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ON THE CAUCHY PROBLEM FOR LINEAR HYPERBOLIC FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. We study the question on the existence, uniqueness, and continuous dependence on parameters of Carathéodory solutions to the Cauchy problem for linear partial functional-differential equations of hyperbolic type. A theorem on the Fredholm alternative is proved, as well.

Keywords: functional-differential equation of hyperbolic type, Cauchy problem, Fredholm alternative, well-posedness, existence of solutions

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1. Introduction

On the rectangle $\mathcal{D} = [a, b] \times [c, d]$, we consider the linear hyperbolic functional-differential equation

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell(u)(t,x) + q(t,x), \tag{1.1}$$

where $\ell : C(\mathcal{D}; \mathbb{R}) \to L(\mathcal{D}; \mathbb{R})$ is a linear bounded operator and $q \in L(\mathcal{D}; \mathbb{R})$. Under a solution to the equation (1.1) is understood a function $u \in C^*(\mathcal{D}; \mathbb{R})$ which satisfies the equality (1.1) almost everywhere on the set $\mathcal{D}$.

For the hyperbolic equation

$$u_{tx} = p(t,x)u + q(t,x), \tag{1.2}$$

which is a particular case of (1.1), a number of results is known namely in the case where the coefficients p and q are continuous and the solution u of (1.2) is supposed to have continuous derivatives up to the second order (see, e.g., [7, 8, 10, 16, 18, 24–26, 28] and references therein). If the coefficients p and q in (1.2) are discontinuous, the concept of Carathéodory solutions (i.e., solutions from the class $C^*(\mathcal{D}; \mathbb{R})$) was used and the results generalized those known in the classical case were obtained (see, e.g., [1, 4, 13–15, 27, 28]).

Various initial and boundary value problems are studied for the hyperbolic equations and their systems (see, e.g., [1, 4, 7, 8, 10, 13–16, 18, 24–28] and references therein). In this paper, we consider the Cauchy problem for the equation (1.1). Let $\mathcal{H}$ be a strictly monotone curve connecting the corners (a, d) and (b, c) of the rectangle $\mathcal{D}$, which is defined as the graph of a decreasing continuous function $h : [a, b] \to [c, d]$ such that $h(a) = d$ and $h(b) = c$. The values u, $u'_{|1}$, and $u'_{|2}$[1] are prescribed on $\mathcal{H}$ as follows:

$$u\,(t, h(t)) = g(t) \quad \text{for } t \in [a, b], \tag{1.3}$$

$$u'_{|1}\,(t, h(t)) = \varphi(t) \quad \text{for a. e. } t \in [a, b], \tag{1.4}$$

$$u'_{|2}\,\big(h^{-1}(x), x\big) = \psi(x) \quad \text{for a. e. } x \in [c, d], \tag{1.5}$$

where $g \in C\big([a, b]; \mathbb{R}\big)$, $\varphi \in L\big([a, b]; \mathbb{R}\big)$, and $\psi \in L\big([c, d]; \mathbb{R}\big)$. The functions g, φ, and ψ cannot be chosen arbitrarily, they must satisfy the so-called consistency condition (see Section 3).

The aim of the paper is to prove the Fredholm alternative and theorems on the continuous dependence of solutions to the problem (1.1), (1.3)–(1.5) on the initial conditions and parameters (see Sections 5 and 8). Moreover, some solvability conditions for the problem considered are given in Section 7, the equations with the so-called Volterra operators are studied, as well.

The result obtained are applied for the equation with argument deviations

$$\frac{\partial^2 u(t, x)}{\partial t\, \partial x} = p(t, x) u\big(\tau(t, x), \mu(t, x)\big) + q(t, x), \tag{1.1$'$}$$

where $p, q \in L\big(\mathcal{D}; \mathbb{R}\big)$ and $\tau : \mathcal{D} \to [a, b]$, $\mu : \mathcal{D} \to [c, d]$ are measurable functions.

Note also that analogous results for the "ordinary" functional-differential equations and their systems are given in [2, 9, 11, 12] and the results dealing with the Darboux problem for the equation (1.1) can be found in [22].

2. Notation and Definitions

The following notation is used throughout the paper.

(1) $\mathbb{N}$ is the set of all natural numbers. $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$. $\mathrm{Ent}(x)$ denotes the entire part of the number $x \in \mathbb{R}$.
(2) $\mathcal{D} = [a, b] \times [c, d]$, where $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$.
(3) The first and the second order partial derivatives of the function $v : \mathcal{D} \to \mathbb{R}$ at the point $(t, x) \in \mathcal{D}$ are denoted by $v_t(t, x)$ (or $v'_{|1}(t, x)$, $\frac{\partial v(t,x)}{\partial t}$), $v_x(t, x)$ (or $v'_{|2}(t, x)$, $\frac{\partial v(t,x)}{\partial x}$), and $v_{tx}(t, x)$ (or $v''_{|12}(t, x)$, $\frac{\partial^2 v(t,x)}{\partial t\, \partial x}$).
(4) $C\big(\mathcal{D}; \mathbb{R}\big)$ is the Banach space of continuous functions $v : \mathcal{D} \to \mathbb{R}$ equipped with the norm $\|v\|_C = \max\big\{|v(t, x)| : (t, x) \in \mathcal{D}\big\}$.
(5) $CD\big([a, b]; [c, d]\big)$ is the set of continuous decreasing functions $v : [a, b] \to [c, d]$ such that $v(a) = d$ and $v(b) = c$.
(6) $\widetilde{C}([\alpha, \beta]; \mathbb{R})$, where $-\infty < \alpha < \beta < +\infty$, is the set of absolutely continuous functions $u : [\alpha, \beta] \to \mathbb{R}$.

[1]Symbols $u'_{|1}$ and $u'_{|2}$ stand for the partial derivatives of u with regard to the first and the second argument, respectively.

(7) $C^*\big(\mathcal{D};\mathbb{R}\big)$ is the set of functions $v:\mathcal{D}\to\mathbb{R}$ admitting the representation

$$v(t,x)=z_1(t)+z_2(x)+\int_a^t\int_c^x f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D},$$

where $z_1\in\widetilde{C}\big([a,b];\mathbb{R}\big)$, $z_2\in\widetilde{C}\big([c,d];\mathbb{R}\big)$, $f\in L\big(\mathcal{D};\mathbb{R}\big)$. Equivalent definitions of the class $C^*\big(\mathcal{D};\mathbb{R}\big)$ are given in Remark 2.1 below.

(8) $L\big(\mathcal{D};\mathbb{R}\big)$ is the Banach space of Lebesgue integrable functions $p:\mathcal{D}\to\mathbb{R}$ equipped with the norm $\|p\|_L=\iint\limits_{\mathcal{D}}|p(t,x)|\,\mathrm{d}t\,\mathrm{d}x$.

(9) $\mathcal{L}(\mathcal{D})$ is the set of linear bounded operators $\ell:C\big(\mathcal{D};\mathbb{R}\big)\to L\big(\mathcal{D};\mathbb{R}\big)$.

(10) $\operatorname{mes}A$ denotes the Lebesgue measure of the set $A\subset R^m$, $m=1,2$.

(11) If X, Y are Banach spaces and $T:X\to Y$ is a linear bounded operator then $\|T\|$ denotes the norm of the operator T, i.e.,

$$\|T\|=\sup\big\{\|T(z)\|_Y : z\in X,\ \|z\|_X\le 1\big\}.$$

(12) $A\div B$ stands for the symmetric difference of the sets A and B, i.e., $A\div B=(A\setminus B)\cup(B\setminus A)$.

Proposition 2.1 ([21, Thm. 2.1])**.** *The following three statements are equivalent:*

(1) $v\in C^*\big(\mathcal{D};\mathbb{R}\big)$;

(2) *the function* $v:\mathcal{D}\to\mathbb{R}$ *satisfies:*

  (a) $v(\cdot,x)\in\widetilde{C}\big([a,b];\mathbb{R}\big)$ *for every* $x\in[c,d]$, $v(a,\cdot)\in\widetilde{C}\big([c,d];\mathbb{R}\big)$;

  (b) $v'_{|1}(t,\cdot)\in\widetilde{C}\big([c,d];\mathbb{R}\big)$ *for almost every* $t\in[a,b]$;

  (c) $v''_{|12}\in L\big(\mathcal{D};\mathbb{R}\big)$;

(3) *the function* v *is absolutely continuous on* $\mathcal{D}$ *in the sense of Carathéodory.*

Remark 2.1. It is clear that the conditions (a)–(c) stated in the previous proposition can be replaced by the symmetric ones:

(A) $v(\cdot,c)\in\widetilde{C}\big([a,b];\mathbb{R}\big)$, $v(t,\cdot)\in\widetilde{C}\big([c,d];\mathbb{R}\big)$ for every $t\in[a,b]$;

(B) $v'_{|2}(\cdot,x)\in\widetilde{C}\big([a,b];\mathbb{R}\big)$ for almost every $x\in[c,d]$;

(C) $v''_{|21}\in L\big(\mathcal{D};\mathbb{R}\big)$.

Notation 2.1. Having $h\in CD\big([a,b];[c,d]\big)$, we put

$$H(t,x)\overset{\text{def}}{=}\Big\{(s,\eta)\in\mathbb{R}^2:\min\{h^{-1}(x),t\}\le s\le\max\{h^{-1}(x),t\},\\ \min\{h(s),x\}\le\eta\le\max\{h(s),x\}\Big\}\quad\text{for } (t,x)\in\mathcal{D}. \tag{2.1}$$

It is clear that, for any $(t,x)\in\mathcal{D}$, the set $H(t,x)$ is a measurable subset of $\mathcal{D}$.

3. Consistency Condition

We first mention that the formulation of the Cauchy problem for the equation (1.1) in the form of the conditions (1.3)–(1.5) is rather natural. Indeed, if u is a function of the class $C^*(\mathcal{D};\mathbb{R})$ then, using conditions (a)–(c) of Proposition 2.1, we get

$$u(\cdot,h(\cdot))\in C([a,b];\mathbb{R}),\quad u'_{|1}(\cdot,h(\cdot))\in L([a,b];\mathbb{R}),\quad u'_{|2}\left(h^{-1}(\cdot),\cdot\right)\in L([c,d];\mathbb{R})$$

provided $h\in CD([a,b];[c,d])$. As it was said above, the functions g, φ, and ψ appearing in the conditions (1.3)–(1.5) cannot be chosen arbitrarily. The following definition is motivated by the notion of a consistency condition presented in [28].

Definition 3.1. Let $h\in CD([a,b];[c,d])$, $g\in C([a,b];\mathbb{R})$, $\varphi\in L([a,b];\mathbb{R})$, and $\psi\in L([c,d];\mathbb{R})$. We say that a triplet (g,φ,ψ) is h–consistent (in the space $C^*(\mathcal{D};\mathbb{R})$) if there exists a function $u\in C^*(\mathcal{D};\mathbb{R})$ satisfying the conditions (1.3)–(1.5).

Now we give several conditions sufficient and necessary for a triplet (g,φ,ψ) to be h–consistent; their proofs are postponed till Section 3.1 below.

Proposition 3.1. *Let $h\in CD([a,b];[c,d])$, $g\in C([a,b];\mathbb{R})$, $\varphi\in L([a,b];\mathbb{R})$, and $\psi\in L([c,d];\mathbb{R})$. Then the triplet (g,φ,ψ) is h–consistent if, and only if the condition*

$$g(t)+\int\limits_{h(t)}^{x}\psi(\eta)\,\mathrm{d}\eta=g\left(h^{-1}(x)\right)+\int\limits_{h^{-1}(x)}^{t}\varphi(s)\,\mathrm{d}s\quad\text{for } (t,x)\in\mathcal{D}\tag{3.1}$$

holds.

Remark 3.1. If the function h is absolutely continuous and $u\in C^*(\mathcal{D};\mathbb{R})$ satisfies the initial condition (1.3) then the function g is also absolutely continuous (see Lemma 3.5 below). Consequently, the assumption on g to be absolutely continuous in Proposition 3.2 is necessary.

Let us consider the following assumption

$$(S)\qquad \operatorname{mes} h^{-1}(E)=b-a,\quad\text{where}\quad E=\left\{x\in[c,d]:\frac{d}{dx}\int_c^x\psi(\eta)\,\mathrm{d}\eta=\psi(x)\right\}.$$

Remark 3.2. The functions h and ψ satisfy the assumption (S), in particular, if $h^{-1}\in\widetilde{C}([c,d];\mathbb{R})$ or $\psi\in C([a,b];\mathbb{R})$.

Proposition 3.2. *Let $h\in CD([a,b];[c,d])$ be an absolutely continuous function, $g\in\widetilde{C}([a,b];\mathbb{R})$, $\varphi\in L([a,b];\mathbb{R})$, and $\psi\in L([c,d];\mathbb{R})$. Then*

(a) *the condition*

$$\varphi(t)+\psi(h(t))h'(t)=g'(t)\quad\text{for a. e. } t\in[a,b]\tag{3.2}$$

is sufficient for the triplet (g,φ,ψ) to be h–consistent;

(b) *the condition (3.2) is necessary for the triplet (g,φ,ψ) to be h–consistent, if the functions h and ψ satisfy the additional assumption (S).*

Remark 3.3. Note that the assumption $h\in\widetilde{C}([a,b];\mathbb{R})$ is not necessary for the existence of a h–consistent triplet. Indeed, let $g\in\widetilde{C}([a,b];\mathbb{R})$. Then the triplet $(g,g',0)$ is h–consistent for an arbitrary $h\in CD([a,b];[c,d])$. To see this it is sufficient to set $u(t,x)=g(t)$ for $(t,x)\in\mathcal{D}$.

A consistent triplet can be characterized in terms of the unique solvability of the problem (1.1), (1.3)–(1.5) with the zero operator ℓ. More precisely, the following statements is true.

Proposition 3.3. *Let* $h \in CD([a,b];[c,d])$, $g \in C([a,b];\mathbb{R})$, $\varphi \in L([a,b];\mathbb{R})$, *and* $\psi \in L([c,d];\mathbb{R})$. *Then the triplet* (g,φ,ψ) *is* h*–consistent if, and only if the problem* (1.1), (1.3)–(1.5) *with* $\ell \equiv 0$ *has a unique solution for every* $q \in L(\mathcal{D};\mathbb{R})$.

3.1. **Proofs.** In order to prove propositions stated above we need the following lemmas.

Lemma 3.1 ([19, Chap. IX, §3, Thm. 3])**.** *Let* $f \in \widetilde{C}([\alpha,\beta];\mathbb{R})$ *be a decreasing function. Then the relation* $\operatorname{mes} f(E) = f(\alpha) - f(\beta)$ *holds for an arbitrary measurable set* $E \subseteq [\alpha,\beta]$ *such that* $\operatorname{mes} E = \beta - \alpha$.

Lemma 3.2 ([21, Prop. 2.5])**.** *Let* $f \in L(\mathcal{D};\mathbb{R})$ *and*

$$u(t,x) = \int_a^t \int_c^x f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D}.$$

Then:

(i) *there exists a set* $E \subseteq [a,b]$ *such that* $\operatorname{mes} E = b-a$ *and*

$$u_t(t,x) = \int_c^x f(t,\eta)\,\mathrm{d}\eta \quad \text{for } t \in E \text{ and } x \in [c,d];$$

(iii) *there exists a set* $F \subseteq \mathcal{D}$ *such that* $\operatorname{mes} F = (b-a)(d-c)$ *and*

$$u_{tx}(t,x) = f(t,x) \quad \text{for } (t,x) \in F.$$

Lemma 3.3. *Let* $h \in CD([a,b];[c,d])$, $g \in C([a,b];\mathbb{R})$, $\varphi \in L([a,b];\mathbb{R})$, *and* $\psi \in L([c,d];\mathbb{R})$. *Then an arbitrary function* $u \in C^*(\mathcal{D};\mathbb{R})$ *fulfilling the conditions* (1.3)–(1.5) *satisfies*

$$u(t,x) = g\left(h^{-1}(x)\right) + \int_{h^{-1}(x)}^{t} \varphi(s)\,\mathrm{d}s + \iint_{H(t,x)} u_{s\eta}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D} \tag{3.3}$$

and

$$u(t,x) = g(t) + \int_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta + \iint_{H(t,x)} u_{s\eta}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \tag{3.4}$$

where the mapping H *is defined by the formula* (2.1).

Proof. Let a function $u \in C^*(\mathcal{D};\mathbb{R})$ satisfy the conditions (1.3)–(1.5). Then, using properties (a)–(c) of Proposition 2.1, we get

$$\iint_{H(t,x)} u_{s\eta}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = \int_{h^{-1}(x)}^{t} \int_{h(s)}^{x} u_{s\eta}(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s = \int_{h^{-1}(x)}^{t} \left[u'_{|1}(s,x) - u'_{|1}(s,h(s))\right]\mathrm{d}s =$$

$$= u(t,x) - u\big(h^{-1}(x),x\big) - \int\limits_{h^{-1}(x)}^{t} u'_{|1}(s,h(s))\,\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D},$$

and thus, in view of (1.3) and (1.4), the relation (3.3) holds.

On the other hand, using properties (A)–(C) of Remark 2.1, we obtain

$$\iint\limits_{H(t,x)} u_{s\eta}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = \int\limits_{h(t)}^{x}\int\limits_{h^{-1}(\eta)}^{t} u_{\eta s}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = \int\limits_{h(t)}^{x} \big[u'_{|2}(t,\eta) - u'_{|2}(h^{-1}(\eta),\eta)\big]\,\mathrm{d}\eta =$$

$$= u(t,x) - u(t,h(t)) - \int\limits_{h(t)}^{x} u'_{|2}(h^{-1}(\eta),\eta)\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D}.$$

Consequently, by virtue of (1.3) and (1.5), the relation (3.4) is satisfied. □

Lemma 3.4. *Let $h \in CD\big([a,b];[c,d]\big)$, $g \in C\big([a,b];\mathbb{R}\big)$, $\varphi \in L\big([a,b];\mathbb{R}\big)$, and $\psi \in L\big([c,d];\mathbb{R}\big)$ satisfy the relation* (3.1). *Let, moreover,*

$$u(t,x) = g(t) + \int\limits_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta + \iint\limits_{H(t,x)} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \textit{for } (t,x)\in\mathcal{D}, \tag{3.5}$$

where $f \in L\big(\mathcal{D};\mathbb{R}\big)$ and the mapping H is defined by the formula (2.1). *Then $u \in C^*\big(\mathcal{D};\mathbb{R}\big)$ and u satisfies the conditions* (1.3)–(1.5) *and*

$$u_{tx}(t,x) = f(t,x) \quad \textit{for a. e. } (t,x)\in\mathcal{D}. \tag{3.6}$$

Proof. In view of (2.1), it follows immediately from (3.5) that the function u satisfies the condition (1.3). It is clear that the relation (3.5) can be rewritten in the form

$$u(t,x) = g(t) - \int\limits_{c}^{h(t)} \psi(\eta)\,\mathrm{d}\eta - \int\limits_{c}^{h(t)}\int\limits_{h^{-1}(\eta)}^{t} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta +$$

$$+ \int\limits_{c}^{x} \psi(\eta)\,\mathrm{d}\eta - \int\limits_{c}^{x}\int\limits_{a}^{h^{-1}(\eta)} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta + \int\limits_{c}^{x}\int\limits_{a}^{t} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D}.$$

Therefore, $u(t,\cdot) \in \widetilde{C}\big([c,d];\mathbb{R}\big)$ for every $t \in [a,b]$. Moreover, in view of Lemma 3.2(i), there exists a set $E_1 \subseteq [c,d]$, $\operatorname{mes} E_1 = d - c$, such that

$$u_x(t,x) = \psi(x) - \int\limits_{a}^{h^{-1}(x)} f(s,x)\,\mathrm{d}s + \int\limits_{a}^{t} f(s,x)\,\mathrm{d}s \quad \text{for } t\in[a,b],\ x\in E_1,$$

whence we get $u'_{|2}\big(h^{-1}(x),x\big) = \psi(x)$ for $x \in E_1$, i.e., the function u satisfies the condition (1.5).

On the other hand, using the condition (3.1), we get from (3.5) the relation

$$u(t,x) = g\left(h^{-1}(x)\right) - \int\limits_a^{h^{-1}(x)} \varphi(s)\,\mathrm{d}s - \int\limits_a^{h^{-1}(x)} \int\limits_{h(s)}^{x} f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s +$$
$$+ \int\limits_a^t \varphi(s)\,\mathrm{d}s - \int\limits_a^t \int\limits_c^{h(s)} f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s + \int\limits_a^t \int\limits_c^x f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D}.$$

Consequently, $u(\cdot,x) \in \widetilde{C}\big([a,b];\mathbb{R}\big)$ for every $x \in [c,d]$. Moreover, in view of Lemma 3.2(i), there exists a set $E_2 \subseteq [a,b]$, $\operatorname{mes} E_2 = b-a$, such that

$$u_t(t,x) = \varphi(t) - \int\limits_c^{h(t)} f(t,\eta)\,\mathrm{d}\eta + \int\limits_c^x f(t,\eta)\,\mathrm{d}\eta \quad \text{for } t \in E_2,\ x \in [c,d]. \tag{3.7}$$

Therefore, $u'_{|1}(t,h(t)) = \varphi(t)$ for $t \in E_2$, i.e., u satisfies the condition (1.4).

Furthermore, the relation (3.7) implies that $u_t(t,\cdot) \in \widetilde{C}\big([c,d];\mathbb{R}\big)$ for every $t \in E_2$ and, by virtue of Lemma 3.2(ii), there exists $E \subseteq \mathcal{D}$, $\operatorname{mes} E = (b-a)(d-c)$, such that $u_{tx}(t,x) = f(t,x)$ for $(t,x) \in E$. It means that the condition (3.6) is fulfilled and $u''_{|12} \in L\big(\mathcal{D};\mathbb{R}\big)$.

We have shown that the function u satisfies the relations (1.3)–(1.5) and the conditions (a)–(c) of Proposition 2.1, and thus $u \in C^*\big(\mathcal{D};\mathbb{R}\big)$. □

Lemma 3.5. *Let $f \in CD\big([a,b];[c,d]\big)$ be an absolutely continuous function and $w \in C^*\big(\mathcal{D};\mathbb{R}\big)$. Then the function z defined by the formula*

$$z(t) = w(t,f(t)) \quad \textit{for } t \in [a,b] \tag{3.8}$$

is absolutely continuous.

Proof. Let $\varepsilon > 0$ be arbitrary fixed. Then there exists $\delta_1 > 0$ such that

$$\iint\limits_P |w_{s\eta}(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta < \frac{\varepsilon}{6} \quad \text{for } P \subseteq \mathcal{D},\ \operatorname{mes} P < \delta_1^2. \tag{3.9}$$

Moreover, there exists $\delta_2 > 0$, $\delta_2 \le \delta_1$, such that

$$\int\limits_I \left| w'_{|1}(s,f(s)) \right| \mathrm{d}s < \frac{\varepsilon}{3} \quad \text{for } I \subseteq [a,b],\ \operatorname{mes} I < \delta_2, \tag{3.10}$$

$$\int\limits_J \left| w'_{|2}\left(f^{-1}(\eta),\eta\right) \right| \mathrm{d}\eta < \frac{\varepsilon}{3} \quad \text{for } J \subseteq [c,d],\ \operatorname{mes} J < \delta_2. \tag{3.11}$$

Since the function f is absolutely continuous, there exists $\delta > 0$, $\delta \le \delta_2$, such that the relation

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \delta_2 \tag{3.12}$$

holds for an arbitrary system $\{]a_k,b_k[\}_{k=1}^n$ of disjoint intervals in $[a,b]$ with the property

$$\sum_{k=1}^{n}(b_k-a_k)<\delta. \tag{3.13}$$

Now let $\{]a_k,b_k[\}_{k=1}^n$ be a system of disjoint intervals in $[a,b]$ satisfying (3.13). Since the function f is decreasing, $\{]f(b_k),f(a_k)[\}_{k=1}^n$ is a system of disjoint intervals in $[c,d]$ such that (3.12) holds, and $\{[a_k,b_k]\times[f(b_k),f(a_k)]\}_{k=1}^n$ is a system of non-overlapping rectangles in $\mathcal{D}$ fulfilling

$$\sum_{k=1}^{n}(b_k-a_k)(f(a_k)-f(b_k))\le\delta\sum_{k=1}^{n}(f(a_k)-f(b_k))<\delta\delta_2\le\delta_1^2. \tag{3.14}$$

It is not difficult to verify that, for any $k=1,2,\dots,n$, we have

$$\begin{aligned}
&z(b_k)-z(a_k)=\\
&\quad=w(b_k,f(b_k))-w(a_k,f(a_k))=\int_{a_k}^{b_k}w_s(s,f(b_k))\,\mathrm{d}s-\int_{f(b_k)}^{f(a_k)}w_\eta(a_k,\eta)\,\mathrm{d}\eta=\\
&\quad=\int_{a_k}^{b_k}w'_{|1}(s,f(s))\,\mathrm{d}s-\int_{a_k}^{b_k}\int_{f(b_k)}^{f(s)}w_{s\eta}(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s-\\
&\qquad-\int_{f(b_k)}^{f(a_k)}w'_{|2}\left(f^{-1}(\eta),\eta\right)\,\mathrm{d}\eta+\int_{f(b_k)}^{f(a_k)}\int_{a_k}^{f^{-1}(\eta)}w_{\eta s}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta,
\end{aligned}$$

whence we get

$$\begin{aligned}
|z(b_k)-z(a_k)|\le&\int_{a_k}^{b_k}\left|w'_{|1}(s,f(s))\right|\,\mathrm{d}s+\int_{f(b_k)}^{f(a_k)}\left|w'_{|2}\left(f^{-1}(\eta),\eta\right)\right|\,\mathrm{d}\eta+\\
&+2\int_{a_k}^{b_k}\int_{f(b_k)}^{f(a_k)}|w_{s\eta}(s,\eta)|\,\mathrm{d}\eta\,\mathrm{d}s\quad\text{for } k=1,2,\dots,n.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{k=1}^{n}|z(b_k)-z(a_k)|\le&\int_I\left|w'_{|1}(s,f(s))\right|\,\mathrm{d}s+\\
&+\int_J\left|w'_{|2}\left(f^{-1}(\eta),\eta\right)\right|\,\mathrm{d}\eta+2\iint_E|w_{s\eta}(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta,
\end{aligned}$$

where $I = \cup_{k=1}^{n}[a_k, b_k]$, $J = \cup_{k=1}^{n}[f(b_k), f(a_k)]$, and $E = \cup_{k=1}^{n}[a_k, b_k]\times[f(b_k), f(a_k)]$. The last relation, together with (3.9)–(3.14), guarantees

$$\sum_{k=1}^{n} |z(b_k) - z(a_k)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2\,\frac{\varepsilon}{6} = \varepsilon,$$

and thus the function z is absolutely continuous. □

Now we are in position to prove Propositions 3.1–3.3.

Proof of Proposition 3.1. First suppose that the triplet (g, φ, ψ) is h–consistent. Then there exists a function $u \in C^*\big(\mathcal{D}; \mathbb{R}\big)$ satisfying the conditions (1.3)–(1.5). According to Lemma 3.3, the function u admits the representations (3.3) and (3.4), whose comparing we get the condition (3.1).

Now suppose that h, g, φ, and ψ are such that the relation (3.1) holds. Then, by virtue of Lemma 3.4, the function u defined by the formula

$$u(t, x) = g(t) + \int_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta \quad \text{for } (t, x) \in \mathcal{D} \tag{3.15}$$

belongs to the set $C^*\big(\mathcal{D}; \mathbb{R}\big)$ and satisfies the conditions (1.3)–(1.5). Consequently, the triplet (g, φ, ψ) is h–consistent. □

Proof of Proposition 3.2. (a) Let the condition (3.2) hold. Then, for any $(t, x) \in \mathcal{D}$, we get

$$\int_{h^{-1}(x)}^{t} \varphi(s)\,\mathrm{d}s = \int_{h^{-1}(x)}^{t} \Big(g'(s) - \psi\big(h(s)\big)h'(s)\Big)\,\mathrm{d}s = g(t) - g\big(h^{-1}(x)\big) - \int_{x}^{h(t)} \psi(\eta)\,\mathrm{d}\eta,$$

i.e., the condition (3.1) is satisfied. Consequently, applying Proposition 3.1 we conclude that the triplet (g, φ, ψ) is h–consistent.

(b) Suppose that the assumption (S) is satisfied and the triplet (g, φ, ψ) is h–consistent. Then, according to Proposition 3.1, the relation (3.1) holds and thus,

$$g(t) + \int_{h(t)}^{c} \psi(\eta)\,\mathrm{d}\eta = g(b) + \int_{b}^{t} \varphi(s)\,\mathrm{d}s \quad \text{for } t \in [a, b]. \tag{3.16}$$

Since $g, h \in \widetilde{C}\big([a, b]; \mathbb{R}\big)$, there exists a set $E_1 \subseteq [a, b]$, $\operatorname{mes} E_1 = b - a$, such that

$$g'(t),\ h'(t) \quad \text{exist for } t \in E_1 \tag{3.17}$$

and

$$\frac{d}{dt}\int_{t}^{b} \varphi(s)\,\mathrm{d}s = -\varphi(t) \quad \text{for } t \in E_1. \tag{3.18}$$

Therefore, the relation (3.16) yields

$$g'(t) - \psi\big(h(t)\big)h'(t) = \varphi(t) \quad \text{for } t \in E_1 \cap h^{-1}(E),$$

where E is the set appearing in the assumption (S). It however means that the conditions (3.2) is satisfied. □

Proof of Proposition 3.3. If the problem (1.1), (1.3)–(1.5) with $\ell \equiv 0$ has a unique solution for every $q \in L(\mathcal{D};\mathbb{R})$ then it is clear that the triplet (g,φ,ψ) is h–consistent.

Conversely, let the triplet (g,φ,ψ) be h–consistent and let $q \in L(\mathcal{D};\mathbb{R})$. Then, according to Proposition 3.1, the condition (3.1) holds and thus, by virtue of Lemma 3.4, the problem (1.1), (1.3)–(1.5) with $\ell \equiv 0$ has at least one solution. The uniqueness follows from Lemma 3.3. □

4. Auxiliary Statements

The following proposition plays a crucial role in the proofs of statements given in Sections 5, 7, and 8.

Proposition 4.1. *Let $h \in CD([a,b];[c,d])$ and $\ell \in \mathcal{L}(\mathcal{D})$. Then the operator $T : C(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ defined by the formula*

$$T(v)(t,x) \overset{\text{def}}{=} \iint\limits_{H(t,x)} \ell(v)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ v \in C(\mathcal{D};\mathbb{R}), \tag{4.1}$$

where the mapping H is given by (2.1), *is completely continuous.*

The statement stated above can be easily proved in the case where the operator ℓ is strongly bounded, i.e., if there exists a function $\eta \in L(\mathcal{D};\mathbb{R}_+)$ such that

$$|\ell(v)(t,x)| \le \eta(t,x)\|v\|_C \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}). \tag{4.2}$$

Schaefer proved however that there exists an operator $\ell \in \mathcal{L}(\mathcal{D})$, which is not strongly bounded (see [20]). To prove Proposition 4.1 without the additional requirement (4.2) we need a number of notions and statements from functional analysis.

Definition 4.1. Let X be a Banach space, X^* be its dual space.

We say that a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ is weakly convergent if there exists $x \in X$ such that $f(x) = \lim\limits_{n\to+\infty} f(x_n)$ for every $f \in X^*$. The element x is said to be a weak limit of this sequence.

A set $M \subseteq X$ is called weakly relatively compact if every sequence of elements from M contains a subsequence which is weakly convergent in X.

A sequence $\{x_n\}_{n=1}^{+\infty}$ of elements from X is said to be weakly fundamental if the sequence $\{f(x_n)\}_{n=1}^{+\infty}$ is fundamental in $\mathbb{R}$ for every $f \in X^*$.

We say that the space X is weakly complete if every weakly fundamental sequence of elements from X possesses a weak limit in X.

Definition 4.2. Let X and Y be Banach spaces, $T : X \to Y$ be a linear bounded operator. The operator T is said to be weakly completely continuous if it maps a unit ball of X into a weakly relatively compact subset of Y.

Definition 4.3. We say that a set $M \subseteq L(\mathcal{D};\mathbb{R})$ has a property of absolutely continuous integral if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that the relation

$$\left|\iint\limits_E p(t,x)dtdx\right| < \varepsilon \quad \text{for every } p \in M$$

is true whenever a measurable set $E \subseteq \mathcal{D}$ is such that $\operatorname{mes} E < \delta$.

The following three lemmas can be found in [5].

Lemma 4.1 (Theorem IV.8.6)**.** *The space $L(\mathcal{D};\mathbb{R})$ is weakly complete.*

Lemma 4.2 (Theorem VI.7.6)**.** *A linear bounded operator mapping the space $C(\mathcal{D};\mathbb{R})$ into a weakly complete Banach space is weakly completely continuous.*

Lemma 4.3 (Theorem IV.8.11)**.** *If a set $M \subseteq L(\mathcal{D};\mathbb{R})$ is weakly relatively compact then it has a property of absolutely continuous integral.*

Proof of Proposition 4.1. Let $M \subseteq C(\mathcal{D};\mathbb{R})$ be a bounded set. We shall show that the set $T(M) = \{T(v) : v \in M\}$ is relatively compact in the space $C(\mathcal{D};\mathbb{R})$. According to Arzelà-Ascoli's lemma, it is sufficient to show that the set $T(M)$ is bounded and equicontinuous.

Boundedness. It is clear that

$$|T(v)(t,x)| \leq \iint\limits_{H(t,x)} |\ell(v)(s,\eta)| \, \mathrm{d}s \, \mathrm{d}\eta \leq \|\ell(v)\|_L \leq \|\ell\| \, \|v\|_C$$

for $(t,x) \in \mathcal{D}$ and every $v \in M$. Therefore, the set $T(M)$ is bounded in $C(\mathcal{D};\mathbb{R})$.

Equicontinuity. Let $\varepsilon > 0$ be arbitrary but fixed. Lemmas 4.1 and 4.2 yield that the operator ℓ is weakly completely continuous, that is, the set $\ell(M) = \{\ell(v) : v \in M\}$ is weakly relatively compact subset of $L(\mathcal{D};\mathbb{R})$. Therefore, Lemma 4.3 guarantees that there exists $\delta > 0$ such that the relation

$$\left|\iint\limits_E \ell(v)(t,x)dtdx\right| < \frac{\varepsilon}{4} \quad \text{for } v \in M \tag{4.3}$$

holds for every measurable set $E \subseteq \mathcal{D}$ satisfying $\operatorname{mes} E < \max\{b-a, d-c\}\delta$.

On the other hand, for $(t_1,x_1),(t_2,x_2) \in \mathcal{D}$ and $v \in M$, we have

$$|T(v)(t_2,x_2) - T(v)(t_1,x_1)| =$$

$$= \left|\iint\limits_{H(t_2,x_2)} \ell(v)(s,\eta) \, \mathrm{d}s \, \mathrm{d}\eta - \iint\limits_{H(t_1,x_1)} \ell(v)(s,\eta) \, \mathrm{d}s \, \mathrm{d}\eta\right| \leq \sum_{k=1}^{4} \left|\iint\limits_{E_k} \ell(v)(s,\eta) \, \mathrm{d}s \, \mathrm{d}\eta\right|,$$

where measurable sets $E_k \subseteq \mathcal{D}$ $(k = 1, \dots, 4)$ are such that $\operatorname{mes} E_k \leq (d-c)|t_2 - t_1|$ for $k = 1,2$ and $\operatorname{mes} E_k \leq (b-a)|x_2 - x_1|$ for $k = 3,4$. Hence, by virtue of (4.3), we get

$$|T(v)(t_2,x_2) - T(v)(t_1,x_1)| < \varepsilon$$

for $(t_1,x_1),(t_2,x_2)\in\mathcal{D},\ |t_2-t_1|+|x_2-x_1|<\delta,$ and $v\in M,$

i.e., the set $T(M)$ is equicontinuous in $C\big(\mathcal{D};\mathbb{R}\big)$. □

5. Fredholm Alternative

Throughout this section, we fix a function $h\in CD\big([a,b];[c,d]\big)$. Along with the problem (1.1), (1.3)–(1.5) we consider the corresponding homogeneous problem

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x}=\ell(u)(t,x), \tag{1.1$_0$}$$

$$u\,(t,h(t))=0 \quad \text{for } t\in[a,b], \tag{1.3$_0$}$$

$$u'_{|1}\,(t,h(t))=0 \quad \text{for a. e. } t\in[a,b], \tag{1.4$_0$}$$

$$u'_{|2}\,\big(h^{-1}(x),x\big)=0 \quad \text{for a. e. } x\in[c,d]. \tag{1.5$_0$}$$

Now we establish the main result of this section, namely, the statement on the Fredholmity of the problem (1.1), (1.3)–(1.5).

Theorem 5.1. *The problem* (1.1), (1.3)–(1.5) *has a unique solution for an arbitrary h–consistent triplet (g,φ,ψ) and every $q\in L\big(\mathcal{D};\mathbb{R}\big)$ if, and only if the corresponding homogeneous problem* (1.1_0), (1.3_0)–(1.5_0) *has only the trivial solution.*

Proof. Let u be a solution to the problem (1.1), (1.3)–(1.5). According to Lemma 3.3, u is a solution to the equation

$$v=T(v)+f \tag{5.1}$$

in the space $C\big(\mathcal{D};\mathbb{R}\big)$, where the operator T is defined by the formula (4.1),

$$f(t,x)\stackrel{\text{def}}{=}g(t)+\int\limits_{h(t)}^{x}\psi(\eta)\,\mathrm{d}\eta+\iint\limits_{H(t,x)}q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D}, \tag{5.2}$$

and the mapping H is given by the formula (2.1).

Conversely, if the triplet (g,φ,ψ) is h–consistent, $q\in L\big(\mathcal{D};\mathbb{R}\big)$, and $v\in C\big(\mathcal{D};\mathbb{R}\big)$ is a solution to the equation (5.1) with f given by (5.2) then, by virtue of Lemma 3.4, $v\in C^*\big(\mathcal{D};\mathbb{R}\big)$ and v is a solution to the problem (1.1), (1.3)–(1.5). Hence, the problem (1.1), (1.3)–(1.5) and the equation (5.1) are equivalent in this sense.

Note also that u is a solution to the homogeneous problem (1.1_0), (1.3_0)–(1.5_0) if, and only if u is a solution to the homogeneous equation

$$v=T(v) \tag{5.3}$$

in the space $C\big(\mathcal{D};\mathbb{R}\big)$.

According to Proposition 4.1, the operator T is completely continuous. It follows from the Riesz-Schauder theory that the equation (5.1) is uniquely solvable for every $f\in C\big(\mathcal{D};\mathbb{R}\big)$ if, and only if the homogeneous equation (5.3) has only the trivial solution. Therefore, the assertion of the theorem is true. □

Definition 5.1. Let the problem (1.1_0), (1.3_0)–(1.5_0) have only the trivial solution. An operator $\Omega : L(\mathcal{D};\mathbb{R}) \to C(\mathcal{D};\mathbb{R})$ which assigns to every $q \in L(\mathcal{D};\mathbb{R})$ the solution u to the problem (1.1), (1.3_0)–(1.5_0) is called the Cauchy operator of the problem (1.1_0), (1.3_0)–(1.5_0).

Remark 5.1. It is clear that the Cauchy operator is linear.

If the homogeneous problem (1.1_0), (1.3_0)–(1.5_0) has a nontrivial solution then, by virtue of Theorem 5.1, there exist a function q and a h–consistent triplet (g,φ,ψ) such that the problem (1.1), (1.3)–(1.5) has either no solution or infinitely many solutions. However, as it follows from the proof of Theorem 5.1, a stronger assertion can be shown in this case.

Proposition 5.1. *Let the problem* (1.1_0), (1.3_0)–(1.5_0) *have a nontrivial solution. Then, for an arbitrary h–consistent triplet* (g,φ,ψ), *there exists a function* $q \in L(\mathcal{D};\mathbb{R})$ *such that the problem* (1.1), (1.3)–(1.5) *has no solution.*

Proof. Let u_0 be a nontrivial solution to the problem (1.1_0), (1.3_0)–(1.5_0), and let (g,φ,ψ) be an h–consistent triplet.

It follows from the proof of Theorem 5.1 that u_0 is also a nontrivial solution to the homogeneous equation (5.3) in the space $C(\mathcal{D};\mathbb{R})$. Therefore, by the Riesz-Schauder theory, there exists $f \in C(\mathcal{D};\mathbb{R})$ such that the equation (5.1) has no solution.

Then the problem (1.1), (1.3)–(1.5) has no solution for $q \equiv \ell(z)$, where

$$z(t,x) = f(t,x) - g(t) - \int_{h(t)}^{x} \psi(\eta)\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}.$$

Indeed, if the problem indicated has a solution u then the function $u+z$ is a solution to the equation (5.1), which is a contradiction. □

6. Volterra Operators

The following definition gives the notion of a $[t_0,h]$–Volterra operator which is useful in the investigation of the Cauchy problem for the equation (1.1) (see, e.g., Theorem 7.2 below).

Definition 6.1. Let $t_0 \in [a,b]$ and $h \in CD([a,b];[c,d])$. We say that $\ell \in \mathcal{L}(\mathcal{D})$ is a $[t_0,h]$–Volterra operator if the relation

$$\ell(v)(t,x) = 0 \quad \text{for a. e. } (t,x) \in [a_0,b_0] \times [h(b_0),h(a_0)]$$

holds for an arbitrary interval $[a_0,b_0] \subseteq [a,b]$ and every function $v \in C(\mathcal{D};\mathbb{R})$ such that $t_0 \in [a_0,b_0]$ and

$$v(t,x) = 0 \quad \text{for } (t,x) \in [a_0,b_0] \times [h(b_0),h(a_0)].$$

Remark 6.1. If the operator ℓ in the equation (1.1) is a $[t_0,h]$–Volterra one then the Cauchy problem (1.1), (1.3)–(1.5) can be restricted to an arbitrary rectangle $[a_0,b_0] \times [h(b_0),h(a_0)] \subseteq \mathcal{D}$ containing the point $(t_0,h(t_0))$.

Let the operator $\ell \in \mathcal{L}(\mathcal{D})$ be defined by the formula

$$\ell(v)(t,x) \stackrel{\text{def}}{=} p(t,x)v\big(\tau(t,x),\mu(t,x)\big) \quad \text{for a. e. } (t,x) \in \mathcal{D}, \text{ all } v \in C\big(\mathcal{D};\mathbb{R}\big), \tag{6.1}$$

where $p \in L\big(\mathcal{D};\mathbb{R}\big)$ and $\tau : \mathcal{D} \to [a,b]$, $\mu : \mathcal{D} \to [c,d]$ are measurable functions.

The following statement can be immediately derived from Definition 6.1.

Proposition 6.1. *Let $t_0 \in [a,b]$ and $h \in CD\big([a,b];[c,d]\big)$. Then the operator ℓ defined by the formula* (6.1) *is a $[t_0,h]$–Volterra one provided that the conditions*

$$\begin{aligned} |p(t,x)| \min\{t_0, t, h^{-1}(x)\} &\le |p(t,x)|\tau(t,x) \le \\ &\le |p(t,x)| \max\{t_0, t, h^{-1}(x)\} \quad \text{for a. e. } (t,x) \in \mathcal{D} \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} |p(t,x)| \min\{h(t_0), h(t), x\} &\le |p(t,x)|\mu(t,x) \le \\ &\le |p(t,x)| \max\{h(t_0), h(t), x\} \quad \text{for a. e. } (t,x) \in \mathcal{D}. \end{aligned} \tag{6.3}$$

are satisfied.

The previous proposition yields

Corollary 6.1. *Let $t_0 \in [a,b]$ and $h \in CD\big([a,b];[c,d]\big)$. Assume that*

$$\big(\tau(t,x) - t_0\big)\big(\tau(t,x) - t\big) \le 0 \quad \text{for a. e. } (t,x) \in \mathcal{D}$$

and

$$\big(\mu(t,x) - h(t_0)\big)\big(\mu(t,x) - x\big) \le 0 \quad \text{for a. e. } (t,x) \in \mathcal{D}.$$

Then the operator ℓ defined by the formula (6.1) *is a $[t_0,h]$–Volterra one.*

7. Existence and Uniqueness Theorems

In this section, we fix a function $h \in CD\big([a,b];[c,d]\big)$. We give some efficient conditions guaranteeing the unique solvability of the problems (1.1), (1.3)–(1.5) and (1.1′), (1.3)–(1.5). We first formulate all the results, their proofs are postponed till Section 7.1 below.

Introduce the following notation.

Notation 7.1. Let $\ell \in \mathcal{L}(\mathcal{D})$. Define the operators $\vartheta_k : C\big(\mathcal{D};\mathbb{R}\big) \to C\big(\mathcal{D};\mathbb{R}\big)$, $k = 0,1,2,\dots$, by setting

$$\vartheta_0(v) \stackrel{\text{def}}{=} v, \quad \vartheta_k(v) \stackrel{\text{def}}{=} T\big(\vartheta_{k-1}(v)\big) \quad \text{for } v \in C\big(\mathcal{D};\mathbb{R}\big), \ k \in \mathbb{N}, \tag{7.1}$$

where the operator T is given by the formula (4.1).

Theorem 7.1. *Let there exist $m \in \mathbb{N}$ and $\alpha \in [0,1[$ such that the inequality*

$$\|\vartheta_m(u)\|_C \le \alpha \|u\|_C \tag{7.2}$$

is satisfied for every solution u of the homogeneous problem (1.1_0), (1.3_0)–(1.5_0). *Then the problem* (1.1), (1.3)–(1.5) *has a unique solution for an arbitrary h–consistent triplet (g,φ,ψ) and every $q \in L\big(\mathcal{D};\mathbb{R}\big)$.*

Remark 7.1. The assumption $\alpha \in [0,1[$ in the previous theorem cannot be replaced by the assumption $\alpha \in [0,1]$ (see Example 9.1).

Corollary 7.1. *Let there exist* $j \in \mathbb{N}$ *such that*

$$\max\left\{\int_a^b\int_c^{h(t)} p_j(t,x)\,\mathrm{d}x\,\mathrm{d}t,\ \int_a^b\int_{h(t)}^d p_j(t,x)\,\mathrm{d}x\,\mathrm{d}t\right\} < 1, \tag{7.3}$$

where $p_1 \equiv |p|$,

$$p_{k+1}(t,x) = |p(t,x)| \iint\limits_{H(\tau(t,x),\mu(t,x))} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for a. e. } (t,x)\in\mathcal{D},\ k\in\mathbb{N}, \tag{7.4}$$

and the mapping H *is defined by the formula* (2.1). *Then the problem* (1.1′), (1.3)–(1.5) *has a unique solution for an arbitrary* h*–consistent triplet* (g,φ,ψ) *and every* $q \in L(\mathcal{D};\mathbb{R})$.

Remark 7.2. Example 9.1 shows that the strict inequality (7.3) in Corollary 7.1 cannot be replaced by the nonstrict one.

Theorem 7.2. *Let the operator* ℓ *be a* $[t_0,h]$*–Volterra one for some* $t_0 \in [a,b]$. *Then the problem* (1.1), (1.3)–(1.5) *has a unique solution for an arbitrary* h*–consistent triplet* (g,φ,ψ) *and every* $q \in L(\mathcal{D};\mathbb{R})$.

Corollary 7.2. *Let there exist* $t_0 \in [a,b]$ *such that the conditions* (6.2) *and* (6.3) *are satisfied. Then the problem* (1.1′), (1.3)–(1.5) *has a unique solution for an arbitrary* h*–consistent triplet* (g,φ,ψ) *and every* $q \in L(\mathcal{D};\mathbb{R})$.

Corollary 7.3. *Let either*

$$\tau(t,x) \le t, \quad \mu(t,x) \ge x \quad \text{for a. e. } (t,x)\in\mathcal{D},$$

or

$$\tau(t,x) \ge t, \quad \mu(t,x) \le x \quad \text{for a. e. } (t,x)\in\mathcal{D}.$$

Then an arbitrary Cauchy problem subjected to the equation (1.1′) *has a unique solution.*

7.1. **Proofs.** Now we prove the statements formulated above.

Proof of Theorem 7.1. According to Theorem 5.1, it is sufficient to show that the homogeneous problem (1.1_0), (1.3_0)–(1.5_0) has only the trivial solution.

Let u be a solution to the problem (1.1_0), (1.3_0)–(1.5_0). Then, by virtue of Lemma 3.3, u satisfies

$$u(t,x) = \iint\limits_{H(t,x)} \ell(u)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = T(u)(t,x) = \vartheta_1(u)(t,x) \quad \text{for } (t,x)\in\mathcal{D}.$$

Therefore, we get

$$u(t,x) = T\big(\vartheta_1(u)\big)(t,x) = \vartheta_2(u)(t,x) \quad \text{for } (t,x)\in\mathcal{D},$$

and thus $u = \vartheta_k(u)$ for every $k\in\mathbb{N}$. Consequently, the relation (7.2) implies

$$\|u\|_C = \|\vartheta_m(u)\|_C \le \alpha\|u\|_C,$$

which guarantees $u \equiv 0$, because $\alpha \in [0,1[$. □

Proof of Theorem 7.1. It is clear that the equation (1.1′) is a particular case of (1.1) with ℓ given by the formula (6.1). It is not difficult to verify that

$$\big|\vartheta_k(v)(t,x)\big| \le \iint\limits_{H(t,x)} \big|p(s,\eta)\vartheta_{k-1}(v)\big(\tau(s,\eta),\mu(s,\eta)\big)\big|\,\mathrm{d}s\,\mathrm{d}\eta \le$$

$$\le \|v\|_C \iint\limits_{H(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N},\ v\in C\big(\mathcal{D};\mathbb{R}\big).$$

Since the functions p_k are nonnegative, we get, for any $k\in\mathbb{N}$, the relation

$$\max_{(t,x)\in\mathcal{D}}\left\{\iint\limits_{H(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right\} = \max\left\{\iint\limits_{H(a,c)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta,\ \iint\limits_{H(b,d)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right\}.$$

Consequently, the assumptions of Theorem 7.1 are satisfied with $m=j$ and

$$\alpha = \max\left\{\int_a^b\int_c^{h(t)} p_j(t,x)dxdt,\ \int_a^b\int_{h(t)}^d p_j(t,x)dxdt\right\}.$$

□

To prove Theorem 7.2 we need the following lemma.

Lemma 7.1. *Let $\ell\in\mathcal{L}(\mathcal{D})$ be a $[t_0,h]$–Volterra operator for some $t_0\in[a,b]$. Then*

$$\lim_{k\to+\infty}\|\vartheta_k\| = 0, \tag{7.5}$$

where the operators ϑ_k are defined by the formula (7.1).

Proof. Let $\varepsilon\in\,]0,1[\,$. According to Proposition 4.1, the operator ϑ_1 is completely continuous. Therefore, by virtue of Arzelà-Ascoli's lemma, there exists $\delta>0$ such that

$$\left|\iint\limits_{H(t_2,x_2)} \ell(w)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t_1,x_1)} \ell(w)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \le \varepsilon\,\|w\|_C \tag{7.6}$$

$$\text{for } (t_1,x_1),(t_2,x_2)\in\mathcal{D},\ |t_2-t_1|+|x_2-x_1|<\delta,\ w\in C\big(\mathcal{D};\mathbb{R}\big).$$

Since $h\in C\big(\mathcal{D};\mathbb{R}\big)$, there exists $\delta_0>0$ such that $\delta_0<\delta/2$, $\delta_0<\max\{t_0-a,b-t_0\}$, and

$$\big|h(t_2)-h(t_1)\big| < \frac{\delta}{2} \quad \text{for } t_1,t_2\in[a,b],\ |t_2-t_1|\le\delta_0. \tag{7.7}$$

Let

$$n = \max\left\{\mathrm{Ent}\left(\frac{t_0-a)}{\delta_0}\right),\mathrm{Ent}\left(\frac{b-t_0)}{\delta_0}\right)\right\}+1.$$

Choose $y_{n+1} \in [a, t_0]$ and $y_{n+2} \in [t_0, b]$ such that $y_{n+2} - y_{n+1} = \delta_0$, and put

$$y_k = y_{n+1} - (n+1-k)\,\frac{y_{n+1}-a}{n} \quad \text{for } k = 1, 2, \dots, n,$$

$$y_k = y_{n+2} + (k-n-2)\,\frac{b-y_{n+2}}{n} \quad \text{for } k = n+3, n+4, \dots, 2n+2,$$

and

$$\mathcal{D}_k = \big[y_{n+2-k}, y_{n+1+k}\big] \times \big[h(y_{n+1+k}), h(y_{n+2-k})\big] \quad \text{for } k = 1, 2, \dots, n+1.$$

Using the relation (7.7) and the definition of the numbers y_k, for any $j, r = 1, 2, \dots, 2n+1$, we get

$$\text{(7.8)} \quad |t_2 - t_1| + |x_2 - x_1| < \delta \quad \text{for } (t_1, x_1), (t_2, x_2) \in \big[y_j, y_{j+1}\big] \times \big[h(y_{r+1}), h(y_r)\big].$$

Having $w \in C\big(\mathcal{D}; \mathbb{R}\big)$, we denote

$$\|w\|_i = \|w\|_{C(\mathcal{D}_i;\mathbb{R})} \quad \text{for } i = 1, 2, \dots, n+1.$$

Let $v \in C\big(\mathcal{D}; \mathbb{R}\big)$ be arbitrary but fixed. We shall show that the relation

$$\text{(7.9)} \qquad \|\vartheta_k(v)\|_i \le \alpha_i(k)\varepsilon^k \|v\|_C \quad \text{for } k \in \mathbb{N}$$

holds for every $i = 1, 2, \dots, n+1$, where

$$\text{(7.10)} \qquad \alpha_i(k) = \alpha_i k^{i-1} \quad \text{for } k \in \mathbb{N},\ i = 1, 2, \dots, n+1,$$

$$\text{(7.11)} \qquad \alpha_1 = 1, \quad \alpha_{i+1} = i + 1 + i\alpha_i \quad \text{for } i = 1, 2, \dots, n.$$

By virtue of (7.6) and (7.8), it is easy to verify that, for any $w \in C\big(\mathcal{D}; \mathbb{R}\big)$ and $i = 1, 2, \dots, n+1$, we have

$$\text{(7.12)} \qquad \left| \iint\limits_{H(t,x)} \ell(w)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \le i\,\varepsilon\,\|w\|_C \quad \text{for } (t,x) \in \mathcal{D}_i\,.$$

We first note that the previous relation immediately implies

$$\text{(7.13)} \qquad \|\vartheta_1(v)\|_i \le i\,\varepsilon\,\|v\|_C \quad \text{for } i = 1, 2, \dots, n+1.$$

Furthermore, on account of (7.6), (7.8), and the fact that ℓ is a $[t_0, h]$–Volterra operator, we obtain

$$\big|\vartheta_{k+1}(v)(t,x)\big| = \left| \iint\limits_{H(t,x)} \ell\big(\vartheta_k(v)\big)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \le \varepsilon\,\|\vartheta_k(v)\|_1 \quad \text{for } (t,x) \in \mathcal{D}_1,\ k \in \mathbb{N}.$$

Hence, by virtue of (7.13), we get

$$\|\vartheta_k(v)\|_1 \le \varepsilon^k\,\|v\|_C \quad \text{for } k \in \mathbb{N},$$

and thus the relation (7.9) is true for $i = 1$.

Now suppose that the relation (7.9) holds for some $i \in \{1, 2, \dots, n\}$. We shall show that the relation indicated is also true for $i + 1$. With respect to (7.8), we obtain

$$\|\vartheta_{k+1}(v)\|_{i+1} = \max\left\{\left|\iint\limits_{H(t,x)} \ell\big(\vartheta_k(v)\big)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| : (t,x)\in\mathcal{D}_{i+1}\right\} =$$

$$= \left|\iint\limits_{H(t_k^*,x_k^*)} \ell\big(\vartheta_k(v)\big)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \le \left|\iint\limits_{H(\hat t_k,\hat x_k)} \ell\big(\vartheta_k(v)\big)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| +$$

$$+ \left|\iint\limits_{H(t_k^*,x_k^*)} \ell\big(\vartheta_k(v)\big)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(\hat t_k,\hat x_k)} \ell\big(\vartheta_k(v)\big)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \quad \text{for } k\in\mathbb{N},$$

where $(t_k^*,x_k^*)\in\mathcal{D}_{i+1}$, $(\hat t_k,\hat x_k)\in\mathcal{D}_i$, and $|t_k^*-\hat t_k|+|x_k^*-\hat x_k|<\delta$ for $k\in\mathbb{N}$. Therefore, on account of (7.6), (7.12), and the fact that ℓ is a $[t_0,h]$–Volterra operator, we get

$$\|\vartheta_{k+1}(v)\|_{i+1} \le \varepsilon\,\|\vartheta_k(v)\|_{i+1} + i\,\varepsilon\,\|\vartheta_k(v)\|_i \le \varepsilon\,\|\vartheta_k(v)\|_{i+1} + i\,\alpha_i(k)\,\varepsilon^{k+1}\,\|v\|_C$$

for $k\in\mathbb{N}$. Consequently,

$$\|\vartheta_{k+1}(v)\|_{i+1} \le \varepsilon\Big(\varepsilon\,\|\vartheta_{k-1}(v)\|_{i+1} + i\,\alpha_i(k-1)\,\varepsilon^k\,\|v\|_C\Big) + \\ + i\,\alpha_i(k)\,\varepsilon^{k+1}\,\|v\|_C \quad \text{for } k\in\mathbb{N}.$$

To continue this procedure, on account of (7.13), we obtain

$$\|\vartheta_{k+1}(v)\|_{i+1} \le \Big(i+1+i\,\big(\alpha_i(1)+\cdots+\alpha_i(k)\big)\Big)\,\varepsilon^{k+1}\,\|v\|_C \quad \text{for } k\in\mathbb{N}. \tag{7.14}$$

Using (7.10) and (7.11), it is easy to verify that

$$i+1+i\,\big(\alpha_i(1)+\cdots+\alpha_i(k)\big) = i+1+i\,\alpha_i\big(1^{i-1}+\cdots+k^{i-1}\big) \le \\ \le i+1+i\,\alpha_i\,k\,k^{i-1} = i+1+i\,\alpha_i\,k^i \le \\ \le (i+1+i\,\alpha_i)\,k^i = \alpha_{i+1}\,k^i \le \alpha_{i+1}\,(k+1).$$

Therefore, (7.13) and (7.14) imply

$$\|\vartheta_k(v)\|_{i+1} \le \alpha_{i+1}(k)\,\varepsilon^k\,\|v\|_C \quad \text{for } k\in\mathbb{N}.$$

Hence, by induction, we have proved that the relation (7.9) is true for every $i = 1,2,\dots,n+1$.

Now it is already clear that, for any $k\in\mathbb{N}$, the estimate

$$\|\vartheta_k(v)\|_C = \|\vartheta_k(v)\|_{n+1} \le \alpha_{n+1}\,k^n\,\varepsilon^k\,\|v\|_C \quad \text{for } v\in C\big(\mathcal{D};\mathbb{R}\big)$$

holds, and thus

$$\|\vartheta_k\| \le \alpha_{n+1}\,k^n\,\varepsilon^k \quad \text{for } k\in\mathbb{N}.$$

Since we suppose $\varepsilon\in\,]0,1[\,$, the last relation yields the validity of the condition (7.5). □

Proof of Theorem 7.2. According to Lemma 7.1, there exists $m_0 \in \mathbb{N}$ such that $\|\vartheta_{m_0}\| < 1$. Moreover, it is clear that

$$\|\vartheta_{m_0}(v)\|_C \leq \|\vartheta_{m_0}\| \, \|v\|_C \quad \text{for } v \in C(\mathcal{D};\mathbb{R}),$$

because the operator ϑ_{m_0} is bounded. Consequently, the assumptions of Theorem 7.1 are satisfied with $m = m_0$ and $\alpha = \|\vartheta_{m_0}\|$. □

Proof of Corollary 7.2. It is clear that the equation (1.1′) is a particular case of (1.1) with the operator ℓ given by the formula (6.1). By virtue of the assumptions (6.2) and (6.3), Proposition 6.1 guarantees that the operator ℓ is a $[t_0, h]$–Volterra one. Consequently, the validity of the corollary follows from Theorem 7.2. □

Proof of Corollary 7.3. It follows immediately from Corollary 7.2 with $t_0 = a$ and $t_0 = b$, respectively. □

8. Well-posedness

In this section, the well-posedness of the problems (1.1), (1.3)–(1.5) and (1.1′), (1.3)–(1.5) is studied. We first formulate all the results, their proofs are given in Section 8.1 below.

Throughout the section, we fix a function $h \in CD([a,b];[c,d])$ for which the set function H is given by the formula (2.1). On the graph of the function h we consider the Cauchy problem (1.3)–(1.5) for the equation (1.1). Recall that the triplet (g, φ, ψ) is supposed to be h–consistent.

For any $k \in \mathbb{N}$, along with the problem (1.1), (1.3)–(1.5) we consider the perturbed problem

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell_k(u)(t,x) + q_k(t,x), \tag{1.1$_k$}$$

$$u\,(t, h_k(t)) = g_k(t) \quad \text{for } t \in [a,b], \tag{1.3$_k$}$$

$$u'_{|1}\,(t, h_k(t)) = \varphi_k(t) \quad \text{for a. e. } t \in [a,b], \tag{1.4$_k$}$$

$$u'_{|2}\,\big(h_k^{-1}(x), x\big) = \psi_k(x) \quad \text{for a. e. } x \in [c,d], \tag{1.5$_k$}$$

where $\ell_k \in \mathcal{L}(\mathcal{D})$, $q_k \in L(\mathcal{D};\mathbb{R})$, $h_k \in CD([a,b];[c,d])$, and $g_k \in C([a,b];\mathbb{R})$, $\varphi_k \in L([a,b];\mathbb{R})$, and $\psi_k \in L([c,d];\mathbb{R})$ are such that the triplet (g_k, φ_k, ψ_k) is h_k–consistent.

Analogously to Notation 2.1, for given functions h_k, we put

$$\begin{aligned} H_k(t,x) \overset{\text{def}}{=} \Big\{ &(s,\eta) \in \mathbb{R}^2 : \min\{h_k^{-1}(x), t\} \leq s \leq \max\{h_k^{-1}(x), t\}, \\ &\min\{h_k(s), x\} \leq \eta \leq \max\{h_k(s), x\} \Big\} \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \end{aligned} \tag{8.1}$$

It is clear that, for any $(t,x) \in \mathcal{D}$ and $k \in \mathbb{N}$, the set $H_k(t,x)$ is a measurable subset of $\mathcal{D}$.

Notation 8.1. Let $p \in \mathcal{L}(\mathcal{D})$ and $\gamma \in CD\big([a,b];[c,d]\big)$. Denote by $M(p,\gamma)$ the set of functions $y \in C^*\big(\mathcal{D};\mathbb{R}\big)$ admitting the representation

$$y(t,x) = \int\limits_{\gamma(t)}^{x} \int\limits_{\gamma^{-1}(\eta)}^{t} p(z)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},$$

where $z \in C\big(\mathcal{D};\mathbb{R}\big)$ and $\|z\|_C = 1$.

Theorem 8.1. *Let the problem* (1.1), (1.3)–(1.5) *have a unique solution* u *and let*

$$\lim_{k\to+\infty} \lambda_k = 0, \tag{8.2}$$

where

$$\lambda_k = \sup_{\substack{(t,x)\in\mathcal{D}\\ y\in M(\ell_k,h_k)}} \left\{ \left| \iint\limits_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \right\} \tag{8.3}$$

for $k \in \mathbb{N}$. *Let, moreover,*

$$\lim_{k\to+\infty} \varrho_k \left[\iint\limits_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right] = 0 \tag{8.4}$$

uniformly on $\mathcal{D}$ *for every* $y \in C^*\big(\mathcal{D};\mathbb{R}\big)$,

$$\lim_{k\to+\infty} \varrho_k \left[\iint\limits_{H_k(t,x)} q_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right] = 0 \quad \textit{uniformly on } \mathcal{D}, \tag{8.5}$$

$$\lim_{k\to+\infty} \varrho_k \int\limits_a^t \big[\varphi_k(s) - \varphi(s)\big]\,\mathrm{d}s = 0 \quad \textit{uniformly on } [a,b], \tag{8.6}$$

$$\lim_{k\to+\infty} \varrho_k \int\limits_c^x \big[\psi_k(\eta) - \psi(\eta)\big]\,\mathrm{d}\eta = 0 \quad \textit{uniformly on } [c,d], \tag{8.7}$$

and

$$\lim_{k\to+\infty} \varrho_k \|g_k - g\|_C = 0, \tag{8.8}$$

where

$$\varrho_k = 1 + \|\ell_k\| \quad \textit{for } k \in \mathbb{N}. \tag{8.9}$$

Then there exists $k_0 \in \mathbb{N}$ *such that, for every* $k > k_0$, *the problem* (1.1_k), (1.3_k)–(1.5_k) *has a unique solution* u_k *and*

$$\lim_{k\to+\infty} \|u_k - u\|_C = 0. \tag{8.10}$$

If we suppose that the operators ℓ_k are "uniformly bounded" in the sense of the relation (8.11) then we obtain the following statement.

Corollary 8.1. *Let the problem* (1.1), (1.3)–(1.5) *have a unique solution* u, *there exist a function* $\omega \in L(\mathcal{D};\mathbb{R}_+)$ *such that*

$$|\ell_k(y)(t,x)| \le \omega(t,x)\|y\|_C \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ and all } y \in C\big(\mathcal{D};\mathbb{R}\big),\ k \in \mathbb{N}, \tag{8.11}$$

and let

$$\lim_{k\to+\infty} \iint\limits_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = \iint\limits_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{uniformly on } \mathcal{D} \text{ for every } y \in C^*\big(\mathcal{D};\mathbb{R}\big). \tag{8.12}$$

Let, moreover,

$$\lim_{k\to+\infty} \iint\limits_{H_k(t,x)} q_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = \iint\limits_{H(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{uniformly on } \mathcal{D}, \tag{8.13}$$

$$\lim_{k\to+\infty} \int_a^t \big[\varphi_k(s) - \varphi(s)\big]\,\mathrm{d}s = 0 \quad \text{uniformly on } [a,b], \tag{8.14}$$

$$\lim_{k\to+\infty} \int_c^x \big[\psi_k(\eta) - \psi(\eta)\big]\,\mathrm{d}\eta = 0 \quad \text{uniformly on } [c,d], \tag{8.15}$$

and

$$\lim_{k\to+\infty} \|g_k - g\|_C = 0. \tag{8.16}$$

Then the conclusion of Theorem 8.1 is true.

Remark 8.1. The assumption (8.11) in the previous corollary is essential and cannot be omitted (see Example 9.2).

Corollary 8.2. *Let the problem* (1.1), (1.3)–(1.5) *have a unique solution* u *and there exist a function* $\omega \in L(\mathcal{D};\mathbb{R}_+)$ *such that the relation* (8.11) *holds. Let, moreover, the conditions* (8.13), (8.15), *and* (8.16) *be satisfied,*

$$\lim_{k\to+\infty} \iint\limits_{H(t,x)} \big[\ell_k(y)(s,\eta) - \ell(y)(s,\eta)\big]\,\mathrm{d}s\,\mathrm{d}\eta = 0 \quad \text{uniformly on } \mathcal{D} \text{ for every } y \in C^*\big(\mathcal{D};\mathbb{R}\big), \tag{8.17}$$

and

$$\lim_{k\to+\infty} \|h_k - h\|_C = 0. \tag{8.18}$$

Then the conclusion of Theorem 8.1 is true.

Corollary 8.2 immediately yields

Corollary 8.3. *Let the homogeneous problem* (1.1_0), (1.3_0)–(1.5_0) *have only the trivial solution. Then the Cauchy operator*[2] *of the problem* (1.1_0), (1.3_0)–(1.5_0) *is continuous.*

Now we give a statement on the well-posedness of the problem $(1.1')$, (1.3)–(1.5). For any $k \in \mathbb{N}$, along with the equation $(1.1')$ we consider the perturbed equation

$$(1.1'_k) \qquad \frac{\partial^2 u(t,x)}{\partial t\, \partial x} = p_k(t,x) u\big(\tau_k(t,x), \mu_k(t,x)\big) + q_k(t,x),$$

where $p_k, q_k \in L\big(\mathcal{D}; \mathbb{R}\big)$ and $\tau_k : \mathcal{D} \to [a,b]$, $\mu_k : \mathcal{D} \to [c,d]$ are measurable functions.

Corollary 8.4. *Let the problem* $(1.1')$, (1.3)–(1.5) *have a unique solution* u*, there exist a function* $\omega \in L(\mathcal{D}; \mathbb{R}_+)$ *such that*

$$(8.19) \qquad |p_k(t,x)| \le \omega(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D},\ k \in \mathbb{N},$$

and let

$$(8.20) \qquad \lim_{k \to +\infty} \iint\limits_{H(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big]\, \mathrm{d}s\, \mathrm{d}\eta = 0 \quad \text{uniformly on } \mathcal{D}.$$

Let, moreover, the conditions (8.13), (8.15), (8.16), *and* (8.18) *be satisfied, and*

$$(8.21) \qquad \lim_{k \to +\infty} \operatorname{ess\,sup} \Big\{ \big|\tau_k(t,x) - \tau(t,x)\big| : (t,x) \in \mathcal{D} \Big\} = 0,$$

$$(8.22) \qquad \lim_{k \to +\infty} \operatorname{ess\,sup} \Big\{ \big|\mu_k(t,x) - \mu(t,x)\big| : (t,x) \in \mathcal{D} \Big\} = 0.$$

Then there exists $k_0 \in \mathbb{N}$ *such that, for every* $k > k_0$*, the problem* $(1.1'_k)$, (1.3_k)–(1.5_k) *has a unique solution* u_k *and the relation* (8.10) *is true.*

Remark 8.2. The assumption (8.19) in the previous theorem is essential and cannot be omitted (see Example 9.2).

Finally, we consider the hyperbolic equation without argument deviations (1.2) in which $p, q \in L\big(\mathcal{D}; \mathbb{R}\big)$. For any $k \in \mathbb{N}$, along with the equation (1.2) we consider the perturbed equation

$$(1.2_k) \qquad u_{tx} = p_k(t,x) u + q_k(t,x)$$

where $p_k, q_k \in L\big(\mathcal{D}; \mathbb{R}\big)$.

The following statement can be derived from Theorem 8.1.

Corollary 8.5. *Let the problem* (1.2)–(1.5) *have a unique solution* u*. Let, moreover, the conditions* (8.5)–(8.8) *be satisfied,*

$$(8.23) \qquad \lim_{k \to +\infty} \varrho_k \left[\iint\limits_{H_k(t,x)} p_k(s,\eta)\, \mathrm{d}s\, \mathrm{d}\eta - \iint\limits_{H(t,x)} p(s,\eta)\, \mathrm{d}s\, \mathrm{d}\eta \right] = 0 \quad \text{uniformly on } \mathcal{D},$$

[2]The notion of the Cauchy operator is given in Definition 5.1.

and

$$(8.24)\qquad \lim_{k\to+\infty} \varrho_k \iint\limits_{H(t,x)\div H_k(t,x)} |p(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta = 0 \quad \textit{uniformly on } \mathcal{D},$$

where

$$(8.25)\qquad \varrho_k = 1 + \iint\limits_{\mathcal{D}} |p_k(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \quad \textit{for } k \in \mathbb{N}.$$

Then there exists $k_0 \in \mathbb{N}$ such that, for every $k > k_0$, the problem (1.2$_k$)–(1.5$_k$) *has a unique solution u_k and the relation* (8.10) *is true.*

Remark 8.3. Note that if the relation $\sup\{\|p_k\|_L : k \in \mathbb{N}\} < +\infty$ holds then the assumption (8.24) of the previous corollary is guaranteed, e.g., by the condition (8.18) (see Lemma 8.2 below).

Corollary 8.5 yields

Corollary 8.6. *Let the problem* (1.2)–(1.5) *have a unique solution u. Let, moreover, the conditions* (8.15), (8.16), *and* (8.18) *be satisfied,*

$$(8.26)\qquad \lim_{k\to+\infty} \|p_k - p\|_L = 0,$$

and

$$(8.27)\qquad \lim_{k\to+\infty} \|q_k - q\|_L = 0.$$

Then the conclusion of Corollary 8.5 is true.

8.1. **Proofs.** In order to prove Theorem 8.1, we need the following lemma.

Lemma 8.1. *Let the problem* (1.1$_0$), (1.3$_0$)–(1.5$_0$) *have only the trivial solution and let the condition* (8.2) *hold, where the numbers λ_k are defined by the formula* (8.3). *Then, for an arbitrary $z \in C^*\big(\mathcal{D};\mathbb{R}\big)$, there exist $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that*

$$(8.28)\qquad \|y - z\|_C \le r_0(1 + \|\ell_k\|)\Big[\|\Delta_k(y) - \Delta(z)\|_C + \|\Gamma_k(y,z)\|_C\Big] \quad \text{for } k > k_0,\ y \in C^*\big(\mathcal{D};\mathbb{R}\big),$$

where

$$(8.29)\qquad \Delta_k(v)(t,x) \overset{\text{def}}{=} v\,(t, h_k(t)) + \int\limits_{h_k(t)}^{x} v'_{|2}\left(h_k^{-1}(\eta),\eta\right)\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ v \in C^*\big(\mathcal{D};\mathbb{R}\big),\ k \in \mathbb{N},$$

$$(8.30)\qquad \Delta(v)(t,x) \overset{\text{def}}{=} v\,(t, h(t)) + \int\limits_{h(t)}^{x} v'_{|2}\left(h^{-1}(\eta),\eta\right)\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ v \in C^*\big(\mathcal{D};\mathbb{R}\big),$$

and

$$\Gamma_k(v,w)(t,x) \overset{\text{def}}{=} \iint\limits_{H_k(t,x)} \big[v_{s\eta}(s,\eta) - \ell_k(v-w)(s,\eta)\big]\,\mathrm{d}s\,\mathrm{d}\eta - \tag{8.31}$$

$$- \iint\limits_{G(t,x)} w_{s\eta}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ v,w\in C^*\big(\mathcal{D};\mathbb{R}\big),\ k\in\mathbb{N}.$$

Proof. Let the operators $T, T_k : C\big(\mathcal{D};\mathbb{R}\big) \to C\big(\mathcal{D};\mathbb{R}\big)$ be defined by the formulae (4.1) and

$$T_k(v)(t,x) \overset{\text{def}}{=} \iint\limits_{H_k(t,x)} \ell_k(v)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ v\in C\big(\mathcal{D};\mathbb{R}\big),\ k\in\mathbb{N}.$$

Obviously,

$$\|T_k(y)\|_C \le \|\ell_k(y)\|_L \le \|\ell_k\|\,\|y\|_C \quad \text{for } y\in C\big(\mathcal{D};\mathbb{R}\big),\ k\in\mathbb{N}.$$

Therefore, the operators T_k $(k\in\mathbb{N})$ are linear bounded ones, and the relation

$$\|T_k\| \le \|\ell_k\| \quad \text{for } k\in\mathbb{N} \tag{8.32}$$

holds. Moreover, the condition (8.2) with λ_k given by (8.3) can be rewritten in the form

$$\sup\Big\{\|T_k(y) - T(y)\|_C : y\in M(\ell_k,h_k)\Big\} \to 0 \quad \text{as } k\to+\infty. \tag{8.33}$$

Assume that, on the contrary, the assertion of the lemma is not true. Then there exist $z\in C^*\big(\mathcal{D};\mathbb{R}\big)$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers, and a sequence $\{y_m\}_{m=1}^{+\infty}$ of functions from $C^*\big(\mathcal{D};\mathbb{R}\big)$ such that, for every $m\in\mathbb{N}$, the relation

$$\|y_m - z\|_C > m(1+\|\ell_{k_m}\|)\Big[\|\Delta_{k_m}(y_m) - \Delta(z)\|_C + \|\Gamma_{k_m}(y_m,z)\|_C\Big] \tag{8.34}$$

holds. For any $m\in\mathbb{N}$ and $(t,x)\in\mathcal{D}$, we put

$$z_m(t,x) = \frac{y_m(t,x) - z(t,x)}{\|y_m - z\|_C}\,, \tag{8.35}$$

$$v_m(t,x) = \frac{1}{\|y_m - z\|_C}\Big[\Delta_{k_m}(y_m)(t,x) - \Delta(z)(t,x) + \Gamma_{k_m}(y_m,z)(t,x)\Big], \tag{8.36}$$

$$z_{0,m}(t,x) = z_m(t,x) - v_m(t,x), \tag{8.37}$$

$$w_m(t,x) = T_{k_m}(z_{0,m})(t,x) - T(z_{0,m})(t,x) + T_{k_m}(v_m)(t,x). \tag{8.38}$$

Obviously,

$$\|z_m\|_C = 1 \quad \text{for } m\in\mathbb{N}. \tag{8.39}$$

Using (8.29)–(8.31) in the relation (8.36) and, by virtue of Lemma 3.3, we get

$$z_{0,m}(t,x) = T_{k_m}(z_m)(t,x) \quad \text{for } (t,x)\in\mathcal{D},\ m\in\mathbb{N}, \tag{8.40}$$

and thus

$$z_{0,m}(t,x) = T(z_{0,m})(t,x) + w_m(t,x) \quad \text{for } (t,x) \in \mathcal{D},\ m \in \mathbb{N}. \tag{8.41}$$

Moreover, it follows from (8.34) and (8.36) that

$$\|v_m\|_C \le \frac{\|\Delta_{k_m}(y_m) - \Delta(z)\|_C + \|\Gamma_{k_m}(y_m, z)\|_C}{\|y_m - z\|_C} < \frac{1}{m(1 + \|\ell_{k_m}\|)} \tag{8.42}$$

for $m \in \mathbb{N}$. Now the relations (8.32) and (8.42) yield

$$\|T_{k_m}(v_m)\|_C \le \|T_{k_m}\|\,\|v_m\|_C \le \frac{\|\ell_{k_m}\|}{m(1 + \|\ell_{k_m}\|)} < \frac{1}{m} \quad \text{for } m \in \mathbb{N}. \tag{8.43}$$

Note that the expression (8.40) and the condition (8.39) guarantee the validity of the inclusion $z_{0,m} \in M(\ell_{k_m}, h_{k_m})$ for $m \in \mathbb{N}$, and thus, in view of (8.33), we obtain

$$\lim_{m \to +\infty} \|T_{k_m}(z_{0,m}) - T(z_{0,m})\|_C = 0. \tag{8.44}$$

According to (8.43) and (8.44), it follows from (8.38) that

$$\lim_{m \to +\infty} \|w_m\|_C = 0, \tag{8.45}$$

and, by virtue of (8.39) and (8.42), the equality (8.37) implies $\|z_{0,m}\|_C < 2$ for $m \in \mathbb{N}$. Since the sequence $\{\|z_{0,m}\|_C\}_{m=1}^{+\infty}$ is bounded and the operator T is completely continuous (see Proposition 4.1), there exists a subsequence of $\{T(z_{0,m})\}_{m=1}^{+\infty}$ which is convergent. We can assume without loss of generality that the sequence $\{T(z_{0,m})\}_{m=1}^{+\infty}$ is convergent, i.e., there exists $z_0 \in C\big(\mathcal{D}; \mathbb{R}\big)$ such that

$$\lim_{m \to +\infty} \|T(z_{0,m}) - z_0\|_C = 0.$$

Then it is clear that

$$\lim_{m \to +\infty} \|z_{0,m} - z_0\|_C = 0, \tag{8.46}$$

because the functions $z_{0,m}$ admit the representation (8.41) and the relation (8.45) holds. However, the estimate (8.42) is true for v_m and thus, the equality (8.37) yields

$$\lim_{m \to +\infty} \|z_m - z_0\|_C = 0,$$

which, together with (8.39), guarantees $\|z_0\|_C = 1$. Since the operator T is continuous and the conditions (8.45) and (8.46) are fulfilled, the relation (8.41) yields $z_0 = T(z_0)$. Consequently, by virtue of Lemma 3.3, $z_0 \in C^*\big(\mathcal{D}; \mathbb{R}\big)$ and z_0 is a nontrivial solution to the homogeneous problem (1.1_0), (1.3_0)–(1.5_0), which is a contradiction. □

Proof of Theorem 8.1. Since the problem (1.1), (1.3)–(1.5) has a unique solution the problem (1.1_0), (1.3_0)–(1.5_0) has only the trivial solution. Therefore, the assumptions of Lemma 8.1 are satisfied, and thus there exist $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|y\|_C \le r_0(1 + \|\ell_k\|)\Big[\|\Delta_k(y)\|_C + \|\Gamma_k(y, 0)\|_C\Big] \quad \text{for } k > k_0,\ y \in C^*\big(\mathcal{D}; \mathbb{R}\big) \tag{8.47}$$

and

$$\|y-u\|_C \le r_0(1+\|\ell_k\|)\Big[\|\Delta_k(y)-\Delta(u)\|_C+\|\Gamma_k(y,u)\|_C\Big] \quad \text{for } k>k_0,\ y\in C^*\big(\mathcal{D};\mathbb{R}\big), \tag{8.48}$$

where the operators Δ_k, Δ, and Γ_k are given by the formulae (8.29)–(8.31), respectively.

If, for some $k\in\mathbb{N}$, u_0 is a solution to the problem

$$\begin{aligned} \frac{\partial^2 u(t,x)}{\partial t\,\partial x} &= \ell_k(u)(t,x),\\ u\left(t,h_k(t)\right) &= 0 \quad \text{for } t\in[a,b],\\ u'_{|1}\left(t,h_k(t)\right) &= 0 \quad \text{for a.e. } t\in[a,b],\\ u'_{|2}\left(h_k^{-1}(x),x\right) &= 0 \quad \text{for a.e. } x\in[c,d] \end{aligned} \tag{8.49}$$

then $\Delta_k(u_0)\equiv 0$ and $\Gamma_k(u_0,0)\equiv 0$. Therefore, the relation (8.47) guarantees that, for every $k>k_0$, the homogeneous problem (8.49) has only the trivial solution. Hence, for every $k>k_0$, the problem (1.1_k), (1.3_k)–(1.5_k) has a unique solution u_k (see Theorem 5.1). Then we get

$$\Delta_k(u_k)(t,x)=g_k(t)+\int_{h_k(t)}^{x}\psi_k(\eta)\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k>k_0,$$

$$\Delta(u)(t,x)=g(t)+\int_{h(t)}^{x}\psi(\eta)\,\mathrm{d}\eta \qquad \text{for } (t,x)\in\mathcal{D},$$

and

$$\begin{aligned}\Gamma_k(u,u_k)(t,x)=&\iint\limits_{H_k(t,x)}\ell_k(u)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta-\iint\limits_{H(t,x)}\ell(u)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta+\\ &+\iint\limits_{H_k(t,x)}q_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta-\iint\limits_{H(t,x)}q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k>k_0.\end{aligned}$$

Note that the assumptions (8.6) and (8.8) yield

$$\lim_{k\to+\infty}(1+\|\ell_k\|)\left[\int_c^{h_k(t)}\psi_k(\eta)\,\mathrm{d}\eta-\int_c^{h(t)}\psi(\eta)\,\mathrm{d}\eta\right]=0 \quad \text{uniformly on } [a,b]. \tag{8.50}$$

Indeed, since we suppose that the triplets (g,φ,ψ) and (g_k,φ_k,ψ_k) are h–consistent and h_k–consistent, respectively, Proposition 3.1 implies

$$g(t)+\int_{h(t)}^{c}\psi(\eta)\,\mathrm{d}\eta=g(b)+\int_b^t\varphi(s)\,\mathrm{d}s \qquad \text{for } t\in[a,b],$$

$$g_k(t) + \int\limits_{h_k(t)}^{c} \psi_k(\eta)\,\mathrm{d}\eta = g_k(b) + \int\limits_{b}^{t} \varphi_k(s)\,\mathrm{d}s \quad \text{for } t \in [a,b],\ k \in \mathbb{N}.$$

Hence, the relations (8.6) and (8.8) yield the condition (8.50).

Now, using (8.4), (8.5), (8.7), (8.8), and (8.50), we get

$$\lim_{k\to+\infty} (1 + \|\ell_k\|)\Big[\|\Delta_k(u_k) - \Delta(u)\|_C + \|\Gamma_k(u_k,u)\|_C\Big] = 0. \tag{8.51}$$

On the other hand, it follows from (8.48) that

$$\|u_k - u\|_C \le r_0(1 + \|\ell_k\|)\Big[\|\Delta_k(u_k) - \Delta(u)\|_C + \|\Gamma_k(u_k,u)\|_C\Big] \quad \text{for } k > k_0 \tag{8.52}$$

and thus, by virtue of (8.51), the condition (8.10) holds. □

Proof of Corollary 8.1. We shall show that the assumptions of Theorem 8.1 are satisfied. Indeed, the relation (8.11) yields $\|\ell_k\| \le \|\omega\|_L$ for $k \in \mathbb{N}$. Therefore, it is clear that, by virtue of the relations (8.12)–(8.16), the assumptions (8.4)–(8.8) of Theorem 8.1 are fulfilled. It remains to show that the condition (8.2) holds, where the numbers λ_k are given by the formula (8.3).

Assume that, on the contrary, the condition (8.2) does not hold. Then there exist $\varepsilon_0 > 0$, an increasing sequence $\{k_m\}_{m=1}^{+\infty}$ of natural numbers, and a sequence $\{y_m\}_{m=1}^{+\infty}$ such that

$$y_m \in M\big(\ell_{k_m}, h_{k_m}\big) \quad \text{for } m \in \mathbb{N} \tag{8.53}$$

and

$$\max_{(t,x)\in\mathcal{D}} \left\{ \left| \iint\limits_{H_{k_m}(t,x)} \ell_{k_m}(y_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \right\} \ge \varepsilon_0 \quad \text{for } m \in \mathbb{N}. \tag{8.54}$$

In view of (8.53) and Notation 8.1, we get

$$y_m(t,x) = \iint\limits_{H_{k_m}(t,x)} \ell_{k_m}(z_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ m \in \mathbb{N},$$

where $z_m \in C\big(\mathcal{D};\mathbb{R}\big)$ and $\|z_m\|_C = 1$ for $m \in \mathbb{N}$. Since we suppose that the operators ℓ_k are uniformly bounded in the sense of condition (8.11), we obtain $\|y_m\|_C \le \|\omega\|_L$ for $m \in \mathbb{N}$, and thus the sequence $\{y_m\}_{m=1}^{+\infty}$ is bounded in the space $C\big(\mathcal{D};\mathbb{R}\big)$. We show that the sequence indicated is also equicontinuous. Let $\varepsilon > 0$ be arbitrary but fixed. Since the function ω is integrable on $\mathcal{D}$, there exists $\delta > 0$ such that the relation

$$\iint\limits_{E} \omega(t,x)dtdx < \frac{\varepsilon}{2} \tag{8.55}$$

holds for every measurable set $E \subseteq \mathcal{D}$ satisfying $\operatorname{mes} E < \max\{b-a, d-c\}\delta$. Using the condition (8.11), for any $(t_1, x_1), (t_2, x_2) \in \mathcal{D}$ and $m \in \mathbb{N}$, we get

$$\left| \iint\limits_{H(t_2,x_2)} \ell_{k_m}(z_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t_1,x_1)} \ell_{k_m}(z_m)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \le \sum_{k=1}^{2} \iint\limits_{E_k} \omega(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta,$$

where the measurable sets $E_1, E_2 \subseteq \mathcal{D}$ are such that $\operatorname{mes} E_1 \le (d-c)|t_2 - t_1|$ and $\operatorname{mes} E_2 \le (b-a)|x_2 - x_1|$. Therefore, by virtue of (8.55), we have

$$|T(v)(t_2,x_2) - T(v)(t_1,x_1)| < \varepsilon \\ \text{for } (t_1,x_1),(t_2,x_2) \in \mathcal{D},\ |t_2-t_1| + |x_2-x_1| < \delta,\ m \in \mathbb{N}.$$

Consequently, the sequence $\{y_m\}_{m=1}^{+\infty}$ is equicontinuous in the space $C(\mathcal{D};\mathbb{R})$. Therefore, according to Arzelà-Ascoli's lemma, we can assume without loss of generality that the sequence indicated is convergent. Hence, there exists $p_0 \in \mathbb{N}$ such that

$$\|y_m - y_{p_0}\|_C < \frac{\varepsilon_0}{2(\|\omega\|_L + \|\ell\| + 1)} \quad \text{for } m \ge p_0. \tag{8.56}$$

Since $y_{p_0} \in C^*(\mathcal{D};\mathbb{R})$ and the relation (8.12) holds, there exists $p_1 \in \mathbb{N}$ such that

$$\max_{(t,x)\in\mathcal{D}} \left\{ \left| \iint\limits_{H_k(t,x)} \ell_k(y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \right\} < \frac{\varepsilon_0}{2} \\ \text{for } k \ge p_1. \tag{8.57}$$

Now we choose a number $M \in \mathbb{N}$ satisfying $M \ge p_0$ and $k_M \ge p_1$. It is clear that

$$\left| \iint\limits_{H_{k_M}(t,x)} \ell_{k_M}(y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \le \\ \le \left| \iint\limits_{H_{k_M}(t,x)} \ell_{k_M}(y_M - y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| + \left| \iint\limits_{H(t,x)} \ell(y_{p_0} - y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| + \\ + \left| \iint\limits_{H_{k_M}(t,x)} \ell_{k_M}(y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y_{p_0})(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \quad \text{for } (t,x) \in \mathcal{D}.$$

Therefore, by virtue of the conditions (8.11), (8.56), and (8.57), the last relation yields

$$\max_{(t,x)\in\mathcal{D}} \left\{ \left| \iint\limits_{H_{k_M}(t,x)} \ell_{k_M}(y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y_M)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \right\} \le \tag{8.58}$$

$$\leq \|\omega\|_L \|y_M - y_{p_0}\|_C + \frac{\varepsilon_0}{2} + \|\ell\| \|y_{p_0} - y_M\|_C < \varepsilon_0,$$

which contradicts the condition (8.54).

The contradiction obtained proves the validity of the condition (8.2), and thus all the assumptions of Theorem 8.1 are satisfied. □

To prove Corollary 8.2 we need the following lemma.

Lemma 8.2. *Let the condition* (8.18) *hold and* $\{\sigma_k\}_{k=1}^{+\infty}$ *be a sequence of functions from* $L(\mathcal{D};\mathbb{R})$ *such that*

$$|\sigma_k(t,x)| \leq \omega(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D},\ k \in \mathbb{N}, \tag{8.59}$$

where $\omega \in L(\mathcal{D};\mathbb{R}_+)$. *Then*

$$\lim_{k\to+\infty} \iint\limits_{H(t,x)\div H_k(t,x)} \big|\sigma_k(s,\eta)\big|\,\mathrm{d}s\,\mathrm{d}\eta = 0 \quad \textit{uniformly on } \mathcal{D} \tag{8.60}$$

and

$$\lim_{k\to+\infty} \left[\,\iint\limits_{H_k(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right] = 0 \quad \textit{uniformly on } \mathcal{D}. \tag{8.61}$$

Proof. Let $\varepsilon > 0$ be arbitrary but fixed. Then there exists $\delta > 0$ such that the relation

$$\iint\limits_{E} \omega(s,\eta)\,\mathrm{d}s\eta < \varepsilon \tag{8.62}$$

is true for every measurable set $E \subseteq \mathcal{D}$ with the property $\operatorname{mes} E < 2(b-a)\delta$. Put $P = \{(t,x) \in \mathcal{D} : |x - h(t)| \leq \delta\}$. It is easy to verify that

$$\operatorname{mes} P < 2(b-a)\delta. \tag{8.63}$$

In view of the condition (8.18), there exists $k_0 \in \mathbb{N}$ such that

$$|h_k(t) - h(t)| < \delta \quad \text{for } t \in [a,b],\ k \geq k_0,$$

and thus

$$\begin{gathered}\big(H(t,x) \setminus P\big) \setminus H_k(t,x) = \emptyset, \quad \big(H_k(t,x) \setminus P\big) \setminus H(t,x) = \emptyset \\ \text{for } (t,x) \in \mathcal{D},\ k \geq k_0.\end{gathered} \tag{8.64}$$

Obviously, for $(t,x) \in \mathcal{D}$ and $k \in \mathbb{N}$, we get

$$\begin{aligned} H(t,x) \div H_k(t,x) &= H(t,x) \setminus H_k(t,x) \cup H_k(t,x) \setminus H(t,x) = \\ &= \Big[\big(H(t,x) \setminus P\big) \setminus H_k(t,x)\Big] \cup \Big[\big(H(t,x) \cap P\big) \setminus H_k(t,x)\Big] \cup \\ &\quad \cup \Big[\big(H_k(t,x) \setminus P\big) \setminus H(t,x)\Big] \cup \Big[\big(H_k(t,x) \cap P\big) \setminus H(t,x)\Big]. \end{aligned}$$

Therefore, by virtue of (8.59) and (8.64), the last relation yields

$$\iint\limits_{H(t,x)\div H_k(t,x)} \big|\sigma_k(s,\eta)\big|\,\mathrm{d}s\,\mathrm{d}\eta =$$

$$= \iint\limits_{(H(t,x)\cap P)\setminus H_k(t,x)} \big|\sigma_k(s,\eta)\big|\,\mathrm{d}s\,\mathrm{d}\eta + \iint\limits_{(H_k(t,x)\cap P)\setminus H(t,x)} \big|\sigma_k(s,\eta)\big|\,\mathrm{d}s\,\mathrm{d}\eta \le$$

$$\le \iint\limits_{P} \big|\sigma_k(s,\eta)\big|\,\mathrm{d}s\,\mathrm{d}\eta \le \iint\limits_{P} \omega(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k\ge k_0,$$

which, together with (8.62) and (8.63), guarantees (8.60).

On the other hand, it is clear that

$$\left| \iint\limits_{H_k(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| =$$

$$= \left| \iint\limits_{H_k(t,x)\setminus H(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)\setminus H_k(t,x)} \sigma_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \le$$

$$\le \iint\limits_{H(t,x)\div H_k(t,x)} \big|\sigma_k(s,\eta)\big|\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}.$$

Consequently, the validity of the condition (8.61) follows immediately from the above-proved relation (8.60). □

Proof of Corollary 8.2. We shall show that the assumptions of Corollary 8.1 are satisfied. Indeed, according to Lemma 8.2, the assumptions (8.11), (8.17), and (8.18) guarantee the validity of the condition (8.12). It remains to verify that the condition (8.14) holds. We first show that

$$\lim_{k\to+\infty} \int\limits_{h_k(t)}^{d} \psi_k(\eta)\,\mathrm{d}\eta = \int\limits_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta \quad \text{uniformly on } [a,b]. \tag{8.65}$$

Let $\varepsilon > 0$ be arbitrary but fixed. By virtue of (8.15), there exists $k_1\in\mathbb{N}$ such that

$$\left| \int\limits_{x}^{d} \big[\psi_k(\eta)-\psi(\eta)\big]\,\mathrm{d}\eta \right| < \frac{\varepsilon}{2} \quad \text{for } x\in[c,d],\ k\ge k_1. \tag{8.66}$$

Moreover, there exists $\delta > 0$ with the property

$$\left| \int\limits_{x_1}^{x_2} \psi(\eta)\,\mathrm{d}\eta \right| < \frac{\varepsilon}{2} \quad \text{for } x_1,x_2\in[c,d],\ |x_2-x_1|<\delta, \tag{8.67}$$

and the assumption (8.18) yields the existence of $k_2\in\mathbb{N}$ such that

$$|h_k(t)-h(t)| < \delta \quad \text{for } t\in[a,b],\ k\ge k_2. \tag{8.68}$$

Therefore, using (8.66)–(8.68), we get

$$\left|\int\limits_{h_k(t)}^{d} \psi_k(\eta)\,\mathrm{d}\eta - \int\limits_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta\right| \le \left|\int\limits_{h_k(t)}^{d} \big[\psi_k(\eta) - \psi(\eta)\big]\,\mathrm{d}\eta\right| + \\ + \left|\int\limits_{h_k(t)}^{h(t)} \psi(\eta)\,\mathrm{d}\eta\right| < \varepsilon \quad \text{for } t \in [a,b],\ k \ge \max\{k_1, k_2\},$$

and thus the condition (8.65) holds.

Since we suppose that the triplets (g, φ, ψ) and (g_k, φ_k, ψ_k) are h–consistent and h_k–consistent, respectively, Proposition 3.1 implies

$$g(t) + \int\limits_{h(t)}^{d} \psi(\eta)\,\mathrm{d}\eta = g(a) + \int\limits_{a}^{t} \varphi(s)\,\mathrm{d}s \qquad \text{for } t \in [a,b],$$

$$g_k(t) + \int\limits_{h_k(t)}^{d} \psi_k(\eta)\,\mathrm{d}\eta = g_k(a) + \int\limits_{a}^{t} \varphi_k(s)\,\mathrm{d}s \quad \text{for } t \in [a,b],\ k \in \mathbb{N}.$$

The last two relations, together with (8.16) and (8.65), guarantee the validity of the condition (8.14).

Consequently, the assertion of the corollary follows from Corollary 8.1. ☐

In order to prove Corollary 8.4, we need the following lemmas.

Lemma 8.3. *Let $f \in L\big(\mathcal{D};\mathbb{R}\big)$, $w \in C^*\big(\mathcal{D};\mathbb{R}\big)$, and $h \in CD\big([a,b];[c,d]\big)$. Then the relation*

$$\begin{aligned} \iint\limits_{H(t,x)} f(s,\eta) w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta =& z(t,x)w(t,x) - \int\limits_{h^{-1}(x)}^{t} z(s,x) w_s(s,x)\,\mathrm{d}s - \\ & - \int\limits_{h(t)}^{x} z(t,\eta) w_\eta(t,\eta)\,\mathrm{d}\eta + \iint\limits_{H(t,x)} z(s,\eta) w_{s\eta}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D} \end{aligned} \tag{8.69}$$

holds, where the mapping H is defined by the formula (2.1) *and*

$$z(t,x) = \iint\limits_{H(t,x)} f(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}. \tag{8.70}$$

Proof. Put

$$\chi(t,x) = \begin{cases} 1 & \text{for } (t,x) \in \mathcal{D},\ x \ge h(t), \\ 0 & \text{for } (t,x) \in \mathcal{D},\ x < h(t) \end{cases}$$

and

$$z_0(t,x)=\int_a^t\int_c^x \chi(s,\eta)f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x)\in\mathcal{D}.$$

Obviously,

$$z(t,x)=z_0(t,x) \quad \text{for } (t,x)\in\mathcal{D},\ x\ge h(t).$$

It can be verified by direct calculation that, for any $(t,x)\in\mathcal{D}$, $x\ge h(t)$, we have

$$\begin{aligned}\iint\limits_{H(t,x)} f(s,\eta)w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta &= \int_a^t\int_c^x \chi(s,\eta)f(s,\eta)w(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s=\\ &= z_0(t,x)w(t,x)-\int_a^t z_0(s,x)w_s(s,x)\,\mathrm{d}s-\\ &\quad -\int_c^x z_0(t,\eta)w_\eta(t,\eta)\,\mathrm{d}\eta+\int_a^t\int_c^x z_0(s,\eta)w_{s\eta}(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s=\\ &= z(t,x)w(t,x)-\int_{h^{-1}(x)}^t z(s,x)w_s(s,x)\,\mathrm{d}s-\\ &\quad -\int_{h(t)}^x z(t,\eta)w_\eta(t,\eta)\,\mathrm{d}\eta+\iint\limits_{H(t,x)} z(s,\eta)w_{s,\eta}(s\eta)\,\mathrm{d}s\,\mathrm{d}\eta.\end{aligned}$$

By analogy, for any $(t,x)\in\mathcal{D}$, $x\le h(t)$, we get

$$\begin{aligned}\iint\limits_{H(t,x)} f(s,\eta)w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta &= \int_t^b\int_x^d \big(1-\chi(s,\eta)\big)f(s,\eta)w(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s=\\ &= z(t,x)w(t,x)+\int_t^{h^{-1}(x)} z(s,x)w_s(s,x)\,\mathrm{d}s+\\ &\quad +\int_x^{h(t)} z(t,\eta)w_\eta(t,\eta)\,\mathrm{d}\eta+\iint\limits_{H(t,x)} z(s,\eta)w_{s\eta}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta.\end{aligned}$$

Consequently, the condition (8.69) holds. □

Using the previous statement, we prove the following Krasnoselski-Krein's type lemma.

Lemma 8.4. *Let $h\in CD\big([a,b];[c,d]\big)$, $p,p_k\in L\big(\mathcal{D};\mathbb{R}\big)$, and $\alpha,\alpha_k:\mathcal{D}\to\mathbb{R}$ be measurable and essentially bounded functions ($k\in\mathbb{N}$). Assume that the relations*

(8.19) *and* (8.20) *with* H *given by* (2.1) *are satisfied, and*

$$\lim_{k\to+\infty} \operatorname{ess\,sup}\Big\{|\alpha_k(t,x)-\alpha(t,x)| : (t,x)\in\mathcal{D}\Big\} = 0. \tag{8.71}$$

Then

$$\lim_{k\to+\infty} \iint\limits_{H(t,x)} \big[p_k(s,\eta)\alpha_k(s,\eta) - p(s,\eta)\alpha(s,\eta)\big]\,\mathrm{d}s\,\mathrm{d}\eta = 0 \quad \text{uniformly on } \mathcal{D}. \tag{8.72}$$

Proof. We can assume without loss of generality that

$$|p(t,x)| \le \omega(t,x) \quad \text{for a. e. } (t,x)\in\mathcal{D}. \tag{8.73}$$

Let $\varepsilon > 0$ be arbitrary but fixed. According to (8.71), there exists $k_0 \in \mathbb{N}$ such that

$$\iint\limits_{\mathcal{D}} \omega(t,x)|\alpha_k(t,x)-\alpha(t,x)|\,\mathrm{d}t\,\mathrm{d}x < \frac{\varepsilon}{4} \quad \text{for } k \ge k_0. \tag{8.74}$$

Since the function α is measurable and essentially bounded, there exists a function $w \in C\big(\mathcal{D};\mathbb{R}\big)$, which has continuous derivatives up to the second order and such that

$$\iint\limits_{\mathcal{D}} \omega(t,x)|\alpha(t,x)-w(t,x)|\,\mathrm{d}t\,\mathrm{d}x < \frac{\varepsilon}{4}\,. \tag{8.75}$$

For any $k \in \mathbb{N}$, we put

$$z_k(t,x) = \iint\limits_{H(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big]\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D}.$$

Clearly, the condition (8.20) can be rewritten in the form

$$\lim_{k\to+\infty} \|z_k\|_C = 0. \tag{8.76}$$

Lemma 8.3 yields

$$\iint\limits_{H(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big]w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = z_k(t,x)w(t,x) - \int\limits_{h^{-1}(x)}^{t} z_k(s,x)w_s(s,x)\,\mathrm{d}s -$$
$$- \int\limits_{h(t)}^{x} z_k(t,\eta)w_\eta(t,\eta)\,\mathrm{d}\eta + \iint\limits_{H(t,x)} z_k(s,\eta)w_{s\eta}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x)\in\mathcal{D},\ k\in\mathbb{N}.$$

Consequently, using (8.76), we get

$$\lim_{k\to+\infty} \iint\limits_{H(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big]w(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = 0 \quad \text{uniformly on } \mathcal{D}.$$

Hence, there exists $k_1 \geq k_0$ such that

$$\left| \iint\limits_{H(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big] w(s,\eta) \, \mathrm{d}s \, \mathrm{d}\eta \right| < \frac{\varepsilon}{4} \quad \text{for } (t,x) \in \mathcal{D}, \ k \geq k_1. \tag{8.77}$$

On the other hand, it is clear that

$$\iint\limits_{H(t,x)} \big[p_k(s,\eta)\alpha_k(s,\eta) - p(s,\eta)\alpha(s,\eta)\big] \, \mathrm{d}s \, \mathrm{d}\eta =$$

$$= \iint\limits_{H(t,x)} p_k(s,\eta)\big[\alpha_k(s,\eta) - \alpha(s,\eta)\big] \, \mathrm{d}s \, \mathrm{d}\eta + \iint\limits_{H(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big] w(s,\eta) \, \mathrm{d}s \, \mathrm{d}\eta +$$

$$+ \iint\limits_{H(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big]\big[\alpha(s,\eta) - w(s,\eta)\big] \, \mathrm{d}s \, \mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \ k \in \mathbb{N}.$$

Therefore, in view of (8.19), (8.73)–(8.75), and (8.77), we get

$$\left| \iint\limits_{H(t,x)} \big[p_k(s,\eta)\alpha_k(s,\eta) - p(s,\eta)\alpha(s,\eta)\big] \, \mathrm{d}s \, \mathrm{d}\eta \right| \leq$$

$$\leq \iint\limits_{\mathcal{D}} \omega(s,\eta) \, |\alpha_k(s,\eta) - \alpha(s,\eta)| \, \mathrm{d}s \, \mathrm{d}\eta + \left| \iint\limits_{H(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big] w(s,\eta) \, \mathrm{d}s \, \mathrm{d}\eta \right| +$$

$$+ 2 \iint\limits_{\mathcal{D}} \omega(s,\eta) \, |\alpha(s,\eta) - w(s,\eta)| \, \mathrm{d}s \, \mathrm{d}\eta < \varepsilon \quad \text{for } (t,x) \in \mathcal{D}, \ k \geq k_1,$$

and thus the relation (8.72) is true. □

Proof of Corollary 8.4. Let the operator ℓ be defined by the formula (6.1). Put

$$\ell_k(v)(t,x) \stackrel{\text{def}}{=} p_k(t,x) v\big(\tau_k(t,x), \mu_k(t,x)\big) \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ and all } v \in C\big(\mathcal{D};\mathbb{R}\big), \ k \in \mathbb{N}. \tag{8.78}$$

We show that the condition (8.17) is satisfied. Indeed, let $y \in C^*\big(\mathcal{D};\mathbb{R}\big)$ be arbitrary but fixed. It is clear that the conditions (8.21) and (8.22) guarantee the validity of the relation (8.71), where

$$\alpha_k(t,x) = y\big(\tau_k(t,x), \mu_k(t,x)\big), \quad \alpha(t,x) = y\big(\tau(t,x), \mu(t,x)\big)$$

for a. e. $(t,x) \in \mathcal{D}$ and all $k \in \mathbb{N}$. Therefore, it follows from Lemma 8.4 that the condition (8.72) holds, i.e., the condition (8.17) is true. Moreover, by virtue of the relation (8.19), the condition (8.11) is satisfied.

Consequently, the assertion of the corollary follows from Corollary 8.2. □

Proof of Corollary 8.5. Let the operators ℓ and ℓ_k be defined by the formulae

$$\ell(v)(t,x) \overset{\text{def}}{=} p(t,x)v(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}), \tag{8.79}$$

and

$$\ell_k(v)(t,x) \overset{\text{def}}{=} p_k(t,x)v(t,x) \quad \text{for a.e. } (t,x) \in \mathcal{D}, \text{ all } v \in C(\mathcal{D};\mathbb{R}), \ k \in \mathbb{N}, \tag{8.80}$$

respectively. Obviously,

$$\|\ell_k\| = \|p_k\|_L \quad \text{for } k \in \mathbb{N}. \tag{8.81}$$

Therefore, it is clear that the assumptions (8.5)–(8.8) of Theorem 8.1 are satisfied. In order to apply Theorem 8.1, it remains to show that the condition (8.2) and (8.4) are fulfilled.

It is easy to see that

$$\left| \iint\limits_{H_k(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big] \,\mathrm{d}s\,\mathrm{d}\eta \right| \leq \left| \iint\limits_{H_k(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| + \\ + \iint\limits_{H(t,x) \div H_k(t,x)} |p(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \ k \in \mathbb{N}.$$

Therefore, the conditions (8.23) and (8.24) guarantee that

$$\lim_{k \to +\infty} \varrho_k \|f_k\|_C = 0, \tag{8.82}$$

where

$$f_k(t,x) = \iint\limits_{H_k(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big]\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \ k \in \mathbb{N}. \tag{8.83}$$

We first note that, for an arbitrary $y \in C(\mathcal{D};\mathbb{R})$, we have

$$\left| \iint\limits_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \leq \\ \leq \left| \iint\limits_{H_k(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big] y(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| + \\ + \iint\limits_{H(t,x) \div H_k(t,x)} |p(s,\eta)y(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \ k \in \mathbb{N}. \tag{8.84}$$

Moreover, for an arbitrary $y \in C^*\big(\mathcal{D};\mathbb{R}\big)$, Lemma 8.3 guarantees

$$\begin{aligned} \iint\limits_{H_k(t,x)} \big[p_k(s,\eta) - p(s,\eta)\big] y(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = f_k(t,x)y(t,x) - \\ - \int\limits_{h_k^{-1}(x)}^{t} f_k(s,x)y_s(s,x)\,\mathrm{d}s - \int\limits_{h_k(t)}^{x} f_k(t,\eta)y_\eta(t,\eta)\,\mathrm{d}\eta + \\ + \iint\limits_{H_k(t,x)} f_k(s,\eta)y_{s\eta}(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \end{aligned} \tag{8.85}$$

Let $k \in \mathbb{N}$ and $y \in M(\ell_k, h_k)$ be arbitrary but fixed. Then, by virtue of Notation 8.1 and the proof of Lemma 3.3, we get

$$|y(t,x)| = \left| \iint\limits_{H_k(t,x)} p_k(s,\eta)z(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \le \varrho_k \quad \text{for } (t,x) \in \mathcal{D}, \tag{8.86}$$

$$\begin{aligned} |y_t(t,x)| = \left| \int\limits_{h_k(t)}^{x} p_k(t,\eta)z(t,\eta)\,\mathrm{d}\eta \right| \le \int\limits_{c}^{d} |p_k(t,\eta)|\,\mathrm{d}\eta \\ \text{for a.e. } t \in [a,b] \text{ and all } x \in [c,d], \end{aligned} \tag{8.87}$$

$$\begin{aligned} |y_x(t,x)| = \left| \int\limits_{h_k^{-1}(x)}^{t} p_k(s,x)z(s,x)\,\mathrm{d}s \right| \le \int\limits_{a}^{b} |p_k(s,x)|\,\mathrm{d}s \\ \text{for all } t \in [a,b] \text{ and a.e. } x \in [c,d], \end{aligned} \tag{8.88}$$

$$|y_{tx}(t,x)| = |p_k(t,x)y(t,x)| \le |p_k(t,x)| \quad \text{for a.e. } (t,x) \in \mathcal{D}. \tag{8.89}$$

Using relations (8.86)–(8.89), it follows from (8.84) and (8.85) that

$$\begin{aligned} \left| \iint\limits_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \le \\ \le 4\varrho_k \|f_k\|_C + \varrho_k \iint\limits_{H(t,x) \div H_k(t,x)} |p(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}. \end{aligned}$$

Therefore, according to (8.24) and (8.82), the condition (8.2) holds, where the numbers λ_k are given by the formula (8.3).

Now let $y \in C^*\big(\mathcal{D};\mathbb{R}\big)$ be arbitrary but fixed. Put

$$\begin{aligned}\varrho_0 = \|y\|_C + \max\left\{\int\limits_a^b |y_s(s,x)|\,\mathrm{d}s : x \in [c,d]\right\} + \\ + \max\left\{\int\limits_c^d |y_\eta(t,\eta)|\,\mathrm{d}\eta : t \in [a,b]\right\} + \|y''_{|12}\|_L.\end{aligned} \tag{8.90}$$

Then, (8.84) and (8.85) imply

$$\left|\iint\limits_{H_k(t,x)} \ell_k(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} \ell(y)(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \leq$$

$$\leq \varrho_0\left[\|f_k\|_C + \iint\limits_{H(t,x)\div H_k(t,x)} |p(s,\eta)|\,\mathrm{d}s\,\mathrm{d}\eta\right] \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}.$$

According to (8.24) and (8.82), the last relation yields the validity of the condition (8.4).

Consequently, the assertion of the corollary follows from Theorem 8.1. □

Proof of Corollary 8.6. It follows from the condition (8.26) that

$$\sup\Big\{\|p_k\|_L : k \in \mathbb{N}\Big\} < +\infty. \tag{8.91}$$

Therefore, in view of the relations (8.15) and (8.16), the assumptions (8.7) and (8.8) of Corollary 8.5 are satisfied. Moreover, analogously to the proof of Corollary 8.2 it can be shown that the conditions (8.15), (8.16), and (8.18) yield the validity of the relation (8.14). Therefore, the assumption (8.6) of Corollary 8.5 is true. Furthermore, by virtue of (8.18) and (8.91), Lemma 8.2 guarantees that the condition (8.24) holds.

On the other hand, it is clear that

$$\begin{aligned}\left|\iint\limits_{H_k(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \leq \|p_k - p\|_L + \\ + \left|\iint\limits_{H_k(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta\right| \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}\end{aligned} \tag{8.92}$$

and

$$
\begin{aligned}
(8.93)\qquad & \left| \iint\limits_{H_k(t,x)} q_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \leq \|q_k - q\|_L + \\
& + \left| \iint\limits_{H_k(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta - \iint\limits_{H(t,x)} q(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \right| \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}.
\end{aligned}
$$

Now, using the conditions (8.18), (8.26), (8.27), and Lemma 8.2, the relations (8.92) and (8.93) imply the validity of the assumptions (8.5) and (8.23) of Corollary 8.5.

Consequently, the assertion of the corollary follows from Corollary 8.5. □

9. Counter-Examples

Example 9.1. Let $p \in L(\mathcal{D};\mathbb{R}_+)$ and $h \in CD([a,b];[c,d])$ be such that the relations

$$
\iint\limits_{H(b,d)} p(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s = 1, \qquad \iint\limits_{H(a,c)} p(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \leq 1
$$

are fulfilled, where the mapping H is defined by the formula (2.1). Let, moreover, the operator ℓ be defined by the formula

$$
\ell(v)(t,x) \overset{\text{def}}{=} p(t,x)v(b,d) \quad \text{for a. e. } (t,x) \in \mathcal{D} \text{ and all } v \in C(\mathcal{D};\mathbb{R}).
$$

Then the condition (7.2) with $\alpha = 1$ is satisfied for every $m \in \mathbb{N}$ and $v \in C(\mathcal{D};\mathbb{R})$. Moreover,

$$
\int_a^b \int_{h(s)}^d p_j(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s = 1, \qquad \int_a^b \int_c^{h(s)} p_j(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \leq 1 \quad \text{for } j \in \mathbb{N},
$$

where the functions p_j are given by the formula (7.4).

On the other hand, the homogeneous problem (1.1_0), (1.3_0)–(1.5_0) has a nontrivial solution

$$
u(t,x) = \iint\limits_{H(t,x)} p(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}.
$$

This example shows that the assumption $\alpha \in [0,1[$ in Theorem 7.1 cannot be replaced by the assumption $\alpha \in [0,1]$, and the strict inequality (7.3) in Corollary 7.1 cannot be replaced by the nonstrict one.

Example 9.2. Let $\mathcal{D} = [0,1] \times [0,1]$,

$$
(9.1)\qquad r_k(t) = k\sin(k^2 t), \quad f_k(t) = k\cos(k^2 t) \quad \text{for } t \in [-1,1],\ k \in \mathbb{N},
$$

$$
(9.2)\qquad y_k(t) = k e^{-\frac{\cos(k^2 t)}{k}} \int_0^t e^{\frac{\cos(k^2 s)}{k}} \cos(k^2 s)\,\mathrm{d}s \quad \text{for } t \in [-1,1],\ k \in \mathbb{N},
$$

and

$$z_k(t) = \int\limits_0^t y_k(s)\,\mathrm{d}s \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}. \tag{9.3}$$

It is not difficult to verify that, for every $k \in \mathbb{N}$,

$$y_k'(t) = r_k(t)y_k(t) + f_k(t) \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}, \tag{9.4}$$

$$|y_k(t)| \le 1 + |t|e^2 \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}, \tag{9.5}$$

and

$$\lim_{k\to+\infty} y_k(t) = \frac{t}{2} \quad \text{for } t \in [-1,1], \tag{9.6}$$

because

$$y_k(t) = \frac{1}{k}\,\sin(k^2t) + \frac{1}{2}\,e^{-\frac{\cos(k^2t)}{k}} \int\limits_0^t e^{\frac{\cos(k^2s)}{k}}\,\mathrm{d}s -$$

$$- \frac{1}{2}\,e^{-\frac{\cos(k^2t)}{k}} \int\limits_0^t e^{\frac{\cos(k^2s)}{k}} \cos(2k^2s)\,\mathrm{d}s \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}.$$

Obviously, the relations (9.2)–(9.6) yield

$$z_k''(t) = -r_k'(t)z_k(t) + w_k'(t) + f_k(t) \quad \text{for } t \in [-1,1],\ k \in \mathbb{N},$$

where

$$w_k(t) = r_k(t)z_k(t) \quad \text{for } t \in [-1,1],\ k \in \mathbb{N}, \tag{9.7}$$

and, moreover,

$$\lim_{k\to+\infty} z_k(t) = \frac{t^2}{4} \quad \text{uniformly on } [-1,1]. \tag{9.8}$$

Furthermore, it follows from (9.1) that

$$\lim_{k\to+\infty} \int\limits_0^t r_k(s)\,\mathrm{d}s = 0 \quad \text{uniformly on } [-1,1], \tag{9.9}$$

$$\lim_{k\to+\infty} \int\limits_0^t f_k(s)\,\mathrm{d}s = 0 \quad \text{uniformly on } [-1,1]. \tag{9.10}$$

The relations (9.3) and (9.7) yield

$$\int\limits_0^t w_k(s)\,\mathrm{d}s = z_k(t) \int\limits_0^t r_k(s)\,\mathrm{d}s - \int\limits_0^t y_k(s) \left(\int\limits_0^t r_k(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s$$

for $t \in [-1,1]$ and $k \in \mathbb{N}$ and thus, using (9.5), (9.8), (9.9), and Krasnoselski-Krein's lemma, we get

$$\lim_{k\to+\infty} \int_0^t w_k(s)\,\mathrm{d}s = 0 \quad \text{uniformly on } [-1,1]. \tag{9.11}$$

Now, let $p \equiv 0$ and $q \equiv 0$ on $\mathcal{D}$, $g \equiv 0$, $\varphi \equiv 0$, and $\psi \equiv 0$ on $[0,1]$,

$$\tau(t,x) = t, \quad \mu(t,x) = x \quad \text{for } (t,x) \in \mathcal{D},$$

and

$$h(t) = 1 - t \quad \text{for } t \in [0,1].$$

Moreover, for any $k \in \mathbb{N}$, we put $g_k \equiv 0$, $\varphi_k \equiv 0$, and $\psi_k \equiv 0$ on $[0,1]$,

$$p_k(t,x) = -r_k'(t+x-1) \quad \text{for } (t,x) \in \mathcal{D},$$
$$q_k(t,x) = w_k'(t+x-1) + f_k(t+x-1) \quad \text{for } (t,x) \in \mathcal{D},$$
$$\tau_k(t,x) = t, \quad \mu_k(t,x) = x \quad \text{for } (t,x) \in \mathcal{D},$$

and

$$h_k(t) = 1 - t \quad \text{for } t \in [0,1].$$

It can be easily verified by direct calculation that

$$\iint_{H(t,x)} p_k(s,\eta)\,\mathrm{d}s\,\mathrm{d}\eta = -\int_{1-t}^{x}\int_{1-\eta}^{t} r_k'(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta =$$
$$= -\int_{1-t}^{x} r_k(t+\eta-1)\,\mathrm{d}\eta = -\int_0^{t+x-1} r_k(\xi)\,\mathrm{d}\xi \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N},$$

$$\iint_{H_k(t,x)} w_k'(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta = \int_{1-t}^{x}\int_{1-\eta}^{t} w_k'(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta =$$
$$= \int_{1-t}^{x} w_k(t+\eta-1)\,\mathrm{d}\eta = \int_0^{t+x-1} w_k(\xi)\,\mathrm{d}\xi \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N},$$

and

$$\iint_{H_k(t,x)} f_k(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta = \int_{1-t}^{x}\int_{1-\eta}^{t} f_k(s+\eta-1)\,\mathrm{d}s\,\mathrm{d}\eta =$$
$$= \int_{1-t}^{x} \left(\int_0^{t+\eta-1} f_k(\xi)\,\mathrm{d}\xi \right) \mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ k \in \mathbb{N}.$$

Therefore, by virtue of the conditions (9.9)–(9.11), the relations (8.13) and (8.20) hold.

Consequently, the assumptions of Corollary 8.4 are satisfied except of the condition (8.19). Let the operators ℓ and ℓ_k be defined by the formulae (6.1) and (8.78), respectively. Then, using Lemma 8.3, it is easy to verify that the assumptions of Corollary 8.1 are fulfilled except of the condition (8.11).

On the other hand,

$$u(t,x) = 0 \quad \text{for } (t,x) \in \mathcal{D}$$

and

$$u_k(t,x) = z_k(t+x-1) \quad \text{for } (t,x) \in \mathcal{D}, \ k \in \mathbb{N}$$

are solutions to the problems (1.1′), (1.3)–(1.5) and ($1.1'_k$), (1.3_k)–(1.5_k), respectively, as well as solutions to the problems (1.1), (1.3)–(1.5) and (1.1_k), (1.3_k)–(1.5_k), respectively. However, in view of (9.8), we get

$$\lim_{k\to+\infty} u_k(t,x) = \lim_{k\to+\infty} z_k(t+x-1) = \frac{(t+x-1)^2}{4} \quad \text{for } (t,x) \in \mathcal{D},$$

that is, the relation (8.10) is not true.

This example shows that the assumption (8.11) in Corollary 8.1 and the assumption (8.19) in Corollary 8.4 are essential and they cannot be omitted.

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