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On a model with hysteresis arising in magnetohydrodynamics

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Abstract

We study the flow of a conducting fluid surrounded by a ferromagnetic solid, under the influence of the hysteretic response of the surrounding medium. We assume that this influence can be represented by the Preisach model and show existence of a solution of the resulting nonlinear system of PDEs in the convexity domain of the Preisach operator.
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1. Introduction

The flow of a conducting fluid surrounded by a ferromagnetic solid is strongly influenced by the hysteretic response of the surrounding medium Ref. [1], part G9. We assume that this influence can be represented by the Preisach model, and show that this assumption is in agreement with thermodynamic principles. Similar problem was recently considered in Ref. [2], where, however, the typical hysteresis magnetization curve is approximated by two linear parts.

Magnetohydrodynamic (MHD) flows have been the subject of an extensive study in the last 40 years (see for instance Ref. [3]). Most of the results assume that no ferromagnetic hysteresis takes place and the magnetic field and the magnetic induction are linked by a linear relation.

In order to take into account the hysteretic effects in MHD, we consider the following problem as a model for MHD flow of a conducting fluid between two ferromagnetic plates

$$\begin{cases} \dfrac{\partial}{\partial t}(u + \mathscr{W}(u)) + \mathbf{v} \cdot \nabla(u + \mathscr{W}(u)) - \Delta u = 0, \\ \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \Delta \mathbf{v} + (u + \mathscr{W}(u))\nabla u + \nabla p = 0, \\ \operatorname{div} \mathbf{v} = 0, \end{cases} \tag{1}$$

in $\Omega \times (0, T)$, coupled with initial conditions and homogeneous Dirichlet boundary conditions, with unknowns u (represents the magnetic field), $\mathbf{v}$ (velocity of the fluid) and p (pressure), where Ω is an open bounded set of $\mathbb{R}^2$ and $\mathscr{W}$ is a Preisach hysteresis operator.

2. Derivation of the model

Let us consider a conducting fluid moving in an electromagnetic field with given velocity $\mathbf{v} = (v_1, v_2, v_3)$ such that

$$\operatorname{div} \mathbf{v} = 0. \tag{2}$$

We recall the Ampère law (due to the *low frequency approximation* the Maxwell term is neglected)

$$c\nabla \times \mathbf{H} = 4\pi \mathbf{j}, \tag{3}$$

the Faraday law

$$c\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

the Ohm law (where the Hall effect is neglected)

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{5}$$

the continuity equation

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0, \tag{6}$$

and the equation of motion

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + \nabla p = \mathbf{j} \times \mathbf{B} + \eta \Delta \mathbf{v} + \frac{1}{3}\eta \nabla \operatorname{div} \mathbf{v}; \tag{7}$$

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here $\mathbf{H}$ is the magnetic field, $\mathbf{j}$ is the electric current, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction, σ is the electric conductivity, ρ is the charge density, p is the pressure, η is the viscosity and c is the speed of light in vacuum; moreover $\frac{\mathrm{D}}{\mathrm{D}t}$ denotes the material derivative.

We further simplify our setting by considering planar waves. More precisely, let Ω be a domain in $\mathbb{R}^2$ and assume that (using orthogonal Cartesian coordinates x, y, z) both $\mathbf{B}$ and $\mathbf{H}$ are parallel to the z-axis and only depend on the coordinates $(x, y) \in \Omega$ i.e.

$$\mathbf{B} = (0, 0, B(x, y)) \quad \text{and} \quad \mathbf{H} = (0, 0, H(x, y)).$$

We assume that $\mathbf{H}$ and $\mathbf{B}$ are linked by a constitutive relation with hysteresis, i.e.

$$B = (I + \mathscr{W})(H), \tag{8}$$

where $\mathscr{W}$ is a scalar Preisach operator in the setting of Refs. [4,5], and I is the identity operator. For more information about modeling and analysis of Preisach-type hysteresis, see [8–13]. As we are considering planar waves, the electric field has the following form

$$\mathbf{E} = (E_1(x, y), E_2(x, y), 0).$$

This implies that

$$\nabla \times \mathbf{E} = \left(0, 0, \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right),$$

$$\nabla \times \mathbf{H} = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}, 0\right).$$

On the other hand

$$\mathbf{v} \times \mathbf{B} = (v_2 B, -v_1 B, 0)$$

and therefore the Ohm law gives

$$\mathbf{j} = \left(\sigma\left(E_1 + \frac{1}{c} v_2 B\right), \sigma\left(E_2 - \frac{1}{c} v_1 B\right), 0\right). \tag{9}$$

Combining Eq. (3) with Eq. (9) and neglecting from now on for simplicity the constants c, 4π, η and σ, we obtain

$$\frac{\partial H}{\partial y} = E_1 + v_2 B \tag{10}$$

and

$$-\frac{\partial H}{\partial x} = E_2 - v_1 B. \tag{11}$$

The Faraday law instead has the following form after our simplifications

$$\frac{\partial B}{\partial t} + \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} = 0. \tag{12}$$

Differentiating Eq. (10) in the y variable and Eq. (11) in the x variable yields

$$\frac{\partial^2 H}{\partial y^2} = \frac{\partial E_1}{\partial y} + \frac{\partial}{\partial y}(v_2 B),$$
$$-\frac{\partial^2 H}{\partial x^2} = \frac{\partial E_2}{\partial x} - \frac{\partial}{\partial x}(v_1 B). \tag{13}$$

Now using Eqs. (12) and (13) we deduce

$$\frac{\partial B}{\partial t} + \left[-\frac{\partial^2 H}{\partial x^2} + \frac{\partial}{\partial x}(v_1 B) - \frac{\partial^2 H}{\partial y^2} + \frac{\partial}{\partial y}(v_2 B)\right] = 0$$

which is equivalent to

$$\frac{\partial B}{\partial t} + \operatorname{div}(\mathbf{v} B) - \Delta H = 0, \tag{14}$$

where we take $\mathbf{v} = \mathbf{v}(x, y, t)$.

We assume ρ to be constant ($\rho = 1$). From the Ampère law,

$$\mathbf{j} = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}, 0\right)$$

so

$$\mathbf{j} \times \mathbf{B} = \left(-\frac{\partial H}{\partial x} B, -\frac{\partial H}{\partial y} B, 0\right) = -B\nabla H.$$

Then, if we express the material derivative in terms of the partial derivative, Eq. (7) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \Delta \mathbf{v} + B\nabla H + \nabla p = 0. \tag{15}$$

Writing the abstract problem obtained by coupling (14), (2) and (15) we have

$$\begin{cases} \dfrac{\partial B}{\partial t} + \mathbf{v} \cdot \nabla B - \Delta H = 0, \\ \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} - \Delta \mathbf{v} + B\nabla H + \nabla p = 0, \\ \operatorname{div} \mathbf{v} = 0 \end{cases} \tag{16}$$

and this, together with Eq. (8) gives nothing but Eq. (1), with $H = u$ and $B = (I + \mathscr{W})(u)$.

3. Main result

Assume that $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain and set $\Omega_T := \Omega \times (0, T)$. For simplicity put $V := H^1_0(\Omega)$ and

$$\tilde{\mathbf{V}} := \left\{\mathbf{v} \in L^2(\Omega; \mathbb{R}^2);\ \int_\Omega \mathbf{v} \cdot \nabla\varphi \,\mathrm{d}x = 0, \forall \varphi \in V\right\},$$
$$\mathbf{V} := \tilde{\mathbf{V}} \cap H^1_0(\Omega; \mathbb{R}^2).$$

We want to solve the following problem.

Problem 1. *Consider given initial data u^0, $\mathbf{v}^0$; we search for functions $(u, \mathbf{v})$ with $u \in L^2(\Omega; \mathscr{C}^0([0, T])) \cap L^2(0, T; V)$, $\mathscr{W}(u) \in L^2(\Omega; \mathscr{C}^0([0, T])) \cap L^4(\Omega_T)$, $\mathbf{v} \in L^2(0, T; \mathbf{V})$ such that*

$$\frac{\partial}{\partial t}(u + \mathscr{W}(u)) \in L^2(\Omega_T), \quad \frac{\partial \mathbf{v}}{\partial t} \in L^2(0, T; L^2(\Omega; \mathbb{R}^2)),$$
$$u(x, 0) = u^0(x), \quad \mathbf{v}(x, 0) = \mathbf{v}^0(x)$$

and for any $z \in V$, any $\mathbf{z} \in \mathbf{V}$ and for a.e. $t \in (0, T)$ we have

$$\int_\Omega \frac{\partial}{\partial t}(u + \mathscr{W}(u)) z \,\mathrm{d}x - \int_\Omega [\mathbf{v} \cdot \nabla z](u + \mathscr{W}(u)) \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla z \,\mathrm{d}x = 0, \tag{17}$$

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$$\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot \mathbf{z}\,\mathrm{d}x + \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{z}\,\mathrm{d}x + \int_\Omega (\nabla\mathbf{v},\nabla\mathbf{z})\,\mathrm{d}x + \int_\Omega (u+\mathscr{W}(u))\,\nabla u\cdot\mathbf{z}\,\mathrm{d}x = 0. \tag{18}$$

Interpretation. If the functions u, $\mathscr{W}(u)$, $\mathbf{v}$ are smooth enough, we may integrate by parts in Eqs. (17) and (18). We see that the function

$$\mathbf{q} := \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \Delta\mathbf{v} + (u+\mathscr{W}(u))\nabla u$$

is orthogonal to every function $\mathbf{z}\in\mathbf{V}$, hence (see Ref. [6]), there exists p such that $\mathbf{q} = -\nabla p$. Formally system (17) and (18) thus reduces to Eq. (1) with homogeneous boundary conditions for both u and $\mathbf{v}$.

The main result of the paper can be stated as follows.

Theorem 2. *Consider given data*

$$u^0 \in V,\quad \mathbf{v}^0\in\mathbf{V},\quad \Delta u^0 \in L^2(\Omega),\quad \Delta\mathbf{v}^0 \in L^2(\Omega;\mathbb{R}^2)$$

and set

$$C_d := \max\{\|u_0\|_V, \|\mathbf{v}_0\|_{\mathbf{V}}, \|\Delta u_0\|_{L^2(\Omega)}, \|\Delta\mathbf{v}_0\|_{L^2(\Omega;\mathbb{R}^2)}\}. \tag{19}$$

Then there exists a constant C_* such that if $C_d \leqslant C_*$, then Problem 1 *admits at least one solution* $(u,\mathbf{v})$ *with additional regularity*

$$u \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;V),$$
$$\mathbf{v} \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^2)) \cap H^1(0,T;\mathbf{V}).$$

We prove this theorem (see Ref. [7] for more details) using standard methods of time discretization, derivation of a priori estimates and passage to the limit; we strongly use the properties of the Preisach hysteresis operator, namely the discrete versions of the first- and second-order energy inequalities.

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References

[1] L. Haňka, Teorie Elektromagnetického Pole, SNTL, Prague, 1975.
[2] L. Bühler, K. Messadek, R. Stieglitz, Fusion Eng. Des. 63–64 (2002) 353.
[3] P.A. Davison, An Introduction to Magnetohydrodynamics, Cambridge University Press, Cambridge, 2001.
[4] M. Brokate, J. Sprekels, Hysteresis and phase transitions, in: Applied Mathematical Sciences, vol. 121, Springer, New York, 1996.
[5] P. Krejčí, Hysteresis, Convexity and Dissipation in Hyperbolic Equations, Gakkotosho, Tokyo, 1996.
[6] G.P. Galdi, An introduction to the mathematical theory of the Navier–Stokes equations, in: Linearized Steady Problems, vol. 1, Springer, New York, 1994.
[7] M. Eleuteri, J. Kopfová, P. Krejčí, An existence result for a model with hysteresis arising in magnetohydrodynamics, in preparation.
[8] G. Bertotti, Hysteresis in Magnetism, Academic Press, Boston, 1998.
[9] E. Della Torre, Magnetic Hysteresis, I.E.E.E. Press, 1999.
[10] M.A. Krasnosel'skiĭ, A.V. Pokrovskiĭ, Systems with hysteresis, Springer, Berlin, 1989 (Russian edition: Nauka, Moscow, 1983).
[11] I.D. Mayergoyz, Mathematical Models of Hysteresis, Springer, New York, 1991.
[12] F. Preisach, Z. Physik 94 (1935) 277 (in German).
[13] A. Visintin, Differential Models of Hysteresis, Springer, Berlin, 1994.