Dissipative solutions to the full Navier-Stokes-Fourier system

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joint work with

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Dedicated to Yoshihiro Shibata on the occasion of his 60th birthday

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Mathematical model

State variables

Mass density

$$\varrho = \varrho(t, x)$$

Absolute temperature

$$\vartheta = \vartheta(t, x)$$

Velocity field

$$\mathbf{u} = \mathbf{u}(t, x)$$

Thermodynamic functions

Pressure

$$p = p(\varrho, \vartheta)$$

Internal energy

$$e = e(\varrho, \vartheta)$$

Entropy

$$s = s(\varrho, \vartheta)$$

Transport

Viscous stress

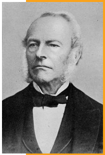
$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Heat flux

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta)$$

Field equations

Claude Louis Marie Henri Navier [1785-1836]



George Gabriel Stokes [1819-1903]

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

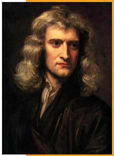
$$\sigma = (\geq) \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Constitutive relations

Joseph Fourier [1768-1830]

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta$$

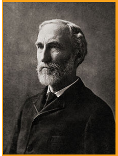


Isaac Newton [1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}$$

Gibbs' relation



Willard Gibbs
[1839-1903]

Gibbs' relation:

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamics stability:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Boundary conditions

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No-slip

$$\mathbf{u}_{\rm tan}|_{\partial\Omega} = 0$$

No-stick

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Weak solutions to the complete system

- Equation of continuity holds in the sense of distributions (renormalized equation also satisfied)
- Momentum balance holds in the sense of distributions
- Entropy production equation holds in the sense of distributions, entropy production rate satisfies the inequality
- The system is augmented by

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x = 0$$

Relative entropy (energy)

Dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = A(t, u(t)), \ u(t) \in X, \ u(0) = u_0$$

Relative entropy

$U : t \mapsto U(t) \in Y$ a “trajectory” in the phase space $Y \subset X$

$$\mathcal{E} \left\{ u(t) \middle| U(t) \right\}, \ \mathcal{E} : X \times Y \to R$$

Basic properties

Positivity(distance)

$\mathcal{E}\{u\,|U\}$ is a "distance" between u, and U, meaning $\mathcal{E}(u|U) \geq 0$ and $\mathcal{E}\{u|U\} = 0$ only if $u = U$

Lyapunov function

$\mathcal{E}\left\{u(t)|\tilde{U}\right\}$ is a Lyapunov function provided $\tilde{U}$ is an equilibrium

$$t \mapsto \mathcal{E}\left\{u(t)\Big|\tilde{U}\right\} \text{ is non-increasing}$$

Gronwall inequality

$$\mathcal{E}\left\{u(\tau)\Big|U(\tau)\right\} \leq \mathcal{E}\left\{u(s)\Big|U(s)\right\} + c(T)\int_s^\tau \mathcal{E}\left\{u(t)\Big|U(t)\right\}\,\mathrm{d}t$$

if U is a solution of the same system (in a "better" space) Y

Applications

Stability of equilibria

Any solution ranging in X stabilizes to an equilibrium belonging to Y (to be proved!)

Weak-strong uniqueness

Solutions ranging in the "better" space Y are unique among solutions in X.

Singular limits

Stability and convergence of a family of solutions u_ε corresponding to A_ε to a solution $U = u$ of the limit problem with generator A.

Navier-Stokes-Fourier system revisited

Total energy balance (conservation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right) \ \mathrm{d}x = 0$$

Total entropy production

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho s(\varrho,\vartheta) \ \mathrm{d}x = \int_\Omega \sigma \ \mathrm{d}x \geq 0$$

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta) - \varrho F \right) \ \mathrm{d}x + \int_\Omega \sigma \ \mathrm{d}x = 0$$

Equilibrium (static) solutions

Equilibrium solutions minimize the entropy production

$$\mathbf{u} \equiv 0, \ \vartheta \equiv \Theta > 0 \text{ a positive constant}$$

Static problem

$$\nabla_x p(\tilde{\varrho}, \Theta) = \tilde{\varrho} \nabla_x F$$

Total mass and energy are constants of motion

$$\int_\Omega \tilde{\varrho} \ \mathrm{d}x = M_0, \ \int_\Omega \tilde{\varrho} e(\tilde{\varrho}, \Theta) - \tilde{\varrho} F \ \mathrm{d}x = E_0$$

Total dissipation balance revisited

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(\tilde\varrho,\Theta)}{\partial\varrho}(\varrho-\tilde\varrho) - H_\Theta(\tilde\varrho,\Theta)\right)\ \mathrm{d}x$$

$$+\int_\Omega \sigma\ \mathrm{d}x = 0$$

Ballistic free energy

$$H_\Theta(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta)\Big)$$

Coercivity of the ballistic free energy

$$\partial^2_{\varrho,\varrho} H_\Theta(\varrho,\Theta) = \frac{1}{\varrho}\partial_\varrho p(\varrho,\Theta)$$

$$\partial_\vartheta H_\Theta(\varrho,\vartheta) = \varrho(\vartheta-\Theta)\partial_\vartheta s(\varrho,\vartheta)$$

Coercivity

$\varrho \mapsto H_\Theta(\varrho,\Theta)$ is convex

$\vartheta \mapsto H_\Theta(\varrho,\vartheta)$ attains its global minimum (zero) at $\vartheta = \Theta$

Relative entropy

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \ \middle| \ r, \Theta, \mathbf{U}\right)$$
$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \ \mathrm{d}x$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \Big(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \Big) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \Big(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta$$

$$+ \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \Big) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Basic properties

- **Global existence.** Weak solutions exist globally in time, under certain constitutive restrictions, for any finite energy initial data.
- **Compatibility.** Any weak solution to the Navier-Stokes-Fourier system is a dissipative solution.
- **Weak-strong uniqueness** Dissipative and strong solutions emanating from the same initial data coincide as long as the latter exists.

Conditional regularity criterion

Theorem (Conditional regularity)

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Under the structural hypotheses specified above, suppose that $\{\varrho, \vartheta, \mathbf{u}\}$ is a dissipative (weak) solution of the Navier-Stokes-Fourier system on the set $(0,T) \times \Omega$ emanating from regular initial data satisfying the relevant compatibility conditions.
Assume, in addition, that

$$\mathrm{ess} \sup_{t \in (0,T)} \|\nabla_x \mathbf{u}(t,\cdot)\|_{L^\infty(\Omega; R^{3\times 3})} < \infty$$

The $\{\varrho, \vartheta, \mathbf{u}\}$ is a classical solution determined uniquely in the class of all dissipative (weak) solutions to the problem.

Other applications

- Inviscid incompressible limits for the system with Navier-type boundary conditions
- Inviscid vanishing viscosity and/or heat conductivity, convergence to (inviscid) Boussinesq system