain” of categories. We can add more information to the results of [39] by adding homogeneity. Namely, every injective functor between small subcategories extends to a bijective functor of the big universal category under consideration. Since we work in the usual Zermelo-Fraenkel set theory (with the axiom of choice), we discuss briefly one obvious class of categories. Other examples (like additive, concrete categories, and so on) can be easily adapted from Trnková’s work [39], restricting the size of categories and adding some cardinal-arithmetic assumptions.

A natural example is the class \mathfrak{K} of all *finite categories*. A category \mathfrak{K} is *finite* if it has finitely many isomorphic types and each hom-set of \mathfrak{K} is finite. It is rather clear that this class has the amalgamation property. It also has the initial object (the empty category), therefore it is directed. Obviously, \mathfrak{K} is dominated by a countable family of injective functors, say, between finite categories whose objects are natural numbers and arrows are taken from a prescribed countable set. By Theorem 3.7, \mathfrak{K} has a unique countable Fraïssé sequence.

Let \mathfrak{U} be the category obtained as the “co-limit” of the Fraïssé sequence in \mathfrak{K} . Then \mathfrak{U} is a category with countably many isomorphic types of objects and each hom-set of \mathfrak{U} is countable. It is an easy exercise to see that every other category with these properties is equivalent to a subcategory of \mathfrak{U} . Finally, \mathfrak{U} has the following homogeneity property: Given an injective functor $F: \mathfrak{S} \rightarrow \mathfrak{T}$, between finite subcategories of \mathfrak{U} , there exists a bijective functor $H: \mathfrak{U} \rightarrow \mathfrak{U}$ extending F .

One can consider a similar construction using injective functors $F: \mathfrak{S} \rightarrow \mathfrak{T}$ such that $F[\mathfrak{S}]$ is a full subcategory of \mathfrak{T} . The Fraïssé limit would be a countable category \mathfrak{U}_f whose all hom-sets are finite, all other countable categories with this property would be equivalent to full subcategories of \mathfrak{U}_f and the obvious variant of homogeneity would hold. We leave the details to interested readers.

5.5 Binary trees: failure of the back-and-forth principle

We describe below the announced example of a category of trees which has many pairwise non-equivalent ω_1 -Fraïssé sequences.

Recall that a tree T is *bounded* if for every $x \in T$ there is $t \in \max T$ such that $x \leq t$. A subset A of T is *closed* if $\sup C \in A$ for every chain $C \subseteq A$. This is equivalent to saying that A is closed with respect to the *interval topology* on T generated by intervals of the form $[0, t]$ and $(s, t]$, where $s < t$.

We define the category \mathfrak{T}_2 as follows. The objects of \mathfrak{T}_2 are nonempty countable bounded binary trees. An arrow from $T \in \mathfrak{T}_2$ into $S \in \mathfrak{T}_2$ is a tree (i.e. strictly order preserving) embedding $f: T \rightarrow S$ such that $f[T]$ is a closed initial segment of S .

A tree T is *healthy* if every element of $T \setminus \max(T)$ has at least two immediate successors and for every $t \in T$ and $\alpha < \text{ht}(T)$ there exists $s \geq t$ such that $\text{Lev}_T(s) \geq \alpha$. An example of a healthy tree of height ω_1 is

$$T = \{x \in 2^{<\omega_1} : |\{\alpha : x(\alpha) = 1\}| < \aleph_0\}.$$

Note that all levels of T are countable. Setting $T_\alpha = \{x \in T: \text{dom}(x) \subseteq \alpha + 1\}$, we obtain an inductive sequence $\{T_\alpha\}_{\alpha < \omega_1}$ in the category \mathfrak{T}_2 . More generally, if S is any binary tree of height ω_1 whose all levels are countable then, setting

$$S_\alpha = \{x \in S: \text{the order type of } [0, x] \text{ is } \leq \alpha\}$$

we obtain an inductive sequence \vec{S} in \mathfrak{T}_2 , where each S_α^β is the inclusion, which is an arrow in \mathfrak{T}_2 . We shall say that \vec{S} is the *natural decomposition* of S .

Lemma 5.8. *Let $V \in \mathfrak{T}_2$ be a healthy tree of height $\alpha + 1$. Then every $T \in \mathfrak{T}_2$ with $\text{ht}(T) \leq \alpha + 1$ is isomorphic to a closed initial segment of V .*

Proof. Denote by \mathfrak{M} the class of all nonempty bounded binary trees T of height $\leq \alpha + 1$ such that $\text{max}(T)$ is finite. Given such a tree T , write $\text{max}(T) = \{w_0, \dots, w_{m-1}\}$ and define inductively $T_0 := [0, w_0]$ and $T_k := [0, w_k] \setminus (T_0 \cup \dots \cup T_{k-1})$. Then $\mathcal{D} = \{T_0, \dots, T_{m-1}\}$ is a natural decomposition of T into connected chains, induced by the enumeration of $\text{max}(T)$.

Let \mathfrak{L} be a category whose objects are pairs $\langle T, \mathcal{D} \rangle$, where $T \in \mathfrak{M}$ and \mathcal{D} is a natural decomposition into connected chains induced by an enumeration of $\text{max}(T)$, as described above. Given $\langle T, \mathcal{D} \rangle, \langle S, \mathcal{F} \rangle \in \mathfrak{L}$, an arrow in \mathfrak{L} is a tree embedding $f: T \rightarrow S$ with the following properties:

- (a) For each $D \in \mathcal{D}$ there is $F \in \mathcal{F}$ such that $f[D]$ is an initial segment of F .
- (b) For each $F \in \mathcal{F}$ there exists at most one D such that $f[D] \subseteq F$.

It is clear that these properties are preserved under the usual composition, so \mathfrak{L} is indeed a category. Now consider the given healthy tree V with $\text{max}(V) = \{e_n\}_{n \in \omega}$ and define $V^n = \bigcup_{i \leq n} [0, e_i]$. Let \mathcal{D}^n be the decomposition of V^n induced by the enumeration $\{e_0, \dots, e_n\}$ of $\text{max}(V^n)$. Then $\langle V^n, \mathcal{D}^n \rangle \in \mathfrak{L}$ and $\vec{V} = \{\langle V^n, \mathcal{D}^n \rangle\}_{n \in \omega}$ is an inductive sequence in \mathfrak{L} . We claim that:

- (1) \vec{V} is a Fraïssé sequence in \mathfrak{L} which has property (E).
- (2) If \vec{T} is an inductive sequence in \mathfrak{L} and $\vec{f} = \{f_n\}_{n \in \omega}$ is an embedding of \vec{T} into \vec{V} then the embedding $f: T \rightarrow V$ induced by \vec{f} has the property that $f[T]$ is a closed initial segment of V .

We first show (2): Fix $y \in V \setminus f[T]$. Find m and $D \in \mathcal{D}_m$ such that $y \in D$. Let \mathcal{S}_n be the natural decomposition of T_n , $\mathcal{S} = \bigcup_{n \in \omega} \mathcal{S}_n$. Then there is at most one $S \in \mathcal{S}$ such that $f[S] \subseteq D$. Moreover $f[S]$ is closed in D , so $D \setminus f[S]$ is a neighborhood of y disjoint from $f[T]$.

For the proof of (1), fix $\langle T, \mathcal{F} \rangle \in \mathfrak{L}$ and assume $f: T \rightarrow V_n$ is an arrow in \mathfrak{L} . Let $T \subseteq T'$ and let $\mathcal{F}' \supseteq \mathcal{F}$ so that the inclusion $T \subseteq T'$ is an arrow between $\langle T, \mathcal{F} \rangle$ and $\langle T', \mathcal{F}' \rangle$. Without loss of generality, we may assume that $\mathcal{F}' = \mathcal{F} \cup \{A\}$, i.e. T' differs from T by only one new branch. Let $a = \min A$. Then, by the definition of

the natural decomposition, a has an immediate predecessor $c \in T$ (note that $0 \in T$ so $a > 0$). Find $F \in \mathcal{F}$ such that $c \in F$. Then $f(c)$ has exactly two immediate successors in V and at most one belongs to $f[T]$, since c has only two immediate successors in T . Let $d \in V \setminus f[T]$ be an immediate of $f(c)$. Find a big enough $m > n$ so that there exists $D \in \mathcal{D}_m$ with $d \in D$. Since each maximal element of V has height α , D is a cofinal branch in V and hence A can be (uniquely) embedded into D as an initial segment. This embedding defines an extension $\bar{f}: T' \rightarrow V_m$ of f . Since \mathfrak{L} has a minimal object, this shows that \vec{V} is Fraïssé and satisfies (E).

Finally, fix $T \in \mathfrak{T}_2$ with $\text{ht}(T) \leq \alpha + 1$. Decompose T into an inductive ω -sequence, according to a fixed enumeration of $\max(T)$. Claims (1) and (2) say that T can be embedded into V as a closed initial subtree. This completes the proof. \square

Theorem 5.9. *Assume U is a healthy binary tree of height ω_1 , whose all levels are countable. Let \vec{U} be the natural \mathfrak{T}_2 -decomposition of U . Then \vec{U} is a Fraïssé sequence in \mathfrak{T}_2 which has the extension property. In particular, \mathfrak{T}_2 is directed and has the amalgamation property.*

Proof. Note that \mathfrak{T}_2 has a minimal object, namely the one-element tree. Clearly, such a tree embeds into U_0 . It suffices to show that \vec{U} satisfies (E), it will then follow that \vec{U} is a Fraïssé sequence. Since there exists a healthy binary tree of height ω_1 with countable levels, we shall be able to conclude that \mathfrak{T}_2 has a Fraïssé sequence satisfying (E) and consequently \mathfrak{T}_2 is directed and has the amalgamation property. Thus, it remains to show that \vec{U} has the extension property.

Fix $\alpha < \omega_1$ and fix an arrow $f: T \rightarrow U_\alpha$ in \mathfrak{T}_2 and assume that T is a closed initial subtree of S , i.e. the inclusion $T \subseteq S$ is an arrow of \mathfrak{T}_2 . Fix $\alpha < \omega_1$ so that $\text{ht}(S) < \alpha$. Let $\{s_n: n \in \omega\}$ enumerate all minimal elements of $S \setminus T$. Let t_n be the immediate predecessor of s_n . Then $t_n \in T$. Recall that $f(t_n)$ has exactly two immediate successors in U_α and at least one of them does not belong to $f[T]$, since otherwise t_n would already have two immediate successors in T . Let y_n be an immediate successor of $f(t_n)$ which does not belong to $f[T]$. Let $V_n = \{y \in U_\alpha: y \geq y_n\}$. Then V_n is a healthy binary tree of countable height. By Lemma 5.8 we can embed $G_n = \{s \in S: s \geq s_n\}$ onto a closed initial segment of V_n . Combining all these embeddings, we obtain an extension $\bar{f}: S \rightarrow U_\alpha$ of f . We claim that $\bar{f}[S]$ is closed in U_α . Indeed, if $C \subseteq S$ is a chain and $C \not\subseteq T$ then $c \geq s_n$ for some $c \in C$ and for some n . Thus $\sup C$ exists in S and hence $\sup \bar{f}[C]$ exists in $\bar{f}[G_n] \subseteq \bar{f}[S]$. \square

Theorem 5.10. *Assume \vec{U} and \vec{V} are ω_1 -Fraïssé sequences in \mathfrak{T}_2 inducing healthy trees U and V respectively. Assume further that $F: \vec{U} \rightarrow \vec{V}$ is an arrow of sequences. Then the trees U, V are isomorphic.*

Proof. Let $f: U \rightarrow V$ be the embedding induced by F , i.e. assuming both \vec{U}, \vec{V} are chains of trees, f is the union of arrows $f_\alpha: U_\alpha \rightarrow V_{\varphi(\alpha)}$ in \mathfrak{T}_2 , where $\varphi: \omega_1 \rightarrow \omega_1$ is an increasing function. We claim that $f[U]$ is closed in $V_{\varphi(\alpha)}$. Suppose otherwise and fix a sequence $x_0 < x_1 < \dots$ in U such that $y = \sup_{n \in \omega} f(x_n) \notin f[U]$. Find $\beta < \omega_1$

such that $\{x_n : n \in \omega\} \subseteq U_\beta$ and $y \in V_{\varphi(\beta)}$. Then $f(x_n) = f_\beta(x_n)$ and $y \notin f_\beta[U_\beta]$, which shows that f_β is not an arrow in \mathfrak{T}_2 , a contradiction.

We finally claim that $V = f[U]$, which of course shows that f is an isomorphism. Suppose $V \neq f[U]$ and fix a minimal element $y \in V \setminus f[U]$. Find $\alpha < \omega_1$ such that $y \in V_{\varphi(\alpha)}$. Since $f[U]$ is closed in $V_{\varphi(\alpha)}$, y has an immediate predecessor, say $v = f(u)$. Let a, b be the two immediate successors of u in U (which exist, because U is healthy). Then either $y = f(a)$ or $y = f(b)$, because V is binary and $f[U]$ is an initial segment of U . This is a contradiction. \square

It can be easily shown that every tree induced by an ω_1 -Fraïssé sequence in \mathfrak{T}_2 is healthy, therefore this assumption can be removed from the statement above.

Recall that sequences \vec{a} and \vec{b} of the same length are *comparable* if there exists an arrow of sequences \vec{f} such that either $\vec{f}: \vec{a} \rightarrow \vec{b}$ or $\vec{f}: \vec{b} \rightarrow \vec{a}$. Otherwise, we say that \vec{a} and \vec{b} are *incomparable*.

Corollary 5.11. *There exist two incomparable ω_1 -Fraïssé sequences in \mathfrak{T}_2 .*

Proof. Let $U = \{x \in 2^{<\omega_1} : |x^{-1}(1)| < \aleph_0\}$ and let V be a healthy binary Aronszajn tree. Clearly, U and V are not isomorphic. Both trees can be naturally decomposed into ω_1 -sequences \vec{U} and \vec{V} respectively. By Theorem 5.9, \vec{U} and \vec{V} are Fraïssé sequences. By Theorem 5.10, these sequences are incomparable. \square

It is not hard to see that there are uncountably many pairwise non-isomorphic healthy binary trees of height ω_1 . Actually, it was proved by Abraham & Shelah [1] that consistently there are 2^{\aleph_1} isomorphic types of Aronszajn trees. In particular, the category \mathfrak{T}_2 has (at least consistently) 2^{\aleph_1} many pairwise incomparable ω_1 -Fraïssé sequences.

5.6 Universal trees

We now discuss certain classes of trees and associated categories, showing how Fraïssé functors can help in finding universal trees. We fix two cardinals $\lambda > 0$, $\kappa \geq \aleph_0$ and we assume that κ is regular. We also fix a linearly ordered set L with $0 = \min L$ and such that

- (0) $0 = \min L$ exists and L is unbounded from above.
- (1) Every bounded increasing sequence of length $< \kappa$ has the supremum in L .
- (2) L contains isomorphic copies of all ordinals $< \kappa$.

Our aim is to find an L -embeddable λ -branching tree of height κ , universal for this class of trees.

Recall that a tree T is *L -embeddable* if it admits a $<$ -preserving function $\varphi: T \rightarrow L$. A standard inductive argument using condition (1) shows that such a function can be “corrected” to a continuous one. More precisely, there exists a continuous

$<$ -preserving function $\psi: T \rightarrow L$ such that $\psi \leq \varphi$. Furthermore, we may require that $\psi(0) = 0$. Recall that ψ is continuous if $\psi(\sup C) = \sup \psi[C]$ whenever C is a bounded chain in T . Given a pair $\langle T, \psi \rangle$, where ψ is as above, we will say that T is ψ -closed if for every unbounded chain $C \subseteq T$ the set $\psi[C]$ is unbounded in L .

Define the category $\mathfrak{T}_{L,\kappa,\lambda}$ as follows. The objects are pairs $\langle T, \psi \rangle$ such that T is a λ -branching tree of height $< \kappa$, φ is a continuous strictly increasing function from T to L such that $\psi(0) = 0$ and T is φ -closed. Recall that T is λ -branching if $|t^+| \leq \lambda$ for every $t \in T$.

A $\mathfrak{T}_{L,\kappa,\lambda}$ -arrow from $\langle T, \psi \rangle$ to $\langle S, \varphi \rangle$ is a tree embedding $f: T \rightarrow S$ such that $\varphi(f(t)) = \psi(t)$ for every $t \in T$, $f[T]$ is closed in S and S is an *end-extension* of $f[T]$. This means, by definition, that for every $t \in T$ and $s \in S \setminus f[T]$ it holds that $\text{Lev}_S(f(t)) < \text{Lev}_S(s)$. It is clear that $\mathfrak{T}_{L,\kappa,\lambda}$ fails the amalgamation property. Indeed, let $T = \{0\}$ be the trivial tree and let $f: T \rightarrow S$, and $g: T \rightarrow S$ be such that S is the 2-element tree $\{0, 1\}$ endowed with functions $\varphi_0, \varphi_1: S \rightarrow L$ such that $\varphi_0(1) \neq \varphi_1(1)$. Then it is not possible to find a pair $\langle R, \psi \rangle$ such that R is the end-extension of both $\langle S, \varphi_0 \rangle$ and $\langle S, \varphi_1 \rangle$ and $\psi: R \rightarrow L$ extends both φ_0 and φ_1 .

We shall now embed $\mathfrak{T}_{L,\kappa,\lambda}$ in a bigger category, as follows. Namely, let $\mathfrak{S}_{L,\kappa,\lambda}$ be the category whose objects are the same as the objects of $\mathfrak{T}_{L,\kappa,\lambda}$, while an $\mathfrak{S}_{L,\kappa,\lambda}$ -arrow from $\langle T, \psi \rangle$ to $\langle S, \varphi \rangle$ is a tree embedding $f: T \rightarrow S$ satisfying

$$(3) \quad \varphi(f(t)) \leq \psi(t) \text{ for every } t \in T.$$

$$(4) \quad f[T] \text{ is an initial segment of } S.$$

Note that the trivial tree $\{0\}$ provides an initial object for both $\mathfrak{T}_{L,\kappa,\lambda}$ and $\mathfrak{S}_{L,\kappa,\lambda}$. The following fact is straightforward.

Lemma 5.12. *The pair $\langle \mathfrak{T}_{L,\kappa,\lambda}, \mathfrak{S}_{L,\kappa,\lambda} \rangle$ has the mixed amalgamation property.*

We now specify an additional property of the linearly ordered set L , needed for the existence of a κ -Fraïssé sequence over the inclusion $\mathfrak{T}_{L,\kappa,\lambda} \subseteq \mathfrak{S}_{L,\kappa,\lambda}$. Namely, assume L satisfies the following condition:

$$(\star) \quad \text{For every } x \in L \text{ there exists } C_x \subseteq L \text{ such that } \inf C_x = \inf(x, \rightarrow) \text{ and } |C_x| \leq \lambda.$$

Let us fix the sets C_x as above and let $p: L \times \lambda \rightarrow L$ be the canonical projection. Define U to be the set of all functions $t: \alpha + 1 \rightarrow L \times \lambda$ such that

- (i) $\alpha < \kappa$ and $p \circ t$ is continuous.
- (ii) $p(t(\xi + 1)) \in C_{p(t(\xi))}$ for every $\xi < \alpha$.

The order of U is extension, that is, $t \leq s$ iff s extends t . Note that U has no maximal elements, because the domain of each $t \in U$ is a successor ordinal and L is unbounded from above. By condition (2), the height of U is κ . We shall see that the canonical decomposition of U provides a Fraïssé sequence over $\mathfrak{T}_{L,\kappa,\lambda} \subseteq \mathfrak{S}_{L,\kappa,\lambda}$. More precisely, let $\theta: U \rightarrow L$ be defined by $\theta(t) = \max \text{dom}(t)$. Then θ is a continuous

strictly increasing function. Let $U_\alpha = \{t \in U : \theta(t) \leq \alpha\}$ and let $\theta_\alpha = \theta \upharpoonright U_\alpha$. Clearly, $\langle U_\alpha, \theta_\alpha \rangle$ is a $\mathfrak{T}_{L,\kappa,\lambda}$ -object and $\vec{u} = \{\langle U_\alpha, \theta_\alpha \rangle\}_{\alpha < \kappa}$ with the obvious embeddings is a continuous sequence in $\mathfrak{T}_{L,\kappa,\lambda}$.

Lemma 5.13. *The sequence \vec{u} is Fraïssé over the inclusion $\mathfrak{T}_{L,\kappa,\lambda} \subseteq \mathfrak{S}_{L,\kappa,\lambda}$.*

Proof. Fix $\alpha < \kappa$ and fix a $\mathfrak{T}_{L,\kappa,\lambda}$ -arrow $f: \langle U_\alpha, \theta_\alpha \rangle \rightarrow \langle S, \psi \rangle$. We may assume that f is “simple” in the sense that $S \setminus f[U_\alpha]$ consists of a single level, since arbitrary $\mathfrak{T}_{L,\kappa,\lambda}$ -arrows are compositions of “simple” arrows and we can later use induction.

We shall find an $\mathfrak{S}_{L,\kappa,\lambda}$ -arrow $g: \langle S, \psi \rangle \rightarrow \langle U_{\alpha+1}, \theta_{\alpha+1} \rangle$ so that $g \circ f$ is the inclusion $U_\alpha \subseteq U_{\alpha+1}$. Namely, let $T = f[U_\alpha]$ and start with $g_0 = f^{-1}$. Let K be the highest level of T and let $T_s = T \cup s^+$ for $s \in K$. We extend g_0 to each T_s independently. Namely, fix $s \in K$ and enumerate s^+ as $\{s_\xi\}_{\xi < \lambda}$ (possibly with repetitions). Let $t = g_0(s)$. Then $t: \alpha + 1 \rightarrow L \times \lambda$ satisfies conditions (i), (ii) above. Fix $\xi < \lambda$. Using (\star) , we can extend t to a function $t_\xi \in U$ whose domain is $\alpha + 2$ and $p(t_\xi(\alpha + 1)) \leq \varphi(s_\xi)$. Define $g_s(s_\xi) = t_\xi$ and $g_s \upharpoonright T = g_0$. This defines $g_s: T_s \rightarrow U_{\alpha+1}$ extending g_0 . The union of all these maps defines the required $\mathfrak{S}_{L,\kappa,\lambda}$ -arrow g , showing that \vec{u} is Fraïssé over $\mathfrak{T}_{L,\kappa,\lambda} \subseteq \mathfrak{S}_{L,\kappa,\lambda}$. \square

Using Lemmata 5.12, 5.13 and Theorem 3.18, we obtain

Theorem 5.14. *Under the assumptions above, there exists a λ -branching tree U of height κ together with a continuous strictly increasing function $\theta: U \rightarrow L$, such that for every λ -branching tree T of height $\leq \kappa$, for every strictly increasing function $\varphi: T \rightarrow L$, there exists a tree embedding $f: T \rightarrow U$ such that $f[T]$ is an initial segment of U and $\theta(f(t)) \leq \varphi(t)$ for every $t \in T$.*

Of course, the above result can be proved directly, without referring to Fraïssé sequences. The key inductive argument is contained in the proof of Lemma 5.13. Our aim was rather showing how the existence and universal properties of certain trees can be explained by means of Fraïssé sequences over a natural inclusion functor.

One of the special cases is $L = \kappa$; then λ could be even finite and “being L -embeddable” is the same as “being of height $\leq \kappa$ ”. When L has the property that $|(x, \rightarrow)| = \lambda$ for every $x \in L$, a natural example of a universal λ -branching L -embeddable tree is the tree of all bounded closed well ordered subsets of L endowed with the “end-extension” ordering. It is worth mentioning that the well-known tree $\sigma\mathbb{Q}$ of bounded well ordered subsets of the rational numbers (again with the “end-extension” ordering) is another example of a universal countably branching \mathbb{R} -embeddable tree (see [36] for a direct argument and more information). It is a folklore fact that $\sigma\mathbb{Q}$ is not \mathbb{Q} -embeddable, however its subtree consisting of all closed sets is, by the arguments above, a universal countably branching \mathbb{Q} -embeddable tree.

Finally, one can consider the category \mathfrak{K} of finite trees with end-extensions and the category \mathfrak{L} of finite trees with arbitrary embeddings. In this case, sequences in \mathfrak{K} lead to finitely-branching trees of height $\leq \omega$ and a sequence is Fraïssé over the inclusion $\mathfrak{K} \subseteq \mathfrak{L}$ if and only if it leads to a perfect tree. Recall that a tree T is

perfect if for every $t \in T$ there are incomparable elements s_0, s_1 such that $s_0 > t$ and $s_1 > t$. As before, the category \mathfrak{K} obviously fails the amalgamation property. On the other hand, \mathfrak{L} has the amalgamation property and since it is countable and directed, it has a unique Fraïssé sequence. This sequence, however, does not lead to a tree: Its natural co-limit is a certain tree-like partially ordered set, in the sense that all intervals (a, b) with $a < b$ are linearly ordered (in fact, isomorphic to \mathbb{Q}) and for every $a < b$ there is $c > a$ not comparable with b .

6 Embedding-projection pairs

In this section we describe a general construction on a given category, which is suitable for applications to the theory of Valdivia compacta and Banach spaces with Markushevich bases. This construction had been first used by Dana Scott in order to get faithful models of untyped λ -calculus. It also appears in Droste & Göbel [11] in the context of Scott domains.

6.1 Definitions and basic properties

We fix a category \mathfrak{K} . Define $\ddagger\mathfrak{K}$ to be the category whose objects are the objects of \mathfrak{K} and a morphism $f: X \rightarrow Y$ is a pair $\langle e, r \rangle$ of arrows in \mathfrak{K} such that $e: X \rightarrow Y$, $r: Y \rightarrow X$ and $r \circ e = \text{id}_X$. We set $e(f) := e$ and $r(f) := r$, so $f = \langle e(f), r(f) \rangle$. Given morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ in $\ddagger\mathfrak{K}$, we define their composition in the obvious way:

$$g \circ f := \langle e(g) \circ e(f), r(f) \circ r(g) \rangle.$$

It is clear that this defines an associative operation on compatible arrows. Further, given an object $a \in \mathfrak{K}$, pair of the form $\langle \text{id}_a, \text{id}_a \rangle$ is the identity morphism in $\ddagger\mathfrak{K}$. Thus, $\ddagger\mathfrak{K}$ is indeed a category. Note that $f \mapsto e(f)$ defines a covariant functor $e: \ddagger\mathfrak{K} \rightarrow \mathfrak{K}$ and $f \mapsto r(f)$ defines a contravariant functor $r: \ddagger\mathfrak{K} \rightarrow \mathfrak{K}$. Following [11], we shall call $\ddagger\mathfrak{K}$ the *category of embedding-projection pairs* or briefly *EP-pairs*. The idea of considering EP-pairs is to obtain more special Fraïssé sequences, namely we would like to obtain a unique cofinal object u in the category of sequences such that every other x can be both “embedded” into u and “projected” from u . In other words, we would like to have arrows $j: x \rightarrow u$ and $r: u \rightarrow x$ satisfying $r \circ j = \text{id}_x$. We shall demonstrate this in Subsection 7.3 below, dealing with the category of finite nonempty sets.

We shall later need the following general facts about sequences of left-invertible arrows.

Lemma 6.1. *Let \mathfrak{K} be a category, let $\delta > 0$ be a limit ordinal and let $a: \delta \rightarrow \ddagger\mathfrak{K}$ be a sequence such that $\langle v, \{i_\alpha\}_{\alpha < \delta} \rangle$ is the co-limit of $e[a]$. Then there exists a sequence $b: (\delta + 1) \rightarrow \ddagger\mathfrak{K}$ such that $b \upharpoonright \delta = a$, $b_\delta = v$ and $e(b_\alpha^\delta) = i_\alpha$ for every $\alpha < \delta$.*

Proof. Fix $\alpha < \delta$. Given $\alpha < \xi < \eta < \delta$, we have

$$r(a_\alpha^\eta) \circ e(a_\xi^\eta) = r(a_\alpha^\xi) \circ r(a_\xi^\eta) \circ e(a_\xi^\eta) = r(a_\alpha^\xi).$$

Using the equation above and the fact that v together with $\{i_\xi\}_{\alpha \leq \xi < \delta}$ is the co-limit of $a \upharpoonright [\alpha, \delta)$, we find a unique arrow $f_\alpha: v \rightarrow a_\alpha$ satisfying

$$f_\alpha \circ i_\xi = r(a_\alpha^\xi)$$

for every $\alpha \leq \xi < \delta$. Setting $\xi := \alpha$ in the equation above, we see that $f_\alpha \circ i_\alpha = \mathbf{id}_{a_\alpha}$. By uniqueness, $f_\xi = r(a_\xi^\eta) \circ f_\eta$ for every $\xi < \eta < \delta$. Finally, define $b_\delta := v$, $b_\alpha^\delta := \langle i_\alpha, f_\alpha \rangle$ and $b \upharpoonright \delta := a$. \square

Lemma 6.2. *Let \vec{x} be a continuous sequence in a category \mathfrak{K} and assume that each bonding arrow x_α^β is left-invertible in \mathfrak{K} . Then there exists a sequence \vec{y} in $\ddagger\mathfrak{K}$ such that*

$$\vec{x} = e[\vec{y}].$$

Proof. Let $\vec{x}: \delta \rightarrow \mathfrak{K}$. Using induction, we shall construct arrows $f_\alpha^\beta: x_\beta \rightarrow x_\alpha$ in \mathfrak{K} so that

$$(*) \quad f_\alpha^\beta \circ x_\alpha^\beta = \mathbf{id}_{x_\alpha} \quad \text{and} \quad f_\alpha^\gamma = f_\beta^\gamma \circ f_\alpha^\beta$$

holds for every $\alpha < \beta < \gamma$. Fix $0 < \beta < \delta$ and suppose f_ξ^η have been constructed for every $\xi < \eta < \beta$. Assume first that β is a successor ordinal, say $\beta = \alpha + 1$. Using the assumption, find $f: x_{\alpha+1} \rightarrow x_\alpha$ such that $f \circ x_\alpha^{\alpha+1} = \mathbf{id}_{x_\alpha}$. Given $\xi < \beta$, define $f_\xi^{\alpha+1} = f_\xi^\alpha \circ f$. It is clear that $(*)$ holds.

Assume now that β is a limit ordinal. Define $a: \beta \rightarrow \ddagger\mathfrak{K}$ by setting $a_\alpha = x_\alpha$ and $a_\xi^\eta = \langle x_\xi^\eta, f_\xi^\eta \rangle$ for $\xi < \eta < \beta$. By $(*)$, a is indeed a sequence in $\ddagger\mathfrak{K}$. By Lemma 6.1, this sequence has an extension $b: \beta + 1 \rightarrow \ddagger\mathfrak{K}$ such that $e(b_\alpha^\beta) = x_\alpha^\beta$ for every $\alpha < \beta$. Setting $f_\alpha^\beta := r(b_\alpha^\beta)$ for $\alpha < \beta$, it is clear that $(*)$ holds.

Finally, setting $y_\alpha := x_\alpha$ and $y_\alpha^\beta := \langle x_\alpha^\beta, f_\alpha^\beta \rangle$, we obtain the desired sequence \vec{y} in the category $\ddagger\mathfrak{K}$. \square

6.2 Semi-continuity and the back-and-forth principle

It turns out that usually $\ddagger\mathfrak{K}$ is not continuous, however one can consider the following weakening of continuity, which is good enough for applications.

We say that a sequence \vec{x} in $\ddagger\mathfrak{K}$ is *semi-continuous* if $e[\vec{x}]$ is continuous in \mathfrak{K} . The dual notion of semi-continuity with respect to the functor r is obtained by considering $\ddagger\mathfrak{K}^{\text{op}}$ instead of $\ddagger\mathfrak{K}$.

Theorem 6.3. *Let κ be an infinite regular cardinal and let \mathfrak{K} be a κ -complete category such that $\ddagger\mathfrak{K}$ is directed and has the amalgamation property. Assume further that $\ddagger\mathfrak{K}$ has a dominating family consisting of at most κ arrows. Then there exists a semi-continuous Fraïssé sequence of length κ in $\ddagger\mathfrak{K}$.*

Proof. By Lemma 6.1, $\ddagger\mathfrak{K}$ is κ -bounded. Thus, we may use Proposition 3.9 applied to $\Phi := e$. \square

We shall prove that semi-continuous Fraïssé sequences satisfy the back-and-forth principle. In order to do it, we need to show that the left-forgetful functor e induces a quasi-limiting operator, as it is suggested by Lemma 6.1. It turns out that this is not completely trivial. Note that we cannot use Proposition 4.1, because the functor e is not faithful.

Lemma 6.4. *Let \mathfrak{K} be a category. Given a sequence \vec{x} in $\ddagger\mathfrak{K}$ such that $e[\vec{x}]$ has the co-limit in \mathfrak{K} , let $\mathcal{L}\vec{x}$ be the co-cone in $\ddagger\mathfrak{K}$, existing by Lemma 6.1, such that $e(\mathcal{L}\vec{x}) = \lim e[\vec{x}]$. Then \mathcal{L} is a partial quasi-limiting operator on $\text{Seq}_{<\kappa}(\mathfrak{K})$.*

Proof. Let $\vec{f}: \vec{x} \rightarrow \vec{y}$ be an isomorphism of sequences in $\ddagger\mathfrak{K}$ and let $\vec{g}: \vec{y} \rightarrow \vec{x}$ be its inverse. We may assume that $\vec{f} = \{f_\xi\}_{\xi < \lambda}$, $\vec{g} = \{g_\xi\}_{\xi < \lambda}$ so that $f_\xi: x_\xi \rightarrow y_\xi$ and $g_\xi: y_\xi \rightarrow x_{\xi+1}$ for $\xi < \lambda$. Let the \mathfrak{K} -objects \bar{x}, \bar{y} be the co-limits of $e[\vec{x}]$ and $e[\vec{y}]$, respectively. The co-limiting co-cones will be denoted by $\{x_\xi^\infty\}_{\xi < \lambda}$ and $\{y_\xi^\infty\}_{\xi < \lambda}$, respectively. Let $f_\infty: \bar{x} \rightarrow \bar{y}$ be the unique arrow satisfying $e(y_\xi^\infty) \circ e(f_\xi) = f_\infty \circ e(x_\xi^\infty)$ for every $\xi < \lambda$. Similarly, let $g_\infty: \bar{y} \rightarrow \bar{x}$ be the unique arrow satisfying $e(x_{\xi+1}^\infty) \circ e(g_\xi) = g_\infty \circ e(y_\xi^\infty)$ for every $\xi < \lambda$. By the universality of the co-limit, $f_\infty \circ g_\infty = \text{id}_{\bar{y}}$ and $g_\infty \circ f_\infty = \text{id}_{\bar{x}}$. In particular, $\langle f_\infty, g_\infty \rangle$ is an isomorphism in $\ddagger\mathfrak{K}$. It remains to check that it commutes with \vec{f} and \vec{g} . By symmetry, it suffices to check that

$$(1) \quad r(f_\alpha) \circ r(y_\alpha^\infty) = r(x_\alpha^\infty) \circ g_\infty$$

holds for all $\alpha < \lambda$.

Fix $\alpha < \lambda$. Given $\xi \geq \alpha$, define $q_\xi = r(f_\alpha) \circ r(y_\alpha^\xi)$. If $\xi < \eta$ then

$$q_\eta \circ e(y_\xi^\eta) = r(f_\alpha) \circ r(y_\alpha^\xi) \circ r(y_\xi^\eta) \circ e(y_\xi^\eta) = q_\xi.$$

It follows that $\{q_\xi\}_{\alpha \leq \xi < \lambda}$ commutes with the sequence $\{e(y_\xi)\}_{\alpha \leq \xi < \lambda}$. As g_∞ is the co-limit of this sequence, there is a unique \mathfrak{K} -arrow $k: \bar{y} \rightarrow x_\alpha$ satisfying

$$(2) \quad k \circ e(y_\xi^\infty) = q_\xi = r(f_\alpha) \circ r(y_\alpha^\xi)$$

for every $\xi \geq \alpha$. Let

$$k_0 = r(f_\alpha) \circ r(y_\alpha^\infty) \quad \text{and} \quad k_1 = r(x_\alpha^\infty) \circ g_\infty.$$

In order to show (1), we check that both k_0 and k_1 satisfy (2) in place of k . Given $\xi \geq \alpha$, we have

$$k_0 \circ e(y_\xi^\infty) = r(f_\alpha) \circ r(y_\alpha^\xi) \circ r(y_\xi^\infty) \circ e(y_\xi^\infty) = r(f_\alpha) \circ r(y_\alpha^\xi).$$

Hence $k_0 = k$. Recalling that g_∞ is the unique \mathfrak{K} -arrow satisfying $e(x_{\xi+1}^\infty) \circ e(g_\xi) = g_\infty \circ e(y_\xi^\infty)$ for every $\xi < \lambda$, we obtain

$$\begin{aligned} k_1 \circ e(y_\xi^\infty) &= r(x_\alpha^\infty) \circ g_\infty \circ e(y_\xi^\infty) = r(x_\alpha^\infty) \circ e(x_{\xi+1}^\infty) \circ e(g_\xi) \\ &= r(x_\alpha^{\xi+1}) \circ r(x_{\xi+1}^\infty) \circ e(x_{\xi+1}^\infty) \circ e(g_\xi) = r(x_\alpha^{\xi+1}) \circ e(g_\xi) \\ &= r(g_\xi \circ y_\alpha^\xi \circ f_\alpha) \circ e(g_\xi) = r(f_\alpha) \circ r(y_\alpha^\xi) \circ r(g_\xi) \circ e(g_\xi) \\ &= r(f_\alpha) \circ r(y_\alpha^\xi). \end{aligned}$$

Hence $k_1 = k$, which completes the proof. \square

Combining the lemma above with Theorem 4.4, we obtain:

Theorem 6.5. *Let \mathfrak{K} be a category. Every two semi-continuous Fraïssé sequences of the same regular length in $\ddagger\mathfrak{K}$ satisfy the back-and-forth principle.*

6.3 Proper amalgamations and proper arrows

We still have not proved that a semi-continuous Fraïssé sequence in a category of EP-pairs is cofinal for uncountable sequences. In fact, even for the countable case, one can consider a better class of arrows between sequences, obtaining stronger cofinality results (see Subsection 7.3 for a simple application in the category of sets). This leads to the following concept.

Let $f: Z \rightarrow X$ and $g: Z \rightarrow Y$ be arrows in $\ddagger\mathfrak{K}$. We say that arrows $h: X \rightarrow W$, $k: Y \rightarrow W$ provide a *proper amalgamation* of f, g if $h \circ f = k \circ g$ and moreover $e(g) \circ r(f) = r(k) \circ e(h)$, $e(f) \circ r(g) = r(h) \circ e(k)$ hold. Translating it back to the original category \mathfrak{K} , this means that the following four diagrams commute:

$$\begin{array}{cccc} \begin{array}{ccc} W & \xleftarrow{e(k)} & Y \\ \uparrow e(h) & & \uparrow e(g) \\ X & \xleftarrow{e(f)} & Z \end{array} & \begin{array}{ccc} W & \xrightarrow{r(k)} & Y \\ \downarrow r(h) & & \downarrow r(g) \\ X & \xrightarrow{r(f)} & Z \end{array} & \begin{array}{ccc} W & \xrightarrow{r(k)} & Y \\ \uparrow e(h) & & \uparrow e(g) \\ X & \xrightarrow{r(f)} & Z \end{array} & \begin{array}{ccc} W & \xleftarrow{e(k)} & Y \\ \downarrow r(h) & & \downarrow r(g) \\ X & \xleftarrow{e(f)} & Z \end{array} \end{array}$$

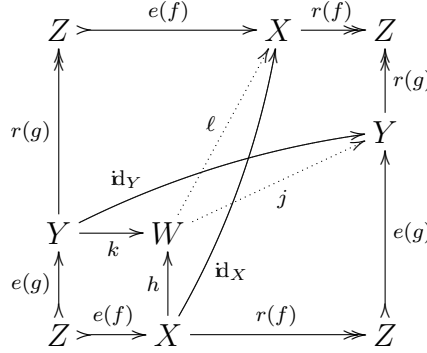
We draw arrows $\xrightarrow{\quad}$ and $\xleftarrow{\quad}$ in order to indicate mono- and epimorphisms respectively. We shall say that $\ddagger\mathfrak{K}$ has *proper amalgamations* if every pair of arrows in $\ddagger\mathfrak{K}$ with common domain can be properly amalgamated in $\ddagger\mathfrak{K}$.

Below is a useful criterion for the existence of proper amalgamations.

Lemma 6.6. *Let \mathfrak{K} be a category and let f, g be arrows in $\ddagger\mathfrak{K}$ with the same domain. If $e(f), e(g)$ have a pushout in \mathfrak{K} then f, g can be properly amalgamated in $\ddagger\mathfrak{K}$.*

Proof. Let $h: X \rightarrow W$ and $k: Y \rightarrow W$ form a pushout of $e(f), e(g)$. Consider the

following diagram:



The dotted arrows indicate unique morphisms completing appropriate diagrams, i.e. j is the unique arrow satisfying equations $j \circ h = e(g) \circ r(f)$, $j \circ k = \text{id}_Y$ and ℓ is the unique arrow satisfying equations $\ell \circ k = e(f) \circ r(g)$, $\ell \circ h = \text{id}_X$. Consequently, $\langle k, j \rangle$ and $\langle h, \ell \rangle$ are morphisms in $\ddagger\mathfrak{K}$. Set $s = r(f) \circ \ell$. Then

$$(1) \quad s \circ k = r(f) \circ \ell \circ k = r(f) \circ e(f) \circ r(g) = r(g) \quad \text{and} \quad s \circ h = r(f) \circ \ell \circ h = r(f).$$

Recall that $r(f) \circ e(f) = \text{id}_Z = r(g) \circ e(g)$. Since k, h is a pushout of $e(f), e(g)$, we deduce that s must be the unique arrow satisfying (1). Now let $t = r(g) \circ j$. Similar computations show that $t \circ k = r(g)$ and $t \circ h = r(f)$, therefore by uniqueness we deduce that $s = t$ or, in other words, $r(f) \circ \ell = r(g) \circ j$. This shows that the full diagram is commutative and hence $\langle k, j \rangle$ and $\langle h, \ell \rangle$ provide a proper amalgamation of f, g in the category $\ddagger\mathfrak{K}$. \square

As an example, if \mathfrak{K} is the category of nonempty sets, then Lemma 6.6 says that category $\ddagger\mathfrak{K}$ has proper amalgamations. We show below that not all amalgamations in $\ddagger\mathfrak{K}$ are proper.

Example 6.7. Consider the category of nonempty finite sets \mathfrak{Set}^+ . Let a, b, c, d be pairwise distinct elements and set $Z = \{a\}$, $X = \{a, b\}$, $Y = \{a, c\}$ and $W = \{a, b, c\}$. We are going to define arrows $f: Z \rightarrow X$, $g: Z \rightarrow Y$, $h: X \rightarrow W$ and $k: Y \rightarrow W$ in the category $\ddagger\mathfrak{Set}^+$. Let $e(f), e(g), e(h)$ and $e(k)$ be the inclusion maps and let $r(f)$ and $r(g)$ be the obvious constant maps. Finally, let $r(h)(c) = a$ and $r(k)(b) = c$. This already defines $r(h)$ and $r(k)$, since these maps must be identity on the ranges of $e(h)$ and $e(k)$ respectively. It is clear that $h \circ f = k \circ g$, i.e. h, k amalgamate f, g in the category $\ddagger\mathfrak{Set}^+$. On the other hand, $e(g) \circ r(f)(b) = a$ and $r(k) \circ e(h)(b) = c$, therefore $e(g) \circ r(f) \neq r(k) \circ e(h)$. Note that actually $e(f) \circ r(g) = r(h) \circ e(k)$ holds, although redefining $r(h)(c)$ to b we can even get $e(f) \circ r(g) \neq r(h) \circ e(k)$.

Lemma 6.8. *Let \mathfrak{K} be a category such that sequences of length $< \kappa$ consisting of left-invertible arrows have co-limits, and assume that $\ddagger\mathfrak{K}$ has proper amalgamations. Let \mathcal{L} be the quasi-limiting operator on $\text{Seq}_{<\kappa}(\ddagger\mathfrak{K})$ associated with the left-forgetful functor e . Let \mathcal{A} denote the class of all proper amalgamations. Then \mathcal{A} is a stable amalgamation structure in $\ddagger\mathfrak{K}$ and $\langle \ddagger\mathfrak{K}, \mathcal{L}, \mathcal{A} \rangle$ is a Fraïssé-Jónsson category.*

Proof. The fact that \mathcal{A} is an \mathcal{L} -continuous amalgamation structure is obvious (assuming of course that $\ddagger\mathfrak{K}$ admits proper amalgamations). We only need to check that \mathcal{A} is stable.

Fix a commutative diagram of the form

$$\begin{array}{ccc} b & \xrightarrow{g} & c \\ f \uparrow & & \uparrow h \\ a & \xrightarrow{\text{id}_a} & a \end{array}$$

where f, g, h are $\ddagger\mathfrak{K}$ -arrows. However, using the fact that $r(h) = r(f) \circ r(g)$, we get $r(h) \circ e(g) = r(f) \circ r(g) \circ e(g) = r(f)$ and, using the fact that $e(h) = e(g) \circ e(f)$, we obtain $r(g) \circ e(h) = r(g) \circ (g) \circ e(f) = e(f)$. \square

An arrow of sequences $\vec{f}: \vec{x} \rightarrow \vec{y}$ that is admissible for the class of proper amalgamations will be called *proper*. In other words, \vec{f} is proper if (up to equivalence) all squares of the form

$$\begin{array}{ccc} x_\alpha & \xrightarrow{x_\alpha^\beta} & x_\beta \\ f_\alpha \downarrow & & \downarrow f_\beta \\ y_{\varphi(\alpha)} & \xrightarrow{y_{\varphi(\alpha)}^{\varphi(\beta)}} & y_{\varphi(\beta)} \end{array}$$

are proper amalgamations, where φ is the increasing ordinal function associated to \vec{f} . As we shall see in Example 7.16, proper arrows are rather restrictive even in the category of countable sequences of nonempty finite sets.

Using Lemma 6.8 and Theorem 4.12, we obtain:

Theorem 6.9. *Assume \mathfrak{K} is a category and \vec{u} is a semi-continuous Fraïssé sequence in $\ddagger\mathfrak{K}$ of regular length $\kappa \geq \aleph_0$. If $\ddagger\mathfrak{K}$ has proper amalgamations then for every semi-continuous sequence $\vec{x} \in \text{Seq}_{\leq \kappa}(\ddagger\mathfrak{K})$ there exists a proper arrow of sequences $\vec{f}: \vec{x} \rightarrow \vec{u}$.*

In Subsection 7.3 we shall present a simple application of the result above (with $\kappa = \aleph_0$) in the category of nonempty finite sets, leading to a classical homogeneity property of the Cantor set.

7 Applications

In this section we collect several applications of our results—mainly those from Section 6—to compact Hausdorff spaces, Banach spaces, the Cantor set, and linearly ordered sets.

7.1 Compact spaces

Let \mathbf{Comp} be the category of all nonempty compact Hausdorff spaces. It is well-known that quotient maps have pullbacks in \mathbf{Comp} . More precisely, given quotient maps $f: X \rightarrow Z$, $g: Y \rightarrow Z$, the two projections from the set

$$P = \{\langle x, y \rangle \in X \times Y : f(x) = g(y)\}$$

from the pullback of f, g . It is obvious that P is compact, as a closed subset of $X \times Y$ and both projections restricted to P are onto. Notice that, assuming the Continuum Hypothesis, the category of nonempty compact metric spaces with quotient maps has \aleph_1 many types of arrows, there it has an inverse Fraïssé sequence of length ω_1 . The limit of this sequence is the well-known space $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, i.e., the remainder of the Čech-Stone compactification of the set of natural numbers. All these things clearly generalize to higher cardinals and were studied by Negrepontis [31], using Jónsson's work [17]. In fact, when restricted to totally disconnected compact spaces, Stone duality allows moving to the category of Boolean algebras, where model-theoretic Fraïssé-Jónsson theory can be applied. The fact that every compact Hausdorff space of weight $\leq \aleph_1$ is a continuous image of \mathbb{N}^* was previously proved by Parovičenko [33].

We now turn to a more special class of spaces, called Valdivia compacta. A topological space K is called *Valdivia compact* if it is homeomorphic to a closed subspace K' of some Tikhonov cube \mathbb{I}^κ , such that $K' \cap \Sigma(\kappa)$ is dense in K' , where

$$\Sigma(\kappa) = \{x \in \mathbb{I}^\kappa : |\text{suppt}(x)| \leq \aleph_0\}$$

is the Σ -product of κ copies of the unit interval, and $\text{suppt}(x) = \{\alpha : x(\alpha) \neq 0\}$ is the *support* of x . This class has been extensively studied by several authors; we refer to Kalenda's surveys [18, 19] for further references. Valdivia compacta are closely related to *Corson compacta* which are defined by strengthening the requirement of $K' \cap \Sigma(\kappa)$ being dense to $K' \subseteq \Sigma(\kappa)$. It is well-known that Corson compacta are precisely those Valdivia compacta that are countably tight. This actually follows easily from the (not completely trivial) fact that Σ -products are countably tight. Recall that a topological space is *countably tight* if the closure of any set is the union of closures of its countable subsets. One has to mention yet another class of compacta: Eberlein compact spaces. By definition, a space is *Eberlein compact* if it is homeomorphic to a weakly compact subset of some Banach space. If this Banach space is Hilbert, the compact is called *uniformly Eberlein*. Every Eberlein compact is Corson and every Corson compact is Valdivia. None of these implications can be reversed.

A result of Bell [4] says that, under the Continuum Hypothesis, there exists a universal uniform Eberlein compact of weight \mathfrak{c} . Another positive result, due to Bell & Marciszewski [5] shows, under the usual assumption $2^{<\kappa} = \kappa$, the existence of a scattered Eberlein compact of weight κ and of Cantor-Bendixson height 3 that is universal for this class in the sense of retractions. Both of the results on Eberlein

compacta are in fact Fraïssé-Jónsson constructions, by putting relevant structures on these spaces.

In contrast, Argyros & Benyamini [2] proved that universal Eberlein compacta for weight κ do not exist whenever $\kappa = \aleph_1$ or $\kappa^{\aleph_0} = \kappa$. Later, Todorčević [37] proved the same negative result for Corson compacta. Both results are negative even when asking for a compact space K such that every other compact in the given class is homeomorphic to a quotient of a closed subset of K .

The results of Section 6 allow us to get, under the Continuum Hypothesis, a Valdivia compact of weight \aleph_1 that is universal for this class in the sense of retractions. More precisely:

Theorem 7.1. *Assume $2^{\aleph_0} = \aleph_1$. There exists a compact space V with the following properties.*

- (1) $w(V) = \aleph_1$ and V is Valdivia compact.
- (2) Every Valdivia compact of weight $\leq \aleph_1$ is a retract of V .
- (3) Given a compact metrizable space K , given retractions $f, g: V \rightarrow K$, there exists a homeomorphism $h: V \rightarrow V$ such that $g = f \circ h$.

Furthermore, properties (1) and (3) describe the space V uniquely, up to a homeomorphism, and every compact space satisfying (3) has weight at least 2^{\aleph_0} .

Before starting the proof, we need one important property of Valdivia compacta, explaining why we actually restrict attention to weight \aleph_1 :

Proposition 7.2 (Kubiś & Michalewski [25]). *A nonempty compact space of weight $\leq \aleph_1$ is Valdivia compact if and only if it is homeomorphic to the limit of a continuous inverse sequence of compact metrizable spaces in which all bonding maps are retractions.*

Proof of Theorem 7.1. As we have mentioned above, quotient maps admit pullbacks in the category of nonempty compact spaces. The same of course holds when restricted to the category \mathfrak{K} of compact metric spaces. It follows that $\ddagger\mathfrak{K}$ has proper amalgamations. The assumption $2^{\aleph_0} = \aleph_1$ implies that $\ddagger\mathfrak{K}$ has a semi-continuous Fraïssé sequence \vec{u} of length ω_1 , where semi-continuity is with respect to quotient maps (obviously, every countable inverse sequence of quotient maps has a limit). Thus, the sequence \vec{u} leads to a continuous inverse sequence of compact metric spaces U_α ($\alpha < \omega_1$) in which all bonding maps are retractions. Denote by V the limit of this sequence. By Proposition 7.2, V is a Valdivia compact of weight \aleph_1 , that is, V satisfies (1). Fix another nonempty Valdivia compact K whose weight is $\leq \aleph_1$. Again by Proposition 7.2, K is the limit of a continuous sequence of retractions between metrizable compacta. By Lemma 6.2, this sequence extends to a semi-continuous sequence \vec{x} in $\ddagger\mathfrak{K}$. Now Theorem 6.9 gives a proper arrow $\vec{f}: \vec{x} \rightarrow \vec{u}$ which gives rise to a retraction $r: V \rightarrow K$. This shows (2).

Property (3) is just a translation of the back-and-forth principle, satisfied by the Fraïssé sequence \vec{u} by Theorem 6.5. The relevant fact needed here is that every map $f: V \rightarrow K$ onto a metrizable compact space factorizes through some U_α , that is, it satisfies $f = f' \circ p_\alpha$, where $p_\alpha: V \rightarrow U_\alpha$ is the retraction obtained from the $\ddagger\mathfrak{K}$ -arrow $u_\alpha^\infty: \vec{u} \rightarrow u_\alpha$.

In order to show the uniqueness of V , let us consider the category \mathfrak{L} of all retractions between nonempty compact metrizable spaces. By Proposition 7.2, the space V is the limit of a continuous inverse sequence \vec{v} in \mathfrak{L} . Property (3) implies that the sequence is Fraïssé in \mathfrak{L} . Even though countable inverse sequences of retractions may have no limits in \mathfrak{L} , the limits “computed” in the bigger category of all continuous maps provide a quasi-limiting operator. Thus, by Theorem 4.4, the sequence \vec{v} is unique, up to an isomorphism of sequences. This clearly translates to the fact that V is unique up to a homeomorphism.

Let us finally see that $w(V) \geq 2^{\aleph_0}$, whenever V satisfies (3). Clearly, there is a retraction $f: V \rightarrow \mathbb{I}$, where \mathbb{I} is the unit interval. Fix a point $\infty \notin \mathbb{I}$ and let $K = \mathbb{I} \cup \{\infty\}$, where ∞ is an isolated point. Let $g: V \rightarrow K$ be a fixed retraction. For each $t \in \mathbb{I}$, let $r_t: K \rightarrow \mathbb{I}$ be such that $r_t \upharpoonright \mathbb{I} = \text{id}_{\mathbb{I}}$ and $r_t(\infty) = t$. Let $h_t: V \rightarrow V$ be a homeomorphism such that $f = r_t \circ g \circ h_t$. Let $U_t = f^{-1}(t)$. Then $U_t = h_t^{-1}(g^{-1}(r_t^{-1}(t))) \supseteq h_t^{-1}(g^{-1}(\infty))$ and, since ∞ is isolated in K , we deduce that U_t has nonempty interior. Obviously, $U_t \cap U_s = \emptyset$ whenever $t \neq s$, which shows that $w(V) \geq |\mathbb{I}| = 2^{\aleph_0}$. \square

Using the full strength of categories of projection-embedding pairs, it is possible to make the result above more precise, by considering a pair of the form $\langle V, D \rangle$, where $D = \Sigma(\omega_1) \cap V$, with respect to the appropriate embedding of V into the Tikhonov cube \mathbb{I}^{ω_1} . In particular, for every retractions $f: V \rightarrow K$, $f: V \rightarrow L$, where K, L are metrizable compacta contained in D , every homeomorphism $h: K \rightarrow L$ extends to an auto-homeomorphism of V . Another fact would be that every Valdivia compact $K \subseteq \mathbb{I}^{\omega_1}$ with a dense set $G = K \cap \mathbb{I}^{\omega_1}$ is homeomorphic to a retract $K' \subseteq V$ of V so that $G = D \cap K'$. We have decided to present the simplified version which is more transparent and less technical.

A result from [25] says that every retract of a Valdivia compact is again Valdivia provided its weight does not exceed \aleph_1 . It is an open problem whether the same holds for arbitrary weight. A category-theoretic description of the whole class of Valdivia compacta is given in [25]. Unfortunately, it is not clear how to use it when the weight exceeds \aleph_1 and, in particular, it is not clear how to construct a retractively universal Valdivia compact space for larger weights.

7.2 Banach spaces

It is well-known that there exist separable Banach spaces that are isometrically universal for the class of all separable Banach spaces. Perhaps the first example is $\mathcal{C}([0, 1])$, the space of continuous functions on the unit interval, endowed with the

maximum norm. This space, however, does not have good homogeneity properties (see Proposition 7.6 for the precise statement). There is another separable Banach space \mathbb{G} , constructed by Gurariĭ [13], that is also isometrically universal for separable spaces and moreover has the following homogeneity property:

- (H) Given an isometry $f: X \rightarrow Y$ between finite-dimensional linear subspaces of \mathbb{G} , given $\varepsilon > 0$, there exists a bijective isometry $h: \mathbb{G} \rightarrow \mathbb{G}$ such that $\|f - h \upharpoonright X\| \leq \varepsilon$.

Furthermore, \mathbb{G} is the unique, up to isometry, separable Banach space containing isometric copies of all finite-dimensional spaces and satisfying (G). For a direct and elementary proof we refer to [26]. Typically, the condition specifying the Gurariĭ space is different from (G), it involves almost isometric embeddings of finite-dimensional spaces. The space \mathbb{G} can be viewed as some sort of an “approximate Fraïssé limit”. On the other hand, there seems to be no natural category whose Fraïssé sequence would lead to \mathbb{G} . In fact, Gurariĭ had already observed that \mathbb{G} does not satisfy the variant of condition (G) with X, Y being one-dimensional and $\varepsilon = 0$.

Below we show the existence, under the Continuum Hypothesis, of a natural variant of Gurariĭ’s space that comes from a Fraïssé sequence in the category of separable Banach spaces. Its existence was not known before³, although it actually could be derived from the results of Droste & Göbel [11], since the Fraïssé sequence in this category can always be made continuous.

Let $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ denote the category whose objects are separable Banach spaces and arrows are linear isometries. The following fact is well-known.

Lemma 7.3. $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ has the amalgamation property.

Proof. Fix $X, Y, Z \in \mathfrak{B}_{\aleph_0}^{\text{iso}}$ and fix linear isometric embeddings $f: Z \rightarrow X$ and $g: Z \rightarrow Y$. Without loss of generality, we may assume that f and g are inclusions, i.e. $Z \subseteq X$ and $Z \subseteq Y$. We may also assume that $X \cap Y = Z$. Now let W be the formal algebraic sum of X and Y , i.e. $W = \{x+y: x \in X, y \in Y\}$ and $x+y = x'+y'$ whenever $x - x' = y' - y \in Z$. The formula

$$\|w\| = \inf\{\|x\|_X + \|y\|_Y: w = x + y\}$$

defines a norm on W such that the canonical embeddings are isometric. Finally, W is a separable Banach space: It can be seen as a suitable quotient of $X \oplus Y$ endowed with the ℓ_1 -norm. \square

Clearly, $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ has an initial object, the zero space. Thus, directedness follows from amalgamation.

Lemma 7.4. $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ is σ -complete.

³See footnote 1 in the last paragraph of Introduction.

Proof. The completion of a countable chain of separable Banach spaces is a separable Banach space. \square

Theorem 7.5. *Assume $2^{\aleph_0} = \aleph_1$. There exists a Banach space V of density \aleph_1 such that every Banach space of density $\leq \aleph_1$ is linearly isometric to a subspace of V and every linear isometry $T: X \rightarrow Y$ between separable subspaces of V can be extended to a linear isometry of V . Moreover, the space V is unique, up to a linear isometry.*

Proof. Assuming the Continuum Hypothesis, there are only \aleph_1 many isometric types of separable Banach spaces and there are only \aleph_1 many types of linear isometries. Thus, by Lemmata 7.3 and 7.4 and by Theorem 3.7, $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ has a Fraïssé sequence \vec{u} of length ω_1 . We may further assume that this sequence is continuous. Let V be the co-limit of \vec{u} in the category of all Banach spaces.

Fix a Banach space X of density $\leq \aleph_1$. We can write $X = \bigcup_{\alpha < \omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha < \omega_1}$ is an increasing chain of closed separable subspaces of X such that $X_\delta = \text{cl}(\bigcup_{\xi < \delta} X_\xi)$ for every limit ordinal $\xi < \omega_1$. Translating it to the language of category theory, we obtain a continuous ω_1 -sequence in $\mathfrak{B}_{\aleph_0}^{\text{iso}}$ whose co-limit, in the category of all Banach spaces, is X . By Theorem 3.12, there is an arrow of sequences $F: \vec{x} \rightarrow \vec{u}$. This arrow has a co-limit in the category of all Banach spaces, which is just a linear isometric embedding of X into V .

The second statement is obtained by the back-and-forth principle, using the continuity of \vec{u} . \square

It has been shown (using non-trivial arguments) in [3] that the space V from the theorem above is not isomorphic to any $\mathcal{C}(K)$ space. The next simple statement provides a short and elementary argument that V cannot be linearly isometric to any $\mathcal{C}(K)$ space.

Proposition 7.6. *Let K be a compact space which contains at least two points. Then there exists a linear isometry $T: X \rightarrow Y$ between 1-dimensional subspaces of $\mathcal{C}(K)$, which cannot be extended to a linear isometry of $\mathcal{C}(K)$.*

Proof. Fix $a \neq b$ in K . Let X consist of all constant functions on K . Let

$$R: \mathcal{C}(\{a, b\}) \rightarrow \mathcal{C}(K)$$

be a regular extension operator for the inclusion $\{a, b\} \subseteq K$. That is, R is a linear operator which assigns to each $f \in \mathcal{C}(\{a, b\})$ its extension $Rf \in \mathcal{C}(K)$ so that $R1 = 1$ and $Rf \geq 0$ whenever $f \geq 0$. For example, let $(Rf)(t) = \varphi(t)f(a) + (1 - \varphi(t))f(b)$, where $\varphi: K \rightarrow [0, 1]$ is a continuous function such that $\varphi(a) = 1$ and $\varphi(b) = 0$ (which exists by Urysohn's Lemma). Note that R is an isometric embedding of $\mathcal{C}(\{a, b\})$ into $\mathcal{C}(K)$.

Now define $T: X \rightarrow \mathcal{C}(K)$ by $T1 = R1_{\{a\}}$, where $1_{\{a\}}$ is the function which takes value 1 at a and value 0 at b . Let $Y = T[X] = \{\lambda R1_{\{a\}}: \lambda \in \mathbb{R}\}$. Then T is an isometry.

Suppose $\bar{T}: \mathcal{C}(K) \rightarrow \mathcal{C}(K)$ is a linear isometry extending T . Let

$$v = (\bar{T})^{-1}[R1_{\{b\}}].$$

Then $\|v\| = 1$. By compactness, there exists $t \in K$ such that $|v(t)| = 1$. Let $\alpha = v(t)$ and consider $u = R(\alpha 1_{\{a\}} + 1_{\{b\}}) = \alpha R1_{\{a\}} + R1_{\{b\}}$. Notice that $\|u\| = 1$, while $\|(\bar{T})^{-1}(u)\| = \|\alpha 1_K + v\| \geq |\alpha + v(t)| = 2$. Hence $\|(\bar{T})^{-1}\| \geq 2$, a contradiction. \square

As we have mentioned above, under the Continuum Hypothesis, the Čech-Stone remainder of the natural numbers \mathbb{N}^* is the inverse limit of the ω_1 -Fräissé sequence in the category of nonempty compact metric spaces with quotient maps. Thus it is natural to expect that our space V is isometric (or at least isomorphic) to $\ell_\infty/c_0 = \mathcal{C}(\mathbb{N}^*)$. This is not the case, because of Proposition 7.6 (and its isomorphic version in [3]).



We now turn to a more special class, namely Banach spaces with projectional resolutions. From this point on, we consider the category \mathfrak{B}_{\aleph_0} whose objects are again all separable Banach spaces and arrows are linear operators of norm ≤ 1 . We shall apply the results of Section 6.

A *projectional resolution of the identity* (briefly: *PRI*) on a Banach space X is a sequence of norm-one projections $\{P_\alpha\}_{\alpha < \kappa}$ of X , satisfying the following conditions:

- (P1) $P_\alpha \circ P_\beta = P_\alpha = P_\beta \circ P_\alpha$ for every $\alpha, \beta < \kappa$.
- (P2) $P_\delta[X] = \text{cl} \bigcup_{\xi < \delta} P_\xi[X]$ for every limit ordinal $\delta < \kappa$.
- (P3) $\text{dens}(P_\alpha[X]) \leq \alpha + \aleph_0$.
- (P4) $X = \text{cl} \bigcup_{\alpha < \kappa} P_\alpha[X]$.

We are interested in the case of $\kappa = \aleph_1$, where the existence of a PRI is equivalent to the existence of the so-called countably 1-norming Markushevich basis, a natural generalization of a Schauder basis. Banach spaces with norming Markushevich bases are often called *Plichko spaces*. We refer to [19] or [21] for details. In view of Lemma 6.2, it is clear that a PRI in a space X of density \aleph_1 is induced by a semi-continuous sequence in $\ddagger\mathfrak{B}_{\aleph_0}$.

Our aim is to obtain a complementably universal Banach space with a projectional resolution of the identity, that is, we want that every other Banach space with a PRI is linearly isometric to its 1-complemented subspace. Recall that a Banach space X is *1-complemented* in Y if there exists a projection $P: Y \rightarrow Y$ such that $\|P\| \leq 1$ and $P[Y] = X$.

Lemma 7.7. *Let $f: Z \rightarrow X$, $g: Z \rightarrow Y$ be left-invertible arrows in \mathfrak{B}_{\aleph_0} . Then f, g have a pushout in \mathfrak{B}_{\aleph_0} .*

Proof. It is well-known and standard to check that the amalgamation described in the proof of Lemma 7.3 is actually the pushout of f, g in the category of Banach spaces with linear operators of norm ≤ 1 . \square

We now have all ingredients needed to show the existence of a universal Banach space with a PRI.

Theorem 7.8. *Assume the Continuum Hypothesis. There exists a Banach space U with a projectional resolution of the identity and of density \aleph_1 , which has the following properties:*

- (1) *The family $\{X \subseteq U : X \text{ is 1-complemented in } U\}$ is, modulo linear isometries, the class of all Banach spaces of density $\leq \aleph_1$ that admit a PRI.*
- (2) *Given separable 1-complemented subspaces $X, Y \subseteq U$, given a bijective linear isometry $T : X \rightarrow Y$, there exists a bijective linear isometry $H : U \rightarrow U$ such that $H \upharpoonright X = T$.*

Moreover, properties (1) and (2) together with the existence of a PRI describe the space U uniquely, up to a linear isometry.

Proof. The proof is completely analogous to that of Theorem 7.1, using the category \mathfrak{B}_{\aleph_0} instead of the category of compact metric spaces, and inductive sequences instead of inverse sequences. Regarding part of property (1), we need to invoke a result from [23] saying that the existence of a PRI in Banach spaces of density \aleph_1 is preserved under 1-complemented subspaces. \square

One has to admit that Theorem 7.8 (except for the uniqueness part) is already contained in the last chapter of the monograph [21], however the construction and arguments were entirely based on the early draft of this work, which existed in a preprint form before the aforementioned monograph appeared.

It is well-known that the Gurariĭ space has a monotone Schauder basis. On the contrary, the space V from Theorem 7.5 cannot have any kind of bases (of length ω_1), because this would imply the existence of an isomorphic variant of a PRI which in turn would imply some isomorphic properties that are not shared by all Banach spaces of density $\leq 2^{\aleph_0} = \aleph_1$. An example of such a property is the existence of a strictly convex renorming. It is well-known that the space ℓ_∞/c_0 fails this property, yet it is obviously isometric to a subspace of V . In particular, the space V is not isomorphic to the space U from Theorem 7.8, since the latter one admits a strictly convex renorming (implied by the PRI). We refer to the book [7] for details on renorming theory of Banach spaces.



One can argue that Markushevich bases in non-separable Banach spaces are not the proper generalization of Schauder bases. In fact, especially when viewed from category-theoretic perspective, the notion of a PRI is the natural generalization of *monote finite-dimensional Schauder decompositions*, not Schauder bases. A monotone finite-dimensional Schauder decomposition in a Banach space X is a sequence of norm one projections $\{P_n\}_{n \in \omega}$ satisfying conditions (P1) and (P4) with $\kappa = \omega$ and such that $P_n[X]$ is finite-dimensional for each $n \in \omega$. On the other hand, the existence of a monotone Schauder basis is equivalent to the existence of a monotone finite-dimensional decomposition $\{P_n\}_{n \in \omega}$ such that $\dim(P_{n+1} - P_n)[X] = 1$ for every $n \in \omega$. The last property obviously generalizes to higher cardinals. In fact, a Banach space X of density \aleph_1 has a *monotone Schauder basis* (of length ω_1) if and only if it has a PRI $\{P_\alpha\}_{\alpha < \omega_1}$ such that the image of $P_{\alpha+1} - P_\alpha$ is 1-dimensional for every $\alpha < \omega_1$. Formally, a *Schauder basis* of type δ (where δ is an ordinal) is a sequence $\{e_\xi\}_{\xi < \delta}$ of vectors in a Banach space X such that for every $x \in X$ there are uniquely determined scalars $\{t_\xi\}_{\xi < \delta}$ such that

$$x = \sum_{\xi < \delta} t_\xi e_\xi,$$

where the convergence of the series is taken with respect to the norm. Once this happens, for each $\alpha < \delta$ there is a canonical projection P_α defined by

$$P_\alpha \left(\sum_{\xi < \delta} t_\xi e_\xi \right) = \sum_{\xi < \alpha} t_\xi e_\xi.$$

The basis is *monotone* if $\|P_\alpha\| \leq 1$ for every $\alpha < \delta$. It is clear that $\{P_\alpha\}_{\alpha < \delta}$ satisfies conditions (P1), (P2), (P4) and

(P5) For each $\alpha < \delta$ the image of $P_{\alpha+1} - P_\alpha$ is 1-dimensional.

On the other hand, given a sequence $\{P_\alpha\}_{\alpha < \delta}$ satisfying (P1), (P2), (P4) and (P5), we can choose for each α a vector e_α from the image of $P_{\alpha+1} - P_\alpha$, obtaining a Schauder basis of type δ .

In order to formulate the result on universal Schauder bases, we need the following notion. Given Banach spaces $Y \subseteq X$, we say that Y is an *initial subspace* of X if there is a sequence of norm one projections $\{P_\alpha\}_{\alpha < \delta}$, where $\delta \leq \omega_1$, satisfying conditions (P1), (P2), (P4), (P5) and such that $Y = P_0[X]$. Typical examples of initial subspaces are linear spans of initial parts of a Schauder basis. Note that an initial subspace is 1-complemented and the trivial space is initial in X if and only if X has a monotone Schauder basis of type $\leq \omega_1$. Given a Schauder basis $\{e_\xi\}_{\xi < \delta}$ in X , given a subset $S \subseteq \delta$, we say that $\{e_\xi\}_{\xi \in S}$ is a *canonically 1-complemented subbasis* if the linear operator $P_S: X \rightarrow X$, defined by conditions $P_S e_\xi = e_\xi$ for $\xi \in S$ and $P_S e_\xi = 0$ for $\xi \notin S$, has norm ≤ 1 . Finally, we say that a basis $\{v_\xi\}_{\xi < \delta}$ is *isometric to* a subbasis of $\{e_\alpha\}_{\alpha < \eta}$ if there is an increasing function $\varphi: \delta \rightarrow \eta$ such that the linear operator f defined by equations $f(v_\xi) = e_{\varphi(\xi)}$ ($\xi < \delta$) is a linear isometric embedding.

Theorem 7.9. *Assume the Continuum Hypothesis. Then there exists a Banach space \mathbf{P} with a monotone Schauder basis $\{e_\alpha\}_{\alpha < \omega_1}$ and with the following properties:*

- (i) *Every monotone Schauder basis of type $\leq \omega_1$ is isometric to a canonically 1-complemented subbasis of $\{e_\alpha\}_{\alpha < \omega_1}$.*
- (ii) *Given separable initial subspaces $X, Y \subseteq \mathbf{P}$, each with a monotone Schauder basis, given a bijective linear isometry $h: X \rightarrow Y$, there exists a bijective linear isometry $H: \mathbf{P} \rightarrow \mathbf{P}$ such that $H \upharpoonright X = h$.*

Furthermore, conditions (i) and (ii) determine the space \mathbf{P} uniquely, up to a linear isometry.

Proof. We shall first consider a pair of categories $\mathfrak{B} \subseteq \mathfrak{B}'$ which has the mixed amalgamation property. Namely, the objects of \mathfrak{B}' are pairs of the form $\langle X, \{x_\xi\}_{\xi < \delta} \rangle$, where $\delta < \omega_1$ and $\{x_\xi\}_{\xi < \delta}$ is a monotone Schauder basis in X . A \mathfrak{B}' -arrow from $\langle X, \{x_\xi\}_{\xi < \delta} \rangle$ into $\langle Y, \{y_\xi\}_{\xi < \eta} \rangle$ is an isometric embedding of the basis $\{x_\xi\}_{\xi < \delta}$ onto a canonically 1-complemented subbasis of $\{y_\xi\}_{\xi < \eta}$. The category \mathfrak{B} has the same objects as \mathfrak{B}' , the arrows are more restrictive. Namely, a \mathfrak{B} -arrow from $\langle X, \{x_\xi\}_{\xi < \delta} \rangle$ into $\langle Y, \{y_\xi\}_{\xi < \eta} \rangle$ is an isometric embedding $f: X \rightarrow Y$ such that $f(x_\xi) = y_\xi$ for $\xi < \delta$ (note that $\{y_\xi\}_{\xi < \delta}$ is canonically 1-complemented in $\{y_\xi\}_{\xi < \eta}$ because the basis is monotone).

Claim 7.10. *The inclusion $\mathfrak{B} \subseteq \mathfrak{B}'$ has the mixed amalgamation property.*

Proof. It suffices to prove the mixed amalgamation involving a \mathfrak{B} -arrow $f: Z \rightarrow X$ such that f is the inclusion $Z \subseteq X$ and $X/f[Z]$ is 1-dimensional. For general \mathfrak{B} -arrows, the mixed amalgamation can later be proved by an easy induction, using continuity at limit steps.

So fix Banach spaces Z, X, Y such that $Z \subseteq X$, $\{x_\xi\}_{\xi \leq \delta}$ is a monotone Schauder basis in X such that $\{x_\xi\}_{\xi < \delta}$ is a Schauder basis for Z . Fix also a \mathfrak{B}' -arrow $i: Z \rightarrow Y$, where $\{y_\xi\}_{\xi < \varrho}$ is a monotone Schauder basis and $\{i(x_\xi)\}_{\xi < \delta}$ is a canonically 1-complemented subbasis of $\{y_\xi\}_{\xi < \varrho}$. Let $i(x_\xi) = y_{\varphi(\xi)}$, where $\varphi: \delta \rightarrow \varrho$ is strictly increasing. Let $P: Y \rightarrow Z$ and $T: X \rightarrow Z$ be the canonical projections.

Forgetting the structure induced by Schauder bases, we move into the category of separable Banach spaces \mathfrak{B}_{\aleph_0} , where by Lemmata 7.7 and 6.6 we conclude that the pair $\langle i, P \rangle, \langle \subseteq, T \rangle$ of \mathfrak{B}_{\aleph_0} -arrows has a proper amalgamation. In other words, there exist a Banach space $W \supseteq Y$ and an isometric embedding $j: X \rightarrow W$, together with norm one projections $S: W \rightarrow Y$ and $R: W \rightarrow X$, satisfying

$$j \upharpoonright Z = i, T \circ R = P \circ S, S \circ j = i \circ T \text{ and } R \upharpoonright Z = P.$$

Replacing W by $Y + j[X]$, we may assume that W/Y is 1-dimensional (in fact, the pushout of the embeddings already has this property). Define $w_\xi = y_\xi$ for $\xi < \varrho$ and define $w_\varrho = j(x_\delta)$. Observe that $Sj(x_\delta) = iT(x_\delta) = 0$, that is, $x_\delta \in \ker S$. It follows that $\{w_\xi\}_{\xi \leq \varrho}$ is a monotone Schauder basis of W and S is the canonical projection,

that is, the inclusion $Y \subseteq W$ becomes a \mathfrak{B}' -arrow. Finally, j is a \mathfrak{B} -arrow, which is witnessed by the projection R . \square

Now, since $2^{\aleph_0} = \aleph_1$, by Theorem 3.17 we conclude that there exists a continuous ω_1 -sequence \vec{U} in \mathfrak{B} that is Fraïssé over the inclusion $\mathfrak{B} \subseteq \mathfrak{B}'$. It is clear that every monotone Schauder basis of length ω_1 is the result of a continuous chain in \mathfrak{B} . Thus, (i) follows from Theorem 3.18.

In order to show (ii), we shall move to another category. Namely, let \mathfrak{K} be the subcategory of \mathfrak{B}_{\aleph_0} consisting of all isometric embeddings $f: X \rightarrow Y$ such that $f[X]$ is an initial subspace of Y and $\{0\}$ is an initial subspace of X (i.e., X has a monotone Schauder basis of a countable type).

Adapting the arguments from Claim 7.10, it is clear that \mathfrak{K} has the amalgamation property. Furthermore, co-limits in the category of Banach spaces provide a quasi-limiting operator on \mathfrak{K} , therefore by Theorem 4.4 there is at most one continuous ω_1 -Fraïssé sequence \vec{u} in \mathfrak{K} and it satisfies the back-and-forth principle. This translates to the fact that the Banach space obtained as the co-limit of \vec{u} in the category of Banach spaces satisfies (ii) and is unique up to a linear isometry.

In order to complete the proof, it suffices to show that \vec{u} is isomorphic to the image of \vec{U} under the forgetful functor from \mathfrak{B}' into \mathfrak{K} . Represent \vec{U} as a suitable continuous chain $\{U_\alpha\}_{\alpha < \omega_1}$ of separable spaces. Fix $\alpha < \omega_1$ and fix an isometric embedding $f: U_\alpha \rightarrow Y$ such that $f[U_\alpha]$ is an initial subspace of Y . We need to find an isometric embedding $g: Y \rightarrow U_\beta$ with $\beta > \alpha$, such that $g[Y]$ is an initial subspace of U_β and $g \circ f$ is the inclusion $U_\alpha \subseteq U_\beta$. Notice that, just by the definition of an initial subspace, Y has a monotone Schauder basis extending the image of the fixed basis of U_α . In other words, f is actually a \mathfrak{B} -arrow. Using the fact that \vec{U} is Fraïssé over the inclusion $\mathfrak{B} \subseteq \mathfrak{B}'$, we find $\beta > \alpha$ and a \mathfrak{B}' -arrow $g: Y \rightarrow U_\beta$ such that $g \circ f$ is the inclusion $U_\alpha \subseteq U_\beta$. Finally, note that a Banach space E is initial in F whenever $E \subseteq F$ and a fixed monotone Schauder basis of E is canonically 1-complemented in a fixed monotone Schauder basis of F . In particular, $g[Y]$ is an initial subspace of U_β , which says that $g \in \mathfrak{K}$. This shows that \vec{u} is a Fraïssé sequence in \mathfrak{K} . \square

Let us mention that in the separable case, Pełczyński [35] constructed a complementably universal (in the sense of isomorphic embeddings) Banach space for Schauder bases of type ω . In the same paper, he also constructed a complementably universal unconditional basis of type ω . Its uncountable counterpart can be obtained by a suitable modification of the proof above. We skip the details.



We finally recall a rather standard argument, going back to Ostrovkii [32], showing that the Continuum Hypothesis is really needed for the existence of the Banach spaces from Theorems 7.5, 7.8 and 7.9. The fact that a space satisfying Theorem 7.5 cannot have density less than the continuum has been already noted in [3, Prop. 4.1].

Proposition 7.11. *Let V be a Banach space containing a fixed isometric copy of c_0 and satisfying the following condition:*

- (\times) *For every Banach space $E \supseteq c_0$ such that $\dim(E/c_0) = 1$ and c_0 is 1-complemented in E , there exists an isometric embedding $f: E \rightarrow V$ such that $f \upharpoonright c_0 = \text{id}_{c_0}$.*

Then the density of V is at least the continuum.

Proof. We assume that $c_0 \subseteq V$. Given an infinite set $A \subseteq \omega$, denote by χ_A its characteristic function, treated as an element of ℓ_∞ . Let $E_A = c_0 + \mathbb{R}\chi_A \subseteq \ell_\infty$ endowed with the norm $\|x + \lambda\chi_A\| = \max\{\|x + \lambda\chi_A\|_\infty, |\lambda|\}$. By this way, the canonical projection onto c_0 that maps χ_A to 0 has norm one and we may use condition (\times). Let $f_A: E_A \rightarrow V$ be an embedding such that $f_A \upharpoonright c_0 = \text{id}_{c_0}$ and let $v_A = f_A(\chi_A)$. Now fix another infinite set $B \subseteq \omega$ such that $A \not\subseteq B$ and fix $m \in A \setminus B$. Then $\|v_A - v_B\| \geq \|v_B - 2\chi_{\{m\}}\| - \|2\chi_{\{m\}} - v_A\| = 3 - 2 = 1$. Finally, taking an independent family of cardinality 2^{\aleph_0} we obtain a 1-discrete subset of V of the same cardinality. \square

Corollary 7.12. *A Banach space satisfying the assertion either of Theorem 7.5 or 7.8 or 7.9 must have density at least 2^{\aleph_0} .*

7.3 The Cantor set

Let \mathfrak{Set}^+ denote the category of nonempty finite sets. It is clear that one-to-one mappings have pushouts in \mathfrak{Set}^+ , therefore it has proper amalgamations. We claim that $\sigma(\ddagger\mathfrak{Set}^+)$ is isomorphic to the following category \mathfrak{L} . The objects of \mathfrak{L} are pairs of the form $\langle K, D \rangle$, where K is a totally disconnected compact metric space and $D \subseteq K$ is a countable dense set. An arrow from $\langle K, D \rangle$ to $\langle L, E \rangle$ is a pair of functions $\langle j, f \rangle$, where $j: D \rightarrow E$, $f: L \rightarrow K$, $f \circ j = \text{id}_D$ and f is continuous.

Given a sequence \vec{x} in $\ddagger\mathfrak{Set}^+$, we take D to be the co-limit of $e[\vec{x}]$ in the category of sets. Clearly, D is a countable set. Now take K to be the inverse limit of $r[\vec{x}]$ in the category of topological spaces. Why do we mention topology here? As we shall see in a moment, arrows correspond to continuous quotients. Clearly, K is compact metrizable and totally disconnected. In view of Lemma 6.1, there is a canonical embedding of D into K . Thus, we may think of D as a subset of K . Notice that every projection from K to an element of the sequence $r[\vec{x}]$ is a continuous quotient; in other words, it corresponds to a partition into clopen sets.

Fix two sequences \vec{x}, \vec{y} in $\ddagger\mathfrak{Set}^+$ and fix an arrow $\vec{f}: \vec{x} \rightarrow \vec{y}$. Let $\langle K, D \rangle$ and $\langle L, E \rangle$ be the \mathfrak{L} -objects corresponding to \vec{x} and \vec{y} , respectively. Notice that the sequence $e(f_n)$ “converges” to a one-to-one map $j: D \rightarrow E$, and the sequence $r(f_n)$ “converges” to a continuous quotient $f: L \rightarrow K$. Clearly, $f \circ j = \text{id}_D$. By this way we have described a functor from $\sigma(\ddagger\mathfrak{Set}^+)$ into \mathfrak{L} .

Now fix an \mathfrak{L} -object $\langle K, D \rangle$ and write $D = \{d_n\}_{n \in \omega}$. Assume D is infinite and its enumeration is one-to-one. Construct inductively partitions \mathcal{U}_n of K into clopen

sets, in such a way that for each $U \in \mathcal{U}_n$ there is a unique index $i \leq n$ such that $d_i \in U$. Let $D_n = \{d_0, \dots, d_n\}$ and define $r_n: K \rightarrow D_n$ so that $r_n^{-1}(d_i) \in \mathcal{U}_n$ and $r_n(d_i) = d_i$ for $i \leq n$. Let $e_n: D_n \rightarrow D$ be the inclusion. Set $x_n^\infty = \langle e_n, r_n \rangle$. Then x_n^∞ is a $\ddagger\mathbf{Set}^+$ -arrow and for each $n < m$ there is a unique $\ddagger\mathbf{Set}^+$ -arrow x_n^m satisfying $x_n^\infty = x_n^m \circ x_n^m$. It is clear that this defines a sequence \vec{x} in $\ddagger\mathbf{Set}^+$ which induces the \mathfrak{L} -object $\langle K, D \rangle$. Call an \mathfrak{L} -arrow *proper* if it corresponds to a proper arrow of sequences in $\ddagger\mathbf{Set}^+$.

Lemma 7.13. *An \mathfrak{L} -arrow $f: \langle K, D \rangle \rightarrow \langle L, E \rangle$ is proper if and only if $f = \langle j, r \rangle$, where $j: K \rightarrow L$, $r: L \rightarrow K$ are continuous maps such that $j[D] \subseteq E$, $r[E] \subseteq D$, and $r \circ j = \mathbf{id}_K$.*

The last three conditions actually imply that $r[E] = D$.

Proof. It is clear that a proper arrow of sequences induces a pair $\langle j, r \rangle$ satisfying the conditions above. Fix continuous maps $j: K \rightarrow L$, $r: L \rightarrow K$ satisfying $r \circ j = \mathbf{id}_K$ and $j[D] \subseteq E$, $r[E] = D$. Without loss of generality, we may assume that $L \subseteq K$, j is the inclusion, and $D = E \cap L$. Choose a chain $\{D_n\}_{n \in \omega}$ of finite subsets of D such that $\bigcup_{n \in \omega} D_n = D$. Choose inductively finite sets $E_n \subseteq E$ so that $D_n \subseteq E_n$, $r[E_n] = D_n$, and $E_n \subseteq E_{n+1}$ for $n \in \omega$. We can do it in such a way that $\bigcup_{n \in \omega} E_n = E$, because $r^{-1}[D] \supseteq E$. Now observe that $f_n := \langle e_n, r_n \rangle$ is a $\ddagger\mathbf{Set}^+$ -arrow, where e_n is the inclusion $D_n \subseteq E_n$ and $r_n = r \upharpoonright E_n$. Finally, $\{f_n\}_{n \in \omega}$ is a proper arrow from the sequence $\{D_n\}_{n \in \omega}$ to the sequence $\{E_n\}_{n \in \omega}$. \square

The category $\ddagger\mathbf{Set}^+$ is certainly countable, therefore it has a Fraïssé sequence. It is easy to check that a sequence \vec{u} is Fraïssé in $\ddagger\mathbf{Set}^+$ if and only if the inverse limit of $r[\vec{u}]$ is the Cantor set 2^ω , together with the canonical projections. Moreover, for every countable dense set D of 2^ω there exists a sequence \vec{u} in $\ddagger\mathbf{Set}^+$ such that 2^ω is the limit of $r[\vec{u}]$ and D is the co-limit of $e[\vec{u}]$ in the category of sets. In particular, we obtain the following properties of the Cantor set which belong to the folklore:

Theorem 7.14. *Let Q denote the set of rational numbers in the standard representation of the Cantor set. Then*

- (1) *Given a countable dense set $D \subseteq 2^\omega$, there exists a homeomorphism $h: 2^\omega \rightarrow 2^\omega$ such that $h[D] = Q$.*
- (2) *Given a compact totally disconnected metric space K together with a countable dense set $D \subseteq K$, there exist a continuous surjection $f: 2^\omega \rightarrow K$ and a topological embedding $i: K \rightarrow 2^\omega$ such that $f \circ i = \mathbf{id}_K$, $i[D] \subseteq Q$ and $f[Q] = D$.*

Homogeneity translates to the following statement, again belonging to the folklore:

Theorem 7.15. *Let Q be a countable dense subset of 2^ω . Given a nonempty finite set s , given continuous mappings $p_i: 2^\omega \rightarrow s$ and embeddings $e_i: s \rightarrow Q$ for $i = 0, 1$, such that $p_0 \circ e_0 = \mathbf{id}_s = p_1 \circ e_1$, there exists a homeomorphism $h: 2^\omega \rightarrow 2^\omega$ such that $h[Q] = Q$, $p_1 \circ h = p_0$ and $h \circ e_0 = e_1$.*

Below we give two examples of improper arrows in $\sigma(\ddagger\mathbf{Set}^+)$.

Example 7.16. (a) Let $x_n = \{0, \dots, n\}$, $y_n = \{0, \dots, n, \infty\}$. Then $x_0 \subseteq x_1 \subseteq \dots$ and $y_0 \subseteq y_1 \subseteq \dots$. Let $K = L = \omega \cup \{\infty\}$, where ω is discrete and $\infty = \lim_{n \rightarrow \infty} n$. Given $n < m$, let $x_n^m \in (\ddagger\mathbf{Set}^+)$ be such that $e(x_n^m)$ is inclusion and $r(x_n^m)$ is the retraction mapping all elements of the set $\{n+1, \dots, m\}$ to n . Let $y_n^m \in (\ddagger\mathbf{Set}^+)$ be such that $e(y_n^m)$ is again inclusion, while $r(y_n^m)$ maps the set $\{n+1, \dots, m, \infty\}$ to ∞ . This defines ω -sequences \vec{x} and \vec{y} in $\ddagger\mathbf{Set}^+$. Now let $f_n: x_n \rightarrow y_n$ be defined by $f_n = \langle e_n, r_n \rangle$, where e_n is inclusion and r_n maps ∞ to n . Notice that, given $n < m$, the equation $y_n^m \circ f_n = f_m \circ x_n^m$ holds in $\ddagger\mathbf{Set}^+$, although this amalgamation is not proper.

The sequences \vec{x} and \vec{y} correspond to pairs $\langle K, \omega \rangle$ and $\langle L, L \rangle$ in \mathfrak{L} , respectively. The sequence $\vec{f} = \{f_n\}_{n \in \omega}$ induces maps $e: \omega \rightarrow L$ and $r: L \rightarrow K$, where e is the inclusion of ω into L and r is the identity. Note that $r[L] \not\subseteq \omega$, therefore $\langle e, r \rangle$ is not proper.

Another example of an improper arrow can be obtained as follows. Consider the compact space $L = \mathbb{Z} \cup \{-\infty, \infty\}$, where \mathbb{Z} is the (discrete!) set of the integers and $\lim_{n \rightarrow \infty} \infty$ and $\lim_{n \rightarrow \infty} -n = -\infty$. Let $K = \mathbb{Z} \cup \{\infty\}$ be a quotient of L obtained by identifying $-\infty$ with ∞ . Let $r: L \rightarrow K$ be the quotient map. Furthermore, let $D = K$, $E = L$, and let $e: D \rightarrow E$ be the inclusion. We claim that $\langle e, r \rangle$ is an arrow from $\langle K, D \rangle$ into $\langle L, E \rangle$. Of course, it cannot be proper, because e is even not continuous.

Let $x_n = \{i \in \mathbb{Z}: |i| \leq n\} \cup \{\infty\}$, $y_n = \{i \in \mathbb{Z}: |i| \leq n\} \cup \{-\infty, \infty\}$. Let x_n^m be such that $e(x_n^m)$ is the inclusion $x_n \subseteq x_m$ and $r(x_n^m)$ is identity on x_n and maps all $j \in x_m \setminus x_n$ to ∞ . Define y_n^m in a similar manner, with the difference that $r(y_n^m)(j) = -\infty$ whenever $j < -n$ and $r(y_n^m)(j) = \infty$ whenever $j > n$. We have defined sequences \vec{x} and \vec{y} corresponding to $\langle K, K \rangle$ and $\langle L, L \rangle$, respectively. Finally, let $f_n = \langle e_n, r_n \rangle$, where e_n is the inclusion $x_n \subseteq y_n$ and r_n is a quotient map of y_n onto x_n that maps $-\infty$ onto ∞ and satisfies $r_n \circ e_n = \text{id}_{x_n}$. Then $\vec{f} = \{f_n\}_{n \in \omega}$ is an improper arrow of sequences inducing $\langle e, r \rangle$.



We have just seen above that the Cantor set 2^ω together with its fixed countable dense subset Q can be seen as the co-limit of a Fraïssé sequence in $\ddagger\mathbf{Set}^+$. Using the concepts from Section 6, we shall now find a universal family of continuous self-functions on 2^ω that preserve Q .

Namely, we define the following category \mathfrak{K} . The objects of \mathfrak{K} are triples of the form $\langle a, b, \{f_i\}_{i \in s} \rangle$, where a, b, s are nonempty finite sets, and $f_i: a \rightarrow b$ is a mapping for each $i \in s$. Given \mathfrak{K} -objects $p = \langle a, b, \{f_i\}_{i \in s} \rangle$, $q = \langle c, d, \{g_i\}_{i \in t} \rangle$, a \mathfrak{K} -arrow from p to q is a triple $\langle k, \ell, \varphi \rangle$ such that $k: a \rightarrow c$, $\ell: b \rightarrow d$ are $\ddagger\mathbf{Set}^+$ -arrows and $\varphi: s \rightarrow t$

is a one-to-one function such that for each $i \in s$ the following diagrams commute:

$$\begin{array}{ccc}
c & \xrightarrow{g_{\varphi(i)}} & d \\
e(k) \uparrow & & \uparrow e(\ell) \\
a & \xrightarrow{f_i} & b
\end{array}
\qquad
\begin{array}{ccc}
c & \xrightarrow{g_{\varphi(i)}} & d \\
r(k) \downarrow & & \downarrow r(\ell) \\
a & \xrightarrow{f_i} & b
\end{array}$$

Recall that also $r(k) \circ e(k) = \text{id}_a$ and $r(\ell) \circ e(\ell) = \text{id}_b$. It is obvious that \mathfrak{K} is countable and has a weakly initial object (a single function between one-element sets). The key point is to show that \mathfrak{K} has the amalgamation property. For this aim, we need the following extension property, whose proof is rather trivial.

Lemma 7.17. *Let $f: a \rightarrow b$ be a \mathfrak{Set}^+ -arrow. Let $i: a \rightarrow a'$, $j: b \rightarrow b'$ be $\ddagger\mathfrak{Set}^+$ -arrows. Then there exists a \mathfrak{Set}^+ -arrow $f': a' \rightarrow b'$ for which the diagrams*

$$\begin{array}{ccc}
a' & \xrightarrow{f'} & b' \\
e(i) \uparrow & & \uparrow e(j) \\
a & \xrightarrow{f} & b
\end{array}
\qquad
\begin{array}{ccc}
a' & \xrightarrow{f'} & b' \\
r(i) \downarrow & & \downarrow r(j) \\
a & \xrightarrow{f} & b
\end{array}$$

commute.

An amalgamation of two \mathfrak{K} -arrows will be called *proper* if the two squares involving $\ddagger\mathfrak{Set}^+$ are proper amalgamations.

Lemma 7.18. *\mathfrak{K} has the proper amalgamation property.*

Proof. Fix \mathfrak{K} -objects $p = \langle a, b, \{f_i\}_{i \in s} \rangle$, $q^j = \langle c^j, d^j, \{g_i^j\}_{i \in t^j} \rangle$, where $j = 0, 1$, and let $\langle k_j, \ell_j, \varphi_j \rangle$ be \mathfrak{K} -arrows from p to q^j . By (the proof of) Lemma 6.6, there exist $\ddagger\mathfrak{Set}^+$ -arrows $k'_j: c^j \rightarrow \bar{c}$ and $\ell'_j: d^j \rightarrow \bar{d}$, where $j = 0, 1$, such that

$$\begin{array}{ccc}
c^0 & \xrightarrow{e(k'_0)} & \bar{c} \\
e(k_0) \uparrow & & \uparrow e(k'_1) \\
a & \xrightarrow{e(k_1)} & c^1
\end{array}
\qquad
\begin{array}{ccc}
d^0 & \xrightarrow{e(\ell'_0)} & \bar{d} \\
e(\ell_0) \uparrow & & \uparrow e(\ell'_1) \\
b & \xrightarrow{e(\ell_1)} & d^1
\end{array}$$

are pushout squares in \mathfrak{Set} . In particular, $\langle k'_0, k'_1 \rangle$ is a proper amalgamation of $\langle k_0, k_1 \rangle$ and $\langle \ell'_0, \ell'_1 \rangle$ is a proper amalgamation of $\langle \ell_0, \ell_1 \rangle$. Without loss of generality, we may assume that $s = t^0 \cap t^1$ and φ_0, φ_1 are inclusions. Let $\bar{t} = t^0 \cup t^1$ and let $\varphi'_j: t^j \rightarrow \bar{t}$ be the inclusion for $j = 0, 1$.

Given $i \in t^0 \setminus s$, using Lemma 7.17, choose $\bar{g}_i: \bar{c} \rightarrow \bar{d}$ satisfying

$$\bar{g}_i \circ e(k'_0) = e(\ell'_0) \circ g_i^0 \quad \text{and} \quad r(\ell'_0) \circ \bar{g}_i = g_i^0 \circ r(k'_0).$$

Similarly, given $i \in t^1 \setminus s$, find \bar{g}_i satisfying analogous conditions. In order to complete the proof, we still need to define \bar{g}_i for $i \in s$, so that $w = \langle \bar{c}, \bar{d}, \{\bar{g}_i\}_{i \in \bar{t}} \rangle$ will

become a \mathfrak{K} -object which together with $k'_0, k'_1, \ell'_0, \ell'_1$ witnesses the proper amalgamation property.

Fix $i \in s$ and define

$$\alpha = g_i^0 \circ r(k'_0), \quad \beta = g_i^1 \circ r(k'_1).$$

Notice that

$$\begin{aligned} r(\ell_0) \circ \alpha &= r(\ell_0) \circ g_i^0 \circ r(k'_0) = f_i \circ r(k_0) \circ r(k'_0) \\ &= f_i \circ r(k_1) \circ r(k'_1) = r(\ell_1) \circ g_i^1 \circ r(k'_1) = r(\ell_1) \circ \beta. \end{aligned}$$

By the property of a pushout, there exists a unique map $\bar{g}_i: \bar{c} \rightarrow \bar{d}$ satisfying

$$r(\ell'_0) \circ \bar{g}_i = \alpha \quad \text{and} \quad r(\ell'_1) \circ \bar{g}_i = \beta.$$

We still need to check that $g_i^0 \circ r(k'_0) = r(\ell'_0) \circ \bar{g}_i$ and $g_i^1 \circ r(k'_1) = r(\ell'_1) \circ \bar{g}_i$. By symmetry, we shall prove the first equality only. Define

$$\delta = r(\ell'_0) \circ \bar{g}_i \circ e(k'_1).$$

We have

$$\begin{aligned} \delta \circ e(k_1) &= r(\ell'_0) \circ \bar{g}_i \circ e(k'_1 \circ k_1) = r(\ell'_0) \circ e(\ell'_1) \circ g_i^1 \circ e(k_1) \\ &= r(\ell'_0) \circ e(\ell'_1) \circ e(\ell_1) \circ f_i = r(\ell'_0) \circ e(\ell'_0) \circ e(\ell_0) \circ f_i \\ &= e(\ell_0) \circ f_i = g_i^0 \circ e(k_0). \end{aligned}$$

This shows that $\langle g_i^0, \delta \rangle$ is an amalgamation of $\langle e(k_0), e(k_1) \rangle$. By the property of a pushout, there is a unique mapping $\gamma: \bar{c} \rightarrow d^0$ satisfying

$$(*) \quad \gamma \circ e(k'_0) = g_i^0 \quad \text{and} \quad \gamma \circ e(k'_1) = \delta.$$

Finally, it remains to check that both $\gamma_1 = g_i^0 \circ r(k'_0)$ and $\gamma_2 = r(\ell'_0) \circ \bar{g}_i$ satisfy $(*)$. That γ_2 satisfies $(*)$ is clear. It is also obvious that $\gamma_1 \circ e(k'_0) = g_i^0$. Finally,

$$\begin{aligned} \gamma_1 \circ e(k'_1) &= g_i^0 \circ r(k'_0) \circ e(k'_1) = g_i^0 \circ e(k_0) \circ r(k_1) \\ &= e(\ell_0) \circ f_i \circ r(k_1) = e(\ell_0) \circ r(\ell_1) \circ g_i^1 \\ &= r(\ell'_0) \circ e(\ell'_1) \circ g_i^1 = r(\ell'_0) \circ \bar{g}_i \circ e(k'_1) = \delta, \end{aligned}$$

which completes the proof. \square

The proof above could be made simpler and shorter, using the point-structure of sets and the explicit form of pushouts of sets. On the other hand, purely category theoretic arguments have the advantage of possible applicability to other categories.

Once we have proper amalgamations, it is natural to consider the category $\sigma\mathfrak{K}$ with *proper arrows*, where the meaning of being proper is rather obvious.

Let \vec{u} be a Fraïssé sequence in \mathfrak{K} , where $u_n = \langle v_n, w_n, \{z_i^n\}_{i \in s_n} \rangle$. We may assume that $s_0 \subseteq s_1 \subseteq \dots$ and that the bonding maps u_m^n are of the form $\langle v_m^n, w_m^n, \psi_m^n \rangle$, where ψ_m^n is the inclusion of s_m into s_n .

Lemma 7.19. \vec{v} and \vec{w} are Fraïssé sequences in $\ddagger\mathfrak{Set}^+$.

Proof. This is an immediate consequence of Lemma 7.17. \square

It follows that both \vec{v} and \vec{w} determine the Cantor set together with a fixed countable dense (as we know, it can be chosen arbitrarily). Finally, in order to show that \vec{u} provides a sequence of continuous self-mappings of the Cantor set that is universal both in the sense of quotients and in the sense of embeddings, we need to decode the information contained in the category $\sigma\mathfrak{K}$.

Namely, let \mathfrak{L}_p be the subcategory of \mathfrak{L} consisting of all proper arrows (see Lemma 7.13). Recall that the objects of \mathfrak{L}_p (as well as of \mathfrak{L}) are pairs $\langle K, D \rangle$, where K is a 0-dimensional compact metrizable space and D is a countable dense subset of K . Let \mathfrak{B} be the category whose objects are the same as the objects of \mathfrak{L} , while a \mathfrak{B} -arrow from $\langle K, D \rangle$ to $\langle L, E \rangle$ is a continuous map $f: K \rightarrow L$ satisfying $f[D] \subseteq E$.

We now define the following category \mathfrak{F} , in the same spirit as the category \mathfrak{K} defined above. The objects of \mathfrak{F} are triples of the form $\langle A, B, \{f_n\}_{n \in S} \rangle$, where A, B are \mathfrak{L} -objects, S is a nonempty countable set and $f_n: A \rightarrow B$ is a \mathfrak{B} -arrow.

An \mathfrak{F} -arrow from $X = \langle A, B, \{f_n\}_{n \in S} \rangle$ to $X' = \langle A', B', \{f'_n\}_{n \in S'} \rangle$ is a triple $f = \langle k, \ell, \varphi \rangle$, where $k: A \rightarrow A'$, $\ell: B \rightarrow B'$ are \mathfrak{L}_p -arrows, $\varphi: S \rightarrow S'$ is a one-to-one mapping such that

$$f'_{\varphi(n)} \circ e(k) = e(\ell) \circ f_n \quad \text{and} \quad r(\ell) \circ f'_{\varphi(n)} = f_n \circ r(k)$$

holds for every $n \in S$.

Lemma 7.20. *The canonical functor from $\sigma\mathfrak{K}$ to \mathfrak{F} , identifying the objects of \mathfrak{K} with constant sequences, is surjective on the class of objects.*

Proof. As usual, we identify \mathfrak{K} with constant sequences. It is obvious how to define a suitable functor $F: \sigma\mathfrak{K} \rightarrow \mathfrak{F}$ which is the identity on \mathfrak{K} . It remains to check that every \mathfrak{F} -arrow is induced by a proper arrow of the corresponding sequences in \mathfrak{K} .

Fix an \mathfrak{F} -object $X = \langle A, B, \{f_n\}_{n \in S} \rangle$, where $A = \langle K, D \rangle$, $B = \langle L, E \rangle$. For convenience, we shall assume that $S = \omega$ (if S is finite, the argument becomes simpler). Let \vec{a}, \vec{b} be two sequences in $\ddagger\mathfrak{Set}^+$ such that $K = \varprojlim r[\vec{a}]$, $L = \varprojlim r[\vec{b}]$, $D = \bigcup_{n \in \omega} a_n$, $E = \bigcup_{n \in \omega} b_n$, assuming $\{a_n\}_{n \in \omega}$ and $\{b_n\}_{n \in \omega}$ are chains of finite sets such that $e(a_m^n)$ and $e(b_m^n)$ are inclusions. For each $n \in \omega$ we can find a strictly increasing function $\varphi_n: \omega \rightarrow \omega$ such that f_n is the limit of $\{f_n^k\}_{k \in \omega}$, or more precisely, for each $k, n \in \omega$ the squares

$$\begin{array}{ccc} D & \xrightarrow{f_n} & E \\ \uparrow & & \uparrow \\ a_k & \xrightarrow{f_n^k} & b_{\varphi_n(k)} \end{array} \quad \begin{array}{ccc} K & \xrightarrow{f_n} & L \\ \downarrow & & \downarrow \\ a_k & \xrightarrow{f_n^k} & b_{\varphi_n(k)} \end{array}$$

are commutative, where the vertical arrows are induced by the sequences \vec{a}, \vec{b} . Define

$$\psi(n) = \max\{\varphi_0(n), \varphi_1(n), \dots, \varphi_n(n)\}$$

and $s_n = \{0, 1, \dots, n\}$. Given $n \leq k$, replace f_n^k by $b_{\varphi(n)}^{\psi(n)} \circ f_n^k$. Now

$$x_k = \langle a_k, b_k, \{f_n^k\}_{k \in s_n} \rangle$$

is a \mathfrak{K} -object and the bonding maps of \vec{a} and \vec{b} can be used to compose a sequence \vec{x} converging to X in the sense that each f_n is the co-limit of $\{f_n^k\}_{k \geq n}$. \square

It is not hard to check that the functor mentioned in Lemma 7.20 witnesses the equivalence of \mathfrak{F} to the category of sequences in \mathfrak{K} with proper arrows.

Summarizing, we obtain the following result concerning continuous functions on the Cantor set.

Theorem 7.21. *Let Q be a fixed countable dense set in the Cantor set 2^ω . Then there exists a sequence $\{u_n\}_{n \in \omega}$ of continuous self-maps of the Cantor set such that $u_n[Q] \subseteq Q$ for every $n \in \omega$ and the following condition is satisfied.*

(\ddagger) *Given a sequence $\{f_n: K \rightarrow L\}_{n \in \omega}$ of continuous functions between 0-dimensional compact metric spaces, given countable dense sets $D \subseteq K, E \subseteq L$ such that $f_n[D] \subseteq E$ for every $n \in \omega$, there exist topological embeddings $i: K \rightarrow 2^\omega, j: L \rightarrow 2^\omega$ together with continuous surjections $p: 2^\omega \rightarrow K, q: 2^\omega \rightarrow L$, satisfying $p \circ i = \text{id}_K, q \circ j = \text{id}_L, i[D] \subseteq Q, p[Q] = D, j[E] \subseteq Q, q[Q] = E$, and there exists a one-to-one function $\theta: \omega \rightarrow \omega$ such that for every $n \in \omega$ the following diagrams commute.*

$$\begin{array}{ccc} 2^\omega & \xrightarrow{u_{\theta(n)}} & 2^\omega \\ i \uparrow & & \uparrow j \\ K & \xrightarrow{f_n} & L \end{array} \qquad \begin{array}{ccc} 2^\omega & \xrightarrow{u_{\theta(n)}} & 2^\omega \\ p \downarrow & & \downarrow q \\ K & \xrightarrow{f_n} & L \end{array}$$

One could also formulate the homogeneity part, describing the sequence $\{u_n\}_{n \in \omega}$ uniquely up to isomorphism. The precise statement seems to be too technical, therefore we have decided to omit it.

7.4 Linear orders

Historically, the set \mathbb{Q} of rational numbers was the first universal homogeneous object, discovered in this context by Cantor at the end of 19th century. In the realm of Fraïssé theory, its properties are rather obvious and easily explained. Since the category of linear orders with embeddings has the amalgamation property, uncountable versions of the set of rational numbers may exist, as usual, subject to a “proper” cardinal arithmetic. For example, the Continuum Hypothesis is equivalent to the

statement “there exists a linearly ordered set \mathbb{Q}_{ω_1} of cardinality \aleph_1 that contains isomorphic copies of all linearly ordered sets of cardinality $\leq \aleph_1$ and is homogeneous with respect to its countable subsets”. Indeed, assuming CH, this is just the consequence of the Fraïssé-Jónsson theory. On the other hand, if such a set \mathbb{Q}_{ω_1} exists then, assuming $\mathbb{Q} \subseteq \mathbb{Q}_{\omega_1}$, every extension $\mathbb{Q} \subseteq \mathbb{Q} \cup \{t\}$ with $t \in \mathbb{R}$ must be realized in \mathbb{Q}_{ω_1} , showing that $|\mathbb{Q}_{\omega_1}| \geq 2^{\aleph_0}$.

Notice that every embedding between finite linearly ordered sets is left-invertible. This may serve as a motivation for studying uncountable versions of the set of rational numbers, where homogeneity is considered with respect to left-invertible embeddings. It turns out that no cardinal arithmetic assumptions are needed for the existence of such a set of size \aleph_1 . In particular, we shall see a natural example of a directed σ -complete category with amalgamations that is dominated by a single arrow.

Namely, consider the category \mathfrak{B} of nonempty countable linearly ordered sets with increasing maps. By “increasing” we mean “preserving the non-strict ordering”. In particular, constant maps are increasing.

Let $f: 1 \rightarrow 2$ be the inclusion map, where, as usual, $1 = \{\emptyset\}$ and $2 = \{\emptyset, 1\}$. It is easy to see that the pair $\langle f, f \rangle$ has no pushout in \mathfrak{B} . However, we shall see that the category $\ddagger\mathfrak{B}$ has proper amalgamations.

Given linearly ordered sets A, B we denote by $A \cdot B$ the set $A \times B$ endowed with the lexicographic ordering. Note that, fixing $p \in B$, the pair $\langle e, r \rangle$ is a $\ddagger\mathfrak{B}$ -arrow from A to $A \cdot B$, where $e(a) = \langle a, p \rangle$ and $r(a, b) = a$ for $a \in A, b \in B$.

In particular, when $B = \mathbb{Q}$ and $p = 0$, the $\ddagger\mathfrak{B}$ -arrow $\ell_A: A \rightarrow A \cdot \mathbb{Q}$ just described above will be called *canonical*.

Lemma 7.22. *The category $\ddagger\mathfrak{B}$ has proper amalgamations.*

Proof. Let $f: Z \rightarrow X, g: Z \rightarrow Y$ be $\ddagger\mathfrak{B}$ -arrows. Without loss of generality, we may assume that $X \cup Y \subseteq \mathbb{Q}$ and $X \cap Y = Z$. Let us identify X and Y with suitable subsets of $Z \cdot \mathbb{Q}$. Since \mathbb{Q} is homogeneous, we may actually assume that the embedding of Z into $Z \cdot \mathbb{Q}$ is of given by the formula $z \mapsto \langle z, 0 \rangle$. Now, consider the $\ddagger\mathfrak{B}$ -arrows $p: Z \cdot \mathbb{Q} \rightarrow Z \cdot \mathbb{Q} \cdot \mathbb{Q}$ and $q: Z \cdot \mathbb{Q} \rightarrow Z \cdot \mathbb{Q} \cdot \mathbb{Q}$ defined by $e(p)(z, s) = \langle z, s, 0 \rangle, r(p)(z, s, t) = \langle z, s \rangle$ and $e(q)(z, s) = \langle z, 0, s \rangle, r(q)(z, t, u) = \langle z, u \rangle$. We shall actually consider p restricted to X and q restricted to Y . We claim that

$$\begin{array}{ccc} Y & \xrightarrow{q} & Z \cdot \mathbb{Q} \cdot \mathbb{Q} \\ g \uparrow & & \uparrow p \\ Z & \xrightarrow{f} & X \end{array}$$

is a proper amalgamation. Fix $x = \langle t, s \rangle \in X$. Then $r(f)(x) = t$ and $e(g)(r(f)(x)) = \langle t, 0 \rangle$. On the other hand, $e(p)(x) = \langle t, s, 0 \rangle$ and $r(q)(e(p)(x)) = \langle t, 0 \rangle$.

Now fix $y = \langle t, s \rangle \in Y$. Then $r(g)(y) = t$ and $e(f)(r(g)(y)) = \langle t, 0 \rangle$. On the other hand, $e(q)(y) = \langle t, 0, s \rangle$ and $r(p)(e(q)(y)) = \langle t, 0 \rangle$. This shows that the amalgamation provided by p and q is proper. \square

Recall that $\ell_{\mathbb{Q}}$ is the canonical $\dagger\mathfrak{B}$ -arrow from \mathbb{Q} to $\mathbb{Q} \cdot \mathbb{Q}$. Note that, up to isomorphism, $\ell_{\mathbb{Q}}$ is a unique $\dagger\mathfrak{B}$ -arrow h with domain \mathbb{Q} and such that all fibers of $r(h)$ are isomorphic to \mathbb{Q} .

Lemma 7.23. *The arrow $\ell_{\mathbb{Q}}$ is dominating in $\dagger\mathfrak{B}$.*

Proof. Condition (D1) in the definition of a dominating family of arrows is obvious, because given a countable linear order X , the lexicographic product $X \cdot \mathbb{Q}$ is isomorphic to \mathbb{Q} . Fix a $\dagger\mathfrak{B}$ -arrow $f: \mathbb{Q} \rightarrow Y$. Let $\ell_Y: Y \rightarrow Y \cdot \mathbb{Q}$ be the canonical $\dagger\mathfrak{B}$ -arrow and let $h = \ell_Y \circ f$. Notice that all fibers of $r(h)$ are isomorphic to \mathbb{Q} , therefore h is isomorphic to $\ell_{\mathbb{Q}}$. \square

Now we know that the category $\dagger\mathfrak{B}$ has an ω -Fraïssé sequence, which we do not find of any particular interest. Since \mathfrak{B} is σ -complete, we can also talk about ω_1 -Fraïssé sequences. By the results of Section 6, there exists a unique semi-continuous ω_1 -sequence in $\dagger\mathfrak{B}$. Its natural co-limit \mathbb{Q}_{ω_1} can be regarded as a natural uncountable variant of the set of rational numbers, existing without any extra set-theoretic assumptions. Note that the ω_1 -Fraïssé sequence has the property that each bonding arrow is isomorphic to $\ell_{\mathbb{Q}}$. With this information at hand, it is straightforward to see that

$$\mathbb{Q}_{\omega_1} = \{x \in \mathbb{Q}^{\omega_1} : |\text{suppt}(x)| < \aleph_0\},$$

endowed with the lexicographic ordering (recall that $\text{suppt}(x) = \{\alpha : x(\alpha) \neq 0\}$).

We now come back to the theory of Valdivia compact spaces, in the context of linearly ordered spaces. Namely, every linearly ordered set induces a natural interval topology. This topology is compact if and only if every subset has the least upper bound (the supremum of the empty set is the minimal element). There is a natural duality (formally: dual equivalence) between the category of linearly ordered sets with increasing maps and the category of compact 0-dimensional linearly ordered spaces with continuous increasing maps preserving the minimum and the maximum. To be more precise, by a *compact line* we mean a compact linearly ordered space K with distinguished elements $0_K = \min K$ and $1_K = \max K$. The arrows of our category are continuous increasing maps preserving the distinguished elements. In particular, compact lines are supposed to be nonempty. The duality can be described briefly as follows. Given a linearly ordered set X , we let $K(X)$ to be the set of all increasing maps from X into $2 = \{0, 1\}$. This is clearly a compact 0-dimensional line. Conversely, given a compact 0-dimensional line K , we define $L(K)$ to be the set of all arrows into $2 = \{0, 1\}$ treated as a compact line. Specifically, $L(K)$ consists of all continuous increasing maps $f: K \rightarrow 2$ satisfying $f(0) = 0$ and $f(1) = 1$. Again, this is a linearly ordered set and we consider it without topology. Both operations canonically extend to contravariant functors, providing the dual equivalence of the two categories. The dual equivalence described above is actually a special case of Priestley duality [6] between partially ordered sets and distributive lattices.

It turns out that Valdivia compact lines cannot be too big. For this aim, we quote the following result from [24]:

Proposition 7.24 ([24, Props. 5.5, 5.7]). *Valdivia compact lines are precisely those compact spaces that can be represented as limits of continuous inverse sequences of compact metrizable lines whose bonding maps are increasing and right-invertible (in the category of continuous maps). In particular, Valdivia compact lines have weight $\leq \aleph_1$.*

It has been furthermore proved in [24] that the class of (nonempty!) Valdivia compact lines has a universal pre-image V_{ω_1} which is 0-dimensional. More precisely, every Valdivia compact is a continuous increasing quotient of V_{ω_1} . As a byproduct, we get the fact that every Valdivia compact line is a continuous increasing quotient of a 0-dimensional one (this can be also proved more directly).

For a moment, let us consider the class of 0-dimensional Valdivia compact lines. In view of Proposition 7.24, this class can be analyzed by looking at continuous inverse sequences in the category \mathfrak{L} of topologically right-invertible increasing quotient maps between 0-dimensional compact metric lines. It has been proved in [24], without referring to Fraïssé sequences, that there is a single increasing quotient that dominates this category. Namely, consider the Cantor set 2^ω endowed with the usual linear order. Actually, all linear orderings of 2^ω compatible with its topology are isomorphic, so by the “usual one” we may consider the lexicographic ordering. Call a point $t \in 2^\omega$ *rational* if it is eventually constant when viewed as a function from ω to 2. In the language of orderings, rational points are precisely those that are isolated from one side (including 0 and 1). Let the map $\pi: 2^\omega \rightarrow 2^\omega$ be defined by the following conditions: $\pi^{-1}(t)$ is the Cantor set, if t is rational and $|\pi^{-1}(t)| = 1$, otherwise. One can easily “extract” from the proof of [24, Thm. 5.8] that this map dominates the category \mathfrak{L} . The Valdivia compact line V_{ω_1} is the inverse limit of an ω_1 -sequence \vec{v} consisting of such maps. Thus, \vec{v} is a Fraïssé sequence in \mathfrak{L} and by this way we obtain the following fact, not discovered in [24]:

Theorem 7.25. *Given an increasing isomorphism $h: A \rightarrow B$, where A, B are closed metrizable subsets of V_{ω_1} , there exists an increasing isomorphism $H: V_{\omega_1} \rightarrow V_{\omega_1}$ such that $H \upharpoonright A = h$.*

Proof. Note that every increasing isomorphism is a homeomorphism and every closed subset of a 0-dimensional compact line is its increasing retract. Thus, after moving to a suitable category of embedding-projection pairs, the statement follows from Theorem 6.5. \square

One has to admit that in the proof of [24, Thm. 5.8] it had been shown that the canonical functor from \mathfrak{L} into the category of all compact metric lines has the mixed amalgamation property and the sequence \vec{v} mentioned above is Fraïssé over this functor. Thus, the argument leading to the fact that every Valdivia compact line is an increasing continuous image of V_{ω_1} can be explained by Theorem 3.18.

Coming back to linearly ordered sets and to the Fraïssé limit \mathbb{Q}_{ω_1} of the category \mathfrak{LO} , let us mention a result from [20], characterizing linearly ordered sets X for which $K(X)$ is Valdivia compact:

Proposition 7.26 ([20, Thm. 3.1]). *Given a linearly ordered set X , the compact line $K(X)$ is Valdivia if and only if X has the following properties:*

- (1) $|X| \leq \aleph_1$.
- (2) *Every bounded monotone ω_1 -sequence in X is convergent.*
- (3) *Given a stationary set $S \subseteq \omega_1$, given a function $f: S \rightarrow X$, there exists a stationary set $T \subseteq S$ such that $f \upharpoonright T$ is monotone.*

Condition (3) perhaps looks somewhat artificial and not natural, however it appears to be one of the proper uncountable versions of the Bolzano-Weierstrass principle, saying that every sequence in a linearly ordered set contains a monotone subsequence.

It is important to note that for a continuous increasing map the property of being right-invertible in the category of topological spaces does not imply being right-invertible in the category of compact lines, because a right inverse may not preserve 0 and 1, although it is always increasing. In fact, it is well-known that every closed set (not necessarily containing 0, 1) of a 0-dimensional compact line is its topological (even increasing) retract. It is easy to check that if $f: K(Y) \rightarrow K(X)$ is a topologically right-invertible increasing quotient map, then $L(f)$ corresponds to an increasing embedding $e: X \rightarrow Y$ such that $e[X]$ is an increasing retract of its convex hull

$$\text{conv}(e[X]) = \{y \in Y : (\exists x_0, x_1 \in X) \ e(x_0) \leq y \leq e(x_1)\}.$$

On the other hand, if $f: X \rightarrow Y$ is left-invertible in the category of linearly ordered sets then $K(f)$ is right-invertible in the category of compact lines with continuous increasing maps. One can easily adapt the arguments of [20] for obtaining the following characterization:

Theorem 7.27. *Let V be a linearly ordered set of cardinality \aleph_1 . Then V is order isomorphic to \mathbb{Q}_{ω_1} if and only if it satisfies the following conditions:*

- (1) *Every monotone ω_1 -sequence in V is convergent.*
- (2) *Given countable sets $A, B \subseteq V$ such that $a < b$ for every $a \in A, b \in B$, there exists $p \in V$ such that $a < p$ for $a \in A$ and $p < b$ for $b \in B$ (we allow the possibility that one of the sets A, B is empty).*
- (3) *Given a stationary set $S \subseteq \omega_1$, given a function $f: S \rightarrow X$, there exists a stationary set $T \subseteq S$ such that $f \upharpoonright T$ is monotone.*

We refer to [20] for details, where one can find all arguments showing that (1) and (3) are necessary and sufficient for the fact that V is the co-limit of a sequence in \mathfrak{L} . Finally, condition (2) says that any ω_1 -sequence with co-limit V is Fraïssé in \mathfrak{L} .

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