WELL-POSEDNESS OF AN EXTENDED MODEL
FOR WATER-ICE PHASE TRANSITIONS

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Abstract. We propose an improved model explaining the occurrence of high
stresses due to the difference in specific volumes during phase transitions be-
tween water and ice. The unknowns of the resulting evolution problem are
the absolute temperature, the volume increment, and the liquid fraction. The
main novelty here consists in including the dependence of the specific heat and
of the speed of sound upon the phase. These additional nonlinearities bring
new mathematical difficulties which require new estimation techniques based
on Moser iteration. We establish the existence of a global solution to the cor-
responding initial-boundary value problem, as well as lower and upper bounds
for the absolute temperature. Assuming constant heat conductivity, we also
prove uniqueness and continuous data dependence of the solution.

1. **Introduction.** In the present contribution we prove the well-posedness of an
initial-boundary value problem associated with the following system coupling a
quasi-linear parabolic internal energy balance (for the absolute temperature θ) with
an integro-differential equation for the relative volume increment U, and a differen-
tial inclusion ruling the evolution of the phase variable χ as follows:

$$c(\chi)e_1(\theta)_t - \operatorname{div}\left(\kappa(\chi)\nabla\theta\right) = c'(\chi)\chi_t(f_1(\theta) - e_1(\theta))$$

$$+ \nu U_t^2 - \beta\theta U_t + \gamma(\theta)\chi_t^2 - L\frac{\theta}{\theta_c}\chi_t, \qquad (1.1)$$

$$\nu U_t + \lambda(\chi)(U - \alpha(1-\chi)) - \beta(\theta - \theta_c) = \varrho_0 g(x_3 - \zeta_\Gamma)$$

$$- K_\Gamma(P_0(t) + U_\Omega(t)) \qquad (1.2)$$

$$-\gamma(\theta)\chi_t - \frac{\lambda'(\chi)}{2}(U - \alpha(1-\chi))^2 \quad \in \quad \partial I(\chi) + \alpha\lambda(\chi)(U - \alpha(1-\chi))$$

$$+c'(\chi)\left(f_1(\theta) - f_1(\theta_c)\right) + L\left(1 - \frac{\theta}{\theta_c}\right) \tag{1.3}$$

with $U_\Omega(t) = \int_\Omega U(x,t)\,\mathrm{d}x$. In the previous paper [12] we have already given a motivation and a complete study of equilibria for this system, which models the water freezing in an elastic container, taking into account differences in the specific volume, specific heat and speed of sound in the solid and liquid phases. The derivation of the system from physical principles and the meaning of the symbols will be explained below in the next Section 2. Here, we describe the mathematical difficulties and comment on previous results related to this type of systems.

There is an abundant classical literature on phase transition processes, see e.g. the monographs [2], [5], [14] and the references therein. It seems, however, that only few publications take into account different mass densities/specific volumes of the phases. In [6], the authors proposed to interpret a phase transition process in terms of a balance equation for macroscopic motions, and to include the possibility of voids. Well-posedness of an initial-boundary value problem associated with the resulting PDE system is proved there and the case of two different densities ϱ_1 and ϱ_2 for the two substances undergoing phase transitions has been pursued in [7].

Here, we deal exclusively with physically measurable quantities. All parameters have a clear physical meaning and the derivation is carried out under the assumption that the displacements are small. This enables us to state the system in Lagrangian coordinates (cf. [7] for a different approach to the subject).

The present model has been previously studied in [10] and [11] under the assumption that the speed of sound and the specific heat are the same in solid and in liquid. In terms of the system (1.1)–(1.3), this corresponds to choose constant functions $\lambda(\chi) \equiv \lambda$ and $c(\chi) \equiv c$. For this particular case, we have proved in [10] and [11] the existence and uniqueness of global solutions, as well as the convergence of the solutions to equilibria. In reality, the specific heat in water is about the double, while the speed of sound in water is less than one half of the one in ice. The main goal of this contribution is to give a well–posedness result for a boundary value problem associated with (1.1)–(1.3) including these dependences into the model. The main result is stated in Section 4. The dependence of speed of sound and of the specific heat on the phase is expressed in terms of additional nonlinearities in the equations which have to be suitably handled. Moreover, here we also generalize the results of [10] and [11] allowing for non constant external pressure and temperature. Finally, we proceed here with a different technique for the proof of existence of solutions with respect to [10] and [11]. Since the contraction argument does not work in our situation, we discretize in time our problem (cf. Subsection 5.1), preparing thus necessary tools for future numerical investigations on this model, and prove the convergence of the scheme. The uniqueness and continuous dependence of solution on the data is proved in Section 6 following the idea already exploited in [4] where we deal with a quasi-linear internal energy balance equation coupled with a vectorial and nonlocal phase dynamic. The main estimates are obtained here by means of the energy inequality which still holds true at the discrete level (cf. Subsection 5.3). Finally, it is worth noting that a time dependent positive lower bound for the θ-component of the solution independent of the time step is established on the time

discrete approximation in Subsection 5.2, while we obtain a uniform in time upper bound on the solution θ by means of a proper Moser estimate (cf. Subsection 5.4).

2. **Balance equations.** Referring to [12] for the complete deduction of the model, we consider a liquid substance contained in a bounded connected container $\Omega \subset \mathbb{R}^3$ with boundary of class $C^{1,1}$. The state variables are the absolute temperature $\theta > 0$, the displacement $\mathbf{u} \in \mathbb{R}^3$, and the phase variable $\chi \in [0, 1]$. The value $\chi = 0$ means solid, $\chi = 1$ means liquid, $\chi \in (0, 1)$ is a mixture of the two.

We make the following modeling hypotheses.

(A1) The displacements are small. Therefore, we state the problem in *Lagrangian coordinates*, in which mass conservation is equivalent to the condition of a constant mass density $\varrho_0 > 0$.

(A2) The substance is isotropic and compressible; the speed of sound and the specific heat may depend on the phase χ.

(A3) The evolution is slow, and we neglect shear viscosity and inertia effects.

(A4) We neglect shear stresses.

(A5) The liquid phase is the reference state, and the specific volume V_i of the solid phase is larger than the specific volume V_w of the liquid phase.

We thus consider the evolution system

$$-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f}_{vol}, \tag{2.1}$$

$$\varrho_0 e_t + \operatorname{div} \mathbf{q} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t, \tag{2.2}$$

$$-\gamma_0(\theta)\chi_t \in \partial_\chi f, \tag{2.3}$$

consisting of a the mechanical equilibrium equation (2.1), energy conservation law (2.2), and a phase dynamic equation (2.3), where the coefficient γ_0 determines the speed of the phase transition. By **(A4)**, the stress has the form $\boldsymbol{\sigma} = -p\,\boldsymbol{\delta}$ and the scalar quantity

$$p := -\nu\boldsymbol{\varepsilon}_t : \boldsymbol{\delta} - \lambda(\chi)(\boldsymbol{\varepsilon} : \boldsymbol{\delta} - \alpha(1 - \chi)) + \beta(\theta - \theta_c) \tag{2.4}$$

is the *pressure*. Here $\nu > 0$ is a volume viscosity coefficient, $\lambda(\chi)$ is the Lamé constant, which may depend on χ by virtue of **(A2)**, $\alpha = (V_i - V_w)/V_w$ is a positive phase expansion coefficient by **(A5)**, while β is the thermal expansion coefficient, which is assumed to be constant, and $\mathbf{f}_{vol}$ is a given volume force density (the gravity force)

$$\mathbf{f}_{vol} = -\varrho_0 g\,\boldsymbol{\delta}_3, \tag{2.5}$$

with standard gravity g and vector $\boldsymbol{\delta}_3 = (0, 0, 1)$.

We denote by e the specific internal energy, s is the specific entropy, and $\mathbf{q}$ is the heat flux vector that we assume for simplicity in the form

$$\mathbf{q} = -\kappa(\chi)\nabla\theta \tag{2.6}$$

with heat conductivity $\kappa(\chi) > 0$ depending possibly on χ.

We assume the specific heat $c_V(\chi, \theta)$ in the form

$$c_V(\chi, \theta) = c_0(\chi)c_1(\theta). \tag{2.7}$$

This is still a rough simplification, and further generalizations are desirable. According to [9, Chapter VI] or [13, Section 5], the purely caloric parts e_{cal} and

s_{cal} of the specific internal energy and specific entropy are given by the formulas $e_{cal}(\chi,\theta) = c_0(\chi)e_1(\theta)$, $s_{cal}(\chi,\theta) = c_0(\chi)s_1(\theta)$, with

$$e_1(\theta) = \int_0^\theta c_1(r)\,\mathrm{d}r\,, \quad s_1(\theta) = \int_0^\theta \frac{c_1(r)}{r}\,\mathrm{d}r\,. \tag{2.8}$$

Then, the specific free energy $f = e - \theta s$ satisfies the conditions $\boldsymbol{\sigma}^e = \varrho_0 \partial_{\boldsymbol{\varepsilon}} f$, $s = -\partial_\theta f$. With a prescribed constant latent heat L_0 and freezing point $\theta_c > 0$ at standard atmospheric pressure P_{stand}, the specific free energy f necessarily has the form

$$f = c_0(\chi)f_1(\theta) + \frac{\lambda(\chi)}{2\varrho_0}(\boldsymbol{\varepsilon}:\boldsymbol{\delta} - \alpha(1-\chi))^2 \tag{2.9}$$

$$- \frac{\beta}{\varrho_0}(\theta - \theta_c)\boldsymbol{\varepsilon}:\boldsymbol{\delta} + L_0\chi\left(1 - \frac{\theta}{\theta_c}\right) + \tilde{f}(\chi)\,,$$

where

$$f_1(\theta) = e_1(\theta) - \theta s_1(\theta) = \int_0^\theta c_1(r)\left(1 - \frac{\theta}{r}\right)\mathrm{d}r\,, \tag{2.10}$$

and $\tilde{f}$ is a arbitrary function of χ (integration "constant" with respect to θ and $\boldsymbol{\varepsilon}$). We choose $\tilde{f}$ so as to ensure that the values of χ remain in the interval $[0,1]$, and that the phase transition under standard pressure takes place at temperature θ_c. More specifically, we set

$$\tilde{f}(\chi) = L_0 I(\chi) - c_0(\chi)f_1(\theta_c)\,.$$

where I is the indicator function of the interval $[0,1]$.

For specific entropy s and specific internal energy e we obtain

$$s = -\partial_\theta f = c_0(\chi)s_1(\theta) + \frac{\beta}{\varrho_0}\boldsymbol{\varepsilon}:\boldsymbol{\delta} + \frac{L_0}{\theta_c}\chi\,, \tag{2.11}$$

$$e = c_0(\chi)(e_1(\theta) - f_1(\theta_c)) + \frac{\lambda(\chi)}{2\varrho_0}(\boldsymbol{\varepsilon}:\boldsymbol{\delta} - \alpha(1-\chi))^2$$

$$+ \frac{\beta}{\varrho_0}\theta_c\boldsymbol{\varepsilon}:\boldsymbol{\delta} + L_0(\chi + I(\chi))\,. \tag{2.12}$$

The equation for the phase χ is obtained by assuming that $-\chi_t$ is proportional to $\partial_\chi f$ with proportionality coefficient (relaxation time) $\gamma_0(\theta) > 0$, where ∂_χ is the partial Clarke subdifferential with respect to χ.

Then, the equilibrium equation (2.1) can be rewritten in the form $\nabla p = \mathbf{f}_{vol}$, hence, as Ω is connected,

$$p(x,t) = P(t) - \varrho_0 g\, x_3\,, \tag{2.13}$$

where P is a function of time only, which is to be determined. Recall that in the reference state $\boldsymbol{\varepsilon}:\boldsymbol{\delta} = \boldsymbol{\varepsilon}_t:\boldsymbol{\delta} = 0$, $\chi = 1$, and at standard pressure P_{stand}, the freezing temperature is θ_c. We thus see from (2.4) that $P(t)$ is in fact the deviation from the standard pressure. We assume also the external pressure in the form $P_{ext} = P_{stand} + p_0$ with a given deviation $p_0(x,t)$. The normal force acting on the boundary is $(P(t) - \varrho_0 g\, x_3 - p_0)\mathbf{n}$, where $\mathbf{n}$ denotes the unit outward normal vector. We assume an elastic response of the boundary, and a heat transfer proportional

to the inner and outer temperature difference. On $\partial\Omega$, we thus prescribe boundary conditions for $\mathbf{u}$ and θ in the form

$$(P(t) - \varrho_0 g\, x_3 - p_0(x,t))\mathbf{n}(x) \;=\; \mathbf{k}(x)\mathbf{u}(x,t)\,, \qquad (2.14)$$

$$\mathbf{q}(x,t)\cdot\mathbf{n}(x) \;=\; h(x)(\theta - \theta_\Gamma(x,t)) \qquad (2.15)$$

with a given symmetric positive definite matrix $\mathbf{k}(x)$ (elasticity of the boundary), positive functions $h(x)$ (heat transfer coefficient), and $\theta_\Gamma(x,t) > 0$ (external temperature). This enables us to find an explicit relation between $\operatorname{div}\mathbf{u}$ and P. Indeed, on $\partial\Omega$ we have by (2.14) that $\mathbf{u}\cdot\mathbf{n} = (P(t) - \varrho_0 g\, x_3 - p_0(x,t))\mathbf{k}^{-1}(x)\mathbf{n}(x)\cdot\mathbf{n}(x)$. Assuming that $\mathbf{k}^{-1}\mathbf{n}\cdot\mathbf{n}$ belongs to $L^1(\partial\Omega)$, we set

$$\frac{1}{K_\Gamma} = \int_{\partial\Omega}\mathbf{k}^{-1}(x)\mathbf{n}(x)\cdot\mathbf{n}(x)\,\mathrm{d}\sigma(x)\,, \quad \zeta_\Gamma = K_\Gamma\int_{\partial\Omega}\mathbf{k}^{-1}(x)\mathbf{n}(x)\cdot\mathbf{n}(x)\,x_3\,\mathrm{d}\sigma(x)\,, \qquad (2.16)$$

and obtain by Gauss' Theorem that

$$U_\Omega(t) := \int_\Omega \operatorname{div}\mathbf{u}(x,t)\,\mathrm{d}x = \frac{1}{K_\Gamma}(P(t) - \varrho_0 g\,\zeta_\Gamma) - P_0(t)\,, \qquad (2.17)$$

where $P_0(t) = \int_{\partial\Omega} p_0(x,t)\mathbf{k}^{-1}(x)\mathbf{n}(x)\cdot\mathbf{n}(x)\,\mathrm{d}\sigma(x)$. Under the small strain hypothesis, the function $\operatorname{div}\mathbf{u}$ describes the local relative volume increment. Hence, Eq. (2.17) establishes a linear relation between the total relative volume increment $U_\Omega(t)$ and the relative pressure $P(t) - p_0(x,t)$. We have $\boldsymbol{\varepsilon}:\boldsymbol{\delta} = \operatorname{div}\mathbf{u}$, and thus the mechanical equilibrium equation (2.13), due to (2.4) and (2.17), reads

$$\nu\operatorname{div}\mathbf{u}_t + \lambda(\chi)(\operatorname{div}\mathbf{u} - \alpha(1-\chi)) - \beta(\theta - \theta_c) + \varrho_0 g(\zeta_\Gamma - x_3) = -K_\Gamma(P_0(t) + U_\Omega(t))\,. \qquad (2.18)$$

As a consequence of (2.6), (2.9), and (2.12), the energy balance and the phase relaxation equation in (2.2)–(2.3) have the form

$$\varrho_0 c_0(\chi)e_1(\theta)_t - \operatorname{div}\left(\kappa(\chi)\nabla\theta\right) + \varrho_0 c_0'(\chi)\chi_t(e_1(\theta) - f_1(\theta))$$

$$= \;\nu(\operatorname{div}\mathbf{u}_t)^2 - \beta\theta\operatorname{div}\mathbf{u}_t + \varrho_0\gamma_0(\theta)\chi_t^2 - \varrho_0 L_0\frac{\theta}{\theta_c}\chi_t\,, \qquad (2.19)$$

$$-\varrho_0\gamma_0(\theta)\chi_t - \frac{\lambda'(\chi)}{2}(\operatorname{div}\mathbf{u} - \alpha(1-\chi))^2 - \alpha\lambda(\chi)(\operatorname{div}\mathbf{u} - \alpha(1-\chi))$$

$$\in \;\varrho_0 c_1'(\chi)\left(f_1(\theta) - f_1(\theta_c)\right) + \varrho_0 L_0\left(1 - \frac{\theta}{\theta_c}\right) + \partial I(\chi)\,. \qquad (2.20)$$

For simplicity, we now set

$$U := \operatorname{div}\mathbf{u}\,, \quad c(\chi) := \varrho_0 c_0(\chi)\,, \quad \gamma(\theta) := \varrho_0\gamma_0(\theta)\,, \quad L := \varrho_0 L_0\,. \qquad (2.21)$$

Note that mathematically, the subdifferential $\partial I(\chi)$ is the same as $\varrho_0 L_0\partial I(\chi)$. The system thus reduces to the system (1.1)–(1.3) of three scalar equations – one PDE and two "ODEs" for three unknown functions θ, χ, and U, with boundary condition (2.15), (2.6). Assuming that a solution to (1.1)–(1.3) is known with $U \in L^2(\Omega \times (0,T))$, we find the vector function $\mathbf{u}$ by defining first Φ to be the solution (unique up to an additive constant) to the Poisson equation $\Delta\Phi = U$ with the Neumann boundary condition $\nabla\Phi\cdot\mathbf{n} = (K_\Gamma U_\Omega(t) + \varrho_0 g(\zeta_\Gamma - x_3))\mathbf{k}^{-1}(x)\mathbf{n}(x)\cdot\mathbf{n}(x)$. With

this Φ, we find $\tilde{\mathbf{u}}$ as a solution to the problem

$$\text{div}\,\tilde{\mathbf{u}} \;=\; 0 \qquad \text{in } \Omega \times (0, T)\,, \quad (2.22)$$

$$\left.\begin{array}{rcl} \tilde{\mathbf{u}} \cdot \mathbf{n} &=& 0 \\[4pt] (\tilde{\mathbf{u}} + \nabla \Phi - (K_\Gamma U_\Omega + \varrho_0 g(\zeta_\Gamma - x_3))\mathbf{k}^{-1}\mathbf{n}) \times \mathbf{n} &=& 0 \end{array}\right\} \quad \text{on } \partial\Omega \times (0,T)\,, (2.23)$$

and set $\mathbf{u} = \tilde{\mathbf{u}} + \nabla\Phi$. Then $\mathbf{u}$ satisfies a.e. in Ω the equation $\text{div}\,\mathbf{u} = U$, together with the boundary condition (2.14), that is, $\mathbf{u} = (K_\Gamma U_\Omega + \varrho_0 g(\zeta_\Gamma - x_3))\mathbf{k}^{-1}\mathbf{n}$ on $\partial\Omega$.

For the solution to (2.22)–(2.23), we refer to [8, Lemma 2.2] which states that for each $\mathbf{g} \in H^{1/2}(\partial\Omega)^3$ satisfying $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}\,\mathrm{d}\sigma(x) = 0$ there exists a function $\tilde{\mathbf{u}} \in H^1(\Omega)^3$, unique up to an additive function $\mathbf{v}$ from the set V of divergence-free $H^1(\Omega)$ functions vanishing on $\partial\Omega$, such that $\text{div}\,\tilde{\mathbf{u}} = 0$ in Ω, $\tilde{\mathbf{u}} = \mathbf{g}$ on $\partial\Omega$. In terms of the system (2.22)–(2.23), it suffices to set $\mathbf{g} = ((\nabla\Phi - (K_\Gamma U_\Omega + \varrho_0 g(\zeta_\Gamma - x_3))\mathbf{k}^{-1}\mathbf{n}) \times \mathbf{n}) \times \mathbf{n}$ and use the identity $(\mathbf{b} \times \mathbf{n}) \times \mathbf{n} = (\mathbf{b} \cdot \mathbf{n})\,\mathbf{n} - \mathbf{b}$ for every vector $\mathbf{b}$. Moreover, the estimate

$$\inf_{\mathbf{v} \in V} \|\tilde{\mathbf{u}} + \mathbf{v}\|_{H^1(\Omega)} \leq C\,\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)} \leq \tilde{C}\|\Phi\|_{H^2(\Omega)} \qquad (2.24)$$

holds with some constants $C, \tilde{C}$. The required regularity is available here by virtue of the assumption that Ω is of class $C^{1,1}$, provided $\mathbf{k}^{-1}$ belongs to $H^{1/2}(\partial\Omega)$. Note that a weaker formulation of problem (2.22)–(2.23) can be found in [1, Section 4].

Due to our hypotheses **(A3)**, **(A4)**, we thus lose any control on possible volume preserving turbulences $\mathbf{v} \in V$. This, however, has no influence on the system (1.1)–(1.3), which is the subject of our interest here. Inequality (2.24) shows that if U is small in agreement with hypothesis **(A1)**, then also $\mathbf{v}$ can be chosen in such a way that hypothesis **(A1)**, interpreted in terms of H^1, is not violated.

3. **Energy and entropy.** In terms of the new variables θ, U, χ, the densities $\varrho_0 e, \varrho_0 s$ of energy and entropy can be written as

$$\varrho_0 e \;=\; c(\chi)(e_1(\theta) - f_1(\theta_c)) + \frac{\lambda(\chi)}{2}(U - \alpha(1 - \chi))^2 + \beta\theta_c U + L(\chi + I(\chi))\,, (3.1)$$

$$\varrho_0 s \;=\; c(\chi)s_1(\theta) + \frac{L}{\theta_c}\chi + \beta U\,. \qquad\qquad (3.2)$$

The energy functional has to be supplemented with the boundary energy term

$$E_\Gamma(t) \;=\; \frac{K_\Gamma}{2}\left(U_\Omega(t) + P_0(t) + \frac{\varrho_0 g\zeta_\Gamma}{K_\Gamma}\right)^2\,, \qquad\qquad (3.3)$$

as well as with the gravity potential $-\varrho_0 g x_3 U$. The energy and entropy balance equations now read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \varrho_0(e(x,t)-gx_3U)\,\mathrm{d}x + E_\Gamma(t)\right) = \int_{\partial\Omega} h(x)(\theta_\Gamma(x,t)-\theta)\,\mathrm{d}\sigma(x)$$

$$+ K_\Gamma(P_0)_t(t)\left(U_\Omega(t)+P_0(t)+\frac{\varrho_0 g\zeta_\Gamma}{K_\Gamma}\right), \tag{3.4}$$

$$\varrho_0 s_t + \mathrm{div}\,\frac{\mathbf{q}}{\theta} = \frac{\kappa(\chi)|\nabla\theta|^2}{\theta^2} + \frac{\gamma(\theta)}{\theta}\chi_t^2 + \frac{\nu}{\theta}U_t^2 \geq 0, \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_0 s(x,t)\,\mathrm{d}x = \int_{\partial\Omega}\frac{h(x)}{\theta}(\theta_\Gamma(x,t)-\theta)\,\mathrm{d}\sigma(x) \tag{3.6}$$

$$+ \int_\Omega\left(\frac{\kappa(\chi)|\nabla\theta|^2}{\theta^2}+\frac{\gamma(\theta)}{\theta}\chi_t^2+\frac{\nu}{\theta}U_t^2\right)\,\mathrm{d}x\,.$$

The entropy balance (3.5) says that the entropy production on the right hand side is nonnegative in agreement with the second principle of thermodynamics. The system is not closed, and the energy supply or the energy loss through the boundary is given by the right hand side of (3.4).

We prescribe the initial conditions

$$\begin{aligned}
\theta(x,0) &= \theta^0(x) && (3.7)\\
U(x,0) &= U^0(x) && (3.8)\\
\chi(x,0) &= \chi^0(x) && (3.9)
\end{aligned}$$

for $x \in \Omega$, and compute from (3.1)–(3.2) the initial values e^0, E_Γ^0, and s^0 for specific energy, boundary energy, and entropy, respectively. Let E^0 and S^0 denote, respectively, $E^0 = \int_\Omega \varrho_0 e^0(x)\,\mathrm{d}x$, $S^0 = \int_\Omega \varrho_0 s^0(x)\,\mathrm{d}x$. We multiply (3.6) by a positive constant $\bar{\theta}_\Gamma$, subtract the result from (3.4), and integrate over time from 0 to t. For the "extended" energy $\varrho_0(e-\bar{\theta}_\Gamma s)$, we use (3.1)–(3.3) to obtain for each admissible time t the following crucial (formal for the moment) balance equation:

$$\int_\Omega\left(c(\chi)(e_1(\theta)-f_1(\theta_c))+\frac{\lambda(\chi)}{2}(U-\alpha(1-\chi))^2\right)(x,t)\,\mathrm{d}x$$

$$+\int_\Omega\left(\beta\theta_c U + L\chi - \varrho_0 g x_3 U\right)(x,t)\,\mathrm{d}x$$

$$+\frac{K_\Gamma}{2}\left(U_\Omega(t)+P_0(t)+\frac{\varrho_0 g\,\zeta_\Gamma}{K_\Gamma}\right)^2$$

$$+\bar{\theta}_\Gamma\int_0^t\int_\Omega\left(\frac{\kappa(\chi)|\nabla\theta|^2}{\theta^2}+\frac{\gamma(\theta)}{\theta}\chi_t^2+\frac{\nu}{\theta}U_t^2\right)(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi$$

$$+\int_0^t\int_{\partial\Omega}\frac{h(x)}{\theta}(\theta-\theta_\Gamma(x,\xi))(\theta-\bar{\theta}_\Gamma)\,\mathrm{d}\sigma(x)\,\mathrm{d}\xi$$

$$= E^0 + E_\Gamma^0 - \bar{\theta}_\Gamma S^0 + \bar{\theta}_\Gamma\int_\Omega\left(c(\chi)s_1(\theta)+\frac{L}{\theta_c}\chi+\beta U\right)(x,t)\,\mathrm{d}x$$

$$+\int_0^t K_\Gamma(P_0)_t(\xi)\left(U_\Omega(\xi)+P_0(\xi)+\frac{\varrho_0 g\zeta_\Gamma}{K_\Gamma}\right)\,\mathrm{d}\xi\,. \tag{3.10}$$

We assume that both $c(\chi)$ and $\lambda(\chi)$ are bounded from above and from below by positive constants. The growth of $s_1(\theta)$ is dominated by $e_1(\theta)$ as a consequence of the inequality

$$\frac{s_1(\theta) - s_1(\theta^*)}{e_1(\theta) - e_1(\theta^*)} \leq \frac{1}{\theta^*} \qquad \forall \theta > \theta^* > 0 \,. \tag{3.11}$$

Let $(0, T)$ for some $0 < T \leq \infty$ be a time interval in which a solution to (1.1)–(1.3) satisfies (3.10). Assume that

$$\int_0^T \int_{\partial\Omega} h(x) \left(\sqrt{\theta_\Gamma(x,t)} - \sqrt{\bar{\theta}_\Gamma} \right)^2 \, d\sigma(x) \, dt < \infty \,, \quad \int_0^T |(P_0)_t(t)| \, dt < \infty \,. \tag{3.12}$$

Using (3.10), (3.11) with $\theta^* = 2\bar{\theta}_\Gamma$, and the identity

$$\frac{1}{\theta}(\theta - a)(\theta - b) = \frac{1}{\theta}(\theta - \sqrt{ab})^2 - (\sqrt{b} - \sqrt{a})^2 \quad \text{for all } \theta, a, b > 0 \tag{3.13}$$

with $a = \theta_\Gamma(x,t)$ and $b = \bar{\theta}_\Gamma$, we find a constant $C > 0$ independent of t such that for all $t \in [0, T)$ we have

$$\int_\Omega \left(e_1(\theta) + U^2 \right)(x,t) \, dx + \int_0^t \int_\Omega \left(\frac{\kappa(\chi)|\nabla\theta|^2}{\theta^2} + \frac{\gamma(\theta)\chi_t^2}{\theta} + \frac{\nu U_t^2}{\theta} \right)(x,\xi) \, dx \, d\xi$$

$$+ \int_0^t \int_{\partial\Omega} \frac{h(x)}{\theta} \left(\theta - \sqrt{\bar{\theta}_\Gamma \theta_\Gamma(x,\xi)} \right)^2 \, d\sigma(x) \, d\xi \ \leq \ C \,. \tag{3.14}$$

4. **Main results.** We construct the solution of (1.1)–(1.3) by a combined truncation and time discretization scheme. The method of proof is independent of the actual values of the material constants, hence we choose for simplicity

$$L = 2, \quad \theta_c = \alpha = \beta = \nu = \varrho_0 = 1 \,. \tag{4.1}$$

We consider the following assumptions on the data.

Hypothesis 4.1. *Assume that there exist positive constants c_*, c^*, $\underline{c}$, $\bar{c}$, $\underline{\lambda}$, $\bar{\lambda}$, κ_*, λ^*, γ_* such that*

  (i) $c \in C^{1,1}([0,1])$, c convex, $0 < c_* \leq c(z)$, $0 < \underline{c} \leq c'(z) \leq \bar{c}$, for all $z \in [0,1]$;
 (ii) $c_1 \in C^0([0,\infty))$, $c_1(0) = 0$, $c_1(\theta) > 0$ for $\theta > 0$, $c_1(\theta) \geq c^*$ for $\theta \geq 1$, $\lim_{\theta\to\infty} c_1(\theta)/\theta = \infty$, $\int_0^1 c_1(r)/r \, dr < \infty$, $\int_0^1 c_1(r)/r^2 \, dr = \infty$;
(iii) λ convex, $\lambda \in C^{1,1}([0,1])$, $0 < \underline{\lambda} \leq \lambda(z) \leq \bar{\lambda}$, $0 \geq \lambda'(z) \geq -\lambda^*$ for all $z \in [0,1]$;
 (iv) $\kappa \in C^{1,1}([0,1])$, $0 < \kappa_* \leq \kappa(z)$ for all $z \in [0,1]$;
  (v) $h \in L^\infty(\partial\Omega)$ is a non-negative function;
 (vi) $\gamma \in C^{0,1}(\mathbb{R}^+)$, $0 < \gamma_* \leq \gamma(r)$ for all $r \in \mathbb{R}^+$.

The liquid phase does not persist for very large temperatures and the behavior of $c_1(\theta)$ as $\theta \to \infty$ thus cannot be experimentally verified. We nevertheless believe that the growth condition (ii) in Hypothesis 4.1 is not completely meaningless taking into account the fact that in the interval between 273 and $373\,K$ (0–$100°C$), the function $c_1(\theta)$ is convex with a minimum at $35°C$ [1].

Keeping the notation introduced in (2.8), (2.10), we define the following symbols:

$$\mathcal{A}(U, \chi, x, t) := \lambda(\chi)(U - 1 + \chi) + K_\Gamma(U_\Omega(t) + P_0(t)) + g(\zeta_\Gamma - x_3) + 1 \,, \tag{4.2}$$

$$\mathcal{B}(\chi, \theta) := c'(\chi)(f_1(\theta) - f_1(1)) - 2\theta \,, \tag{4.3}$$

$$\mathcal{C}(U, \chi) := \frac{\lambda'(\chi)}{2}(U - 1 + \chi)^2 + \lambda(\chi)(U - 1 + \chi) + 2 \,. \tag{4.4}$$

---

[1] see http://www.engineeringtoolbox.com/water-thermal-properties-d_162.html

System (1.1)–(1.3) with boundary condition (2.15) then can be written in the form

$$\int_\Omega c(\chi)e_1(\theta)_t w(x)\,\mathrm{d}x + \int_\Omega \kappa(\chi)\nabla\theta\cdot\nabla w(x)\,\mathrm{d}x = \int_{\partial\Omega} h(x)(\theta_\Gamma(x,t)-\theta)w(x)\,\mathrm{d}\sigma(x)$$

$$- \int_\Omega \big(U_t\mathcal{A}(U,\chi,x,t) + \chi_t\big(\mathcal{C}(U,\chi) + c'(\chi)(e_1(\theta)-f_1(1))\big)\big)w(x)\,\mathrm{d}x, \qquad (4.5)$$

$$U_t - \theta = -\mathcal{A}(U,\chi,x,t), \qquad (4.6)$$

$$\gamma(\theta)\chi_t + \mathcal{B}(\chi,\theta) + \partial I(\chi) \ni -\mathcal{C}(U,\chi), \qquad (4.7)$$

where (4.5) is obtained from (1.1) by using (4.6)–(4.7). Let us note that (4.5) is to be satisfied for all test functions $w \in W^{1,2}(\Omega)$ and a.e. $t > 0$, while (4.6)–(4.7) are supposed to hold a.e. in the space-time cylinder that we denote $\Omega_T := \Omega \times (0,T)$ for $T > 0$, $\Omega_\infty := \Omega \times (0,\infty)$.

In this section we prove the following existence and uniqueness result.

Theorem 4.2. *Let Hypothesis 4.1 be satisfied, and let $\theta_\Gamma \in H^1(0,T;L^2(\partial\Omega))$ such that $0 < \theta_* \le \theta_\Gamma \le \theta^*$, and $P_0 \in W^{1,1}(0,T)$ be given functions. Let the initial conditions in (3.7)–(3.9) be such that*

$$\theta^0 \in W^{1,2}(\Omega)\cap L^\infty(\Omega), \quad 0 < \theta_* \le \theta^0(x) \le \theta^* \quad a.e.,$$
$$U^0,\chi^0 \in W^{1,2}(\Omega)\cap L^\infty(\Omega), \quad 0 \le \chi^0(x) \le 1 \quad a.e.$$

Then there exists at least a solution (θ,U,χ) to (4.5)–(4.7), (3.7)–(3.9), and constants $\theta^\sharp(T) \ge \theta^\flat(T) > 0$ such that

$$\theta^\flat(T) \le \theta(x,t) \le \theta^\sharp(T) \quad for\ a.e.\ (x,t) \in \Omega_T, \qquad (4.8)$$

$\chi \in [0,1]$ a.e., $U,U_t,\chi_t \in L^\infty(\Omega_T)$, $\theta_t \in L^2(\Omega_T)$, $\nabla U, \nabla\chi, \nabla\theta \in L^\infty(0,T;L^2(\Omega))$. Moreover, if the above assumptions hold for all $T > 0$ and condition (3.12) is satisfied for $T = \infty$, then the solution exists globally, and $\theta^\sharp(T)$ can be chosen independently of T. Finally, if $\kappa(\theta) \equiv \bar\kappa \in \mathbb{R}^+$ is constant, then the solution is unique, and its L^2-norm depends continuously on the data.

Remark 4.3. Let us note that we could prove our existence result assuming that $\kappa = \kappa(\theta,\chi) = k_1(\theta)k_2(\chi)$ with the same techniques. Moreover, also uniqueness would hold true in case $\kappa = \kappa_1(\theta)$ with an appropriate modification of the boundary condition by means of the standard Kirchhoff transformation technique.

5. **Existence proof.** We proceed as follows: we truncate from above the functions depending on θ in (4.5)–(4.7), and discretize the system in time. For the discrete system, we derive upper and lower bounds that enable us to let the time step tend to 0 and prove the existence of a solution to the truncated problem. Finally, we prove a time dependent lower bound and a uniform (in time and w.r.t. the truncation parameters) upper bound on θ, so that the truncation can be removed, and this will conclude the proof of existence of solutions.

5.1. **Approximation and discrete energy estimate.** We introduce, for $\theta \in \mathbb{R}$, $R > 0$, the functions

$$Q_R(\theta) = \min\{\theta^+, B(R)\}, \quad B(R) = R^{1/2}(\min\{e_1(R), |f_1(R)|\})^{1/4}, \tag{5.1}$$

$$c_1^R(\theta) = c_1(Q_R(\theta)), \tag{5.2}$$

$$e_1^R(\theta) = \int_0^\theta c_1^R(r) \, dr, \tag{5.3}$$

$$s_1^R(\theta) = \int_0^\theta \frac{c_1^R(r)}{Q_R(r)} \, dr, \tag{5.4}$$

$$f_1^R(\theta) = e_1^R(\theta) - Q_R(\theta)s_1^R(\theta) = \int_0^\theta c_1^R(r) \left(1 - \frac{Q_R(\theta)}{Q_R(r)}\right) \, dr, \tag{5.5}$$

with the convention $f_1^R(\theta) = s_1^R(\theta) = 0$ for $\theta \leq 0$. By Hypo. 4.1 (ii), we have $c_1^R(0) = 0$, and the functions $c_1^R, e_1^R, s_1^R, f_1^R$ are continuous across $\theta = 0$. In the rest of the proof the following relations, which directly follow from the above definitions and from Hypo. 4.1 (ii), play an important role:

- If $0 < \theta \leq B(R)$ then $e_1^R(\theta) = e_1(\theta)$, $s_1^R(\theta) = s_1(\theta)$, $f_1^R(\theta) = f_1(\theta)$, $\tag{5.6}$
- If $\theta > B(R)$ then $e_1^R(\theta) = e_1(B(R)) + c_1(B(R))(\theta - B(R))$,

$$s_1^R(\theta) = s_1(B(R)) + \frac{1}{B(R)}c_1(B(R))(\theta - B(R)), \quad f_1^R(\theta) = f_1(B(R)), \tag{5.7}$$

- If $\theta > R$ then $e_1^R(\theta) > e_1(R) > 0$, $\quad f_1^R(\theta) < f_1(R) < 0$, $\tag{5.8}$
- $\displaystyle \lim_{R\to\infty} \frac{e_1(R)}{R^2} = \lim_{R\to\infty} \frac{c_1(R)}{2R} = \infty$, $\tag{5.9}$
- $\displaystyle \lim_{R\to\infty} \frac{f_1(R)}{R^2} = -\lim_{R\to\infty} \frac{s_1(R)}{2R} = -\lim_{R\to\infty} \frac{c_1(R)}{2R} = -\infty$, $\tag{5.10}$
- $\displaystyle \lim_{R\to\infty} \frac{|f_1(R)|}{B^2(R)} = \lim_{R\to\infty} \frac{e_1(R)}{B^2(R)} = \infty \, , \, \lim_{R\to\infty} \frac{B(R)}{R} = \infty. \tag{5.11}$

We now introduce the time-discrete version of (4.5)–(4.7). For an arbitrary $n \in \mathbb{N}$, we define the time step $\tau = T/n$. Choosing a constant $c_R \in \mathbb{R}^+$ depending on R, which we specify below, we look for a solution $\{(\theta_k, U_k, \chi_k)\}_{k=1}^n$ to the scheme

$$\frac{1}{\tau} \int_\Omega c(\chi_k) \left(e_1^R(\theta_k) - e_1^R(\theta_{k-1})\right) w(x) \, dx + \int_\Omega \kappa(\chi_{k-1})\nabla\theta_k \cdot \nabla w(x) \, dx$$

$$+ \int_\Omega c_R(\theta_k\theta_k^+ - \theta_{k-1}\theta_{k-1}^+)w(x) \, dx + \int_{\partial\Omega} h(x)(\theta_k - \theta_{k\Gamma})w(x) \, d\sigma(x)$$

$$= - \int_\Omega c'(\chi_k) \overset{\bullet}{\chi}_k(e_1^R(\theta_{k-1}) - f_1(1))w(x) \, dx$$

$$- \int_\Omega \left(\overset{\bullet}{U}_k \mathcal{A}_k(U_k, \chi_k, \chi_{k-1}, x) + \overset{\bullet}{\chi}_k \mathcal{C}_k(U_k, \chi_k, \chi_{k-1})\right) w(x) \, dx, \tag{5.12}$$

$$\overset{\bullet}{U}_k - Q_R(\theta_{k-1}) = -\mathcal{A}_k(U_k, \chi_k, \chi_{k-1}, x), \tag{5.13}$$

$$\gamma(\theta_{k-1}) \overset{\bullet}{\chi}_k + \mathcal{B}_k(\chi_k, \theta_{k-1}) + \partial I(\chi_k) \ni -\mathcal{C}_k(U_k, \chi_k, \chi_{k-1}), \tag{5.14}$$

where

$$\mathcal{A}_k(U_k, \chi_k, \chi_{k-1}, x) := \lambda(\chi_{k-1})(U_k - 1 + \chi_k) + K_\Gamma(U_{k\Omega} + p_k) + g(\zeta_\Gamma - x_3) + 1,$$
(5.15)

$$\mathcal{B}_k(\chi_k, \theta_{k-1}) := c'(\chi_k)(f_1^R(\theta_{k-1}) - f_1(1)) - 2Q_R(\theta_{k-1}),$$
(5.16)

$$\mathcal{C}_k(U_k, \chi_k, \chi_{k-1}) := \frac{\lambda'(\chi_k)}{2}(U_k - 1 + \chi_k)^2 + \lambda(\chi_{k-1})(U_k - 1 + \chi_k) + 2, \quad (5.17)$$

with $\theta_{k\Gamma} = \theta_\Gamma(\cdot, k\tau)$, $p_k = P_0(k\tau)$, $U_{k\Omega} = U_\Omega(k\tau)$, and with initial condition $(\theta_0, U_0, \chi_0) = (\theta^0, U^0, \chi^0)$. The symbol $\overset{\bullet}{\varphi}_k$ denotes the discrete time derivative $(\varphi_k - \varphi_{k-1})/\tau$ for a generic sequence $\{\varphi_k\}$. Eq. (5.12) has to be satisfied for all test functions $w \in W^{1,2}(\Omega)$, while (5.13)–(5.14) are supposed to hold a.e. in Ω.

It is easy to see then that the latter semi-implicit scheme has a unique solution. Indeed, at each time step, we assume that $\theta_{k-1}, U_{k-1}, \chi_{k-1}$ are known, and find U_k, χ_k satisfying (5.13)–(5.14). For τ sufficiently small, (5.13)–(5.14) is an algebraic system for (U_k, χ_k) of the form $\Phi(U_k, \chi_k) = Y_k$ with $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ strictly maximal monotone, hence it admits a unique solution. Finally, we insert U_k and χ_k in (5.12) and solve the resulting coercive elliptic equation, obtaining in that way the desired solution (θ_k, U_k, χ_k).

Then, we note that the total energy balance still holds true for the discrete system. Indeed, we take (5.12) with $w = 1$, and denote $E_k = e_1^R(\theta_k) - f_1(1)$. We have

$$\frac{1}{\tau} \int_\Omega \left(c(\chi_k)(e_1^R(\theta_k) - e_1^R(\theta_{k-1})) + (\chi_k - \chi_{k-1})c'(\chi_k)(e_1^R(\theta_{k-1}) - f_1(1)) \right) \, \mathrm{d}x$$

$$= \frac{1}{\tau} \int_\Omega \left(c(\chi_k)(E_k - E_{k-1}) + (\chi_k - \chi_{k-1})c'(\chi_k)E_{k-1} \right) \, \mathrm{d}x$$

$$= \frac{1}{\tau} \int_\Omega \left(c(\chi_k)E_k - (c(\chi_k) - (\chi_k - \chi_{k-1})c'(\chi_k))E_{k-1} \right) \, \mathrm{d}x,$$

and using the fact that $E_k \geq 0$ and that c is convex (cf. Hypo. 4.1 (i)), we get

$$\frac{1}{\tau} \int_\Omega \left(c(\chi_k)(e_1^R(\theta_k) - e_1^R(\theta_{k-1})) + (\chi_k - \chi_{k-1})c'(\chi_k)(e_1^R(\theta_{k-1}) - f_1(1)) \right) \, \mathrm{d}x$$

$$\geq \frac{1}{\tau} \int_\Omega \left(c(\chi_k)E_k - c(\chi_{k-1})E_{k-1} \right) \, \mathrm{d}x.$$
(5.18)

Set now $S_k = U_k + \chi_k - 1$. Then, we obtain

$$\int_\Omega \left((\overset{\bullet}{\chi}_k + \overset{\bullet}{U}_k)\lambda(\chi_{k-1})(U_k + \chi_k - 1) + \overset{\bullet}{\chi}_k \frac{\lambda'(\chi_k)}{2}(U_k + \chi_k - 1)^2 \right) \, \mathrm{d}x$$

$$= \frac{1}{\tau} \int_\Omega \left(S_k(S_k - S_{k-1})\lambda(\chi_{k-1}) + \frac{1}{2}S_k^2(\chi_k - \chi_{k-1})\lambda'(\chi_k) \right) \, \mathrm{d}x$$

$$\geq \frac{1}{2\tau} \int_\Omega \left((S_k^2 - S_{k-1}^2)\lambda(\chi_{k-1}) + S_k^2(\chi_k - \chi_{k-1})\lambda'(\chi_k) \right) \, \mathrm{d}x$$

$$= \frac{1}{2\tau} \int_\Omega \left(S_k^2\left(\lambda(\chi_{k-1}) + \lambda'(\chi_k)(\chi_k - \chi_{k-1}) \right) - S_{k-1}^2\lambda(\chi_{k-1}) \right) \, \mathrm{d}x.$$

Using now the convexity of λ (cf. Hypo. 4.1 (iii)), we get

$$
\int_\Omega \left((\overset{\bullet}{\chi}_k + \overset{\bullet}{U}_k)\lambda(\chi_{k-1})(U_k + \chi_k - 1) + \overset{\bullet}{\chi}_k \frac{\lambda'(\chi_k)}{2}(U_k + \chi_k - 1)^2 \right) \mathrm{d}x
$$
$$
\geq \frac{1}{2\tau} \int_\Omega (S_k^2 \lambda(\chi_k) - S_{k-1}^2 \lambda(\chi_{k-1})) \, \mathrm{d}x \,. \tag{5.19}
$$

Hence, from (5.12), using (5.18) and (5.19), we obtain

$$
\frac{1}{\tau} \int_\Omega \left(c(\chi_k)E_k - c(\chi_{k-1})E_{k-1} + \frac{1}{2}\left(S_k^2 \lambda(\chi_k) - S_{k-1}^2 \lambda(\chi_{k-1})\right) \right) \mathrm{d}x
$$
$$
+ \overset{\bullet}{U}_{k\Omega} \left(K_\Gamma(U_{k\Omega} + p_k) + g\zeta_\Gamma \right) - \int_\Omega g x_3 \overset{\bullet}{U}_k \, \mathrm{d}x
$$
$$
+ \int_\Omega \left(\overset{\bullet}{U}_k + 2\overset{\bullet}{\chi}_k + c_R(\theta_k \theta_k^+ - \theta_{k-1}\theta_{k-1}^+) \right) \mathrm{d}x + \int_{\partial\Omega} h(x)(\theta_k - \theta_{k\Gamma}) \, \mathrm{d}\sigma(x)
$$
$$
\leq 0 \,. \tag{5.20}
$$

Summing now (5.20) over $k = 1, \ldots, m$, $1 \leq m \leq n$, we get

$$
\int_\Omega \left(c(\chi_m)E_m + \frac{1}{2}S_m^2 \lambda(\chi_m) - g x_3 U_m + U_m + \chi_m + c_R \tau \theta_m \theta_m^+ \right) \mathrm{d}x
$$
$$
+ \frac{K_\Gamma}{2}\left(U_{m\Omega} + p_m + \frac{g\zeta_\Gamma}{K_\Gamma} \right)^2 + \tau \sum_{k=1}^m \int_{\partial\Omega} h(x)(\theta_k - \theta_{k\Gamma}) \, \mathrm{d}\sigma(x)
$$
$$
\leq \int_\Omega \left(c(\chi_0)E_0 + \frac{1}{2}S_0^2 \lambda(\chi_0) - g x_3 U_0 + U_0 + \chi_0 + c_R \theta_0 \theta_0^+ \right) \mathrm{d}x
$$
$$
+ \frac{K_\Gamma}{2}\left(U_{0\Omega} + p_0 + \frac{g\zeta_\Gamma}{K_\Gamma} \right)^2 + K_\Gamma \sum_{k=1}^m |p_k - p_{k-1}| \max_{0 \leq k \leq m} \left| U_{k\Omega} + p_k + \frac{g\zeta_\Gamma}{K_\Gamma} \right| .
$$
$$
\tag{5.21}
$$

Using the fact that $\tau \sum_{k=1}^n \int_{\partial\Omega} h(x)\theta_{k\Gamma}(x) \, \mathrm{d}\sigma(x) \leq C(T)$, $\sum_{k=1}^n |p_k - p_{k-1}| \leq C(T)$, with a constant $C(T)$ independent of τ and R, we check that the left hand side of (5.21) is bounded independently of τ and R. Consequently, all terms in Eq. (5.13) are bounded by a multiple of $(1 + B(R))$. Similarly, multiplying (5.14) by $\overset{\bullet}{\chi}_k$ and using the fact that $\overset{\bullet}{\chi}_k \xi_k \geq 0$ for all $\xi_k \in \partial I(\chi_k)$, we obtain the estimates

$$
\left.
\begin{array}{rcl}
|U_k| + |\overset{\bullet}{U}_k| & \leq & C(T)(1 + B(R)) \\[4pt]
|\overset{\bullet}{\chi}_k| & \leq & C(T)(1 + B(R) + B^2(R) + |f_1(B(R))|)
\end{array}
\right\} \quad \text{a.e.} \tag{5.22}
$$

5.2. **Lower bound for θ_k.** Here we derive a lower bound for the approximated absolute temperature θ_k. We first rewrite (5.12) for $w \in W^{1,2}(\Omega)$, $w \geq 0$ a.e., using

(5.13)–(5.14), in the form

$$\frac{1}{\tau} \int_\Omega c(\chi_k) \left(e_1^R(\theta_k) - e_1^R(\theta_{k-1})\right) w(x)\,\mathrm{d}x + \int_\Omega \kappa(\chi_{k-1})\nabla\theta_k \cdot \nabla w(x)\,\mathrm{d}x$$

$$+ \int_\Omega \left(c_R(\theta_k\theta_k^+ - \theta_{k-1}\theta_{k-1}^+)\right) w(x)\,\mathrm{d}x + \int_{\partial\Omega} h(x)(\theta_k - \theta_{k\Gamma})w(x)\,\mathrm{d}\sigma(x)$$

$$\geq \int_\Omega \overset{\bullet}{U}_k(\overset{\bullet}{U}_k - Q_R(\theta_{k-1}))w(x)\,\mathrm{d}x$$

$$+ \int_\Omega \overset{\bullet}{\chi}_k \big(\gamma(\theta_{k-1}) \overset{\bullet}{\chi}_k + c'(\chi_k)(f_1^R(\theta_{k-1}) - e_1^R(\theta_{k-1})) - 2Q_R(\theta_{k-1})\big)w(x)\,\mathrm{d}x\,,$$

$$(5.23)$$

where we have used again the fact that

$$- \overset{\bullet}{\chi}_k \mathcal{C}(U_k, \chi_k, \chi_{k-1}) \; \geq \overset{\bullet}{\chi}_k \big(\gamma(\theta_{k-1}) \overset{\bullet}{\chi}_k + \mathcal{B}(\chi_k, \theta_{k-1})\big)$$

by definition of the subdifferential. The right hand side of (5.23) is bounded from below by a negative multiple (depending on R) of $\theta_{k-1}\theta_{k-1}^+$. We can now choose c_R in (5.12) sufficiently large in order to get the following inequality for all $w \in W^{1,2}(\Omega)$, $w \geq 0$ a.e.:

$$\frac{1}{\tau} \int_\Omega c(\chi_k) \left(e_1^R(\theta_k) - e_1^R(\theta_{k-1})\right) w(x)\,\mathrm{d}x + \int_\Omega \kappa(\chi_{k-1})\nabla\theta_k \cdot \nabla w(x)\,\mathrm{d}x$$

$$+ \int_{\partial\Omega} h(x)(\theta_k - \theta_{k\Gamma})w(x)\,\mathrm{d}\sigma(x) \geq -c_R \int_\Omega \theta_k\theta_k^+\, w(x)\,\mathrm{d}x\,. \qquad (5.24)$$

We now compare this inequality with the constant decreasing sequence $\{v_k\}$ defined recurrently as

$$\frac{1}{\tau}c_* \left(e_1^R(v_k) - e_1^R(v_{k-1})\right) = -c_R v_k^2, \quad v_0 := \theta_*\,. \qquad (5.25)$$

We write (5.25), adding the zero term $-\mathrm{div}\,(k(\chi_{k-1})\nabla v_k)$, in the form

$$\frac{1}{\tau} \int_\Omega c_* \left(e_1^R(v_k) - e_1^R(v_{k-1})\right) w(x)\,\mathrm{d}x + \int_\Omega \kappa(\chi_{k-1})\nabla v_k \cdot \nabla w(x)\,\mathrm{d}x$$

$$= -c_R \int_\Omega v_k^2\, w(x)\,\mathrm{d}x\,. \qquad (5.26)$$

Subtracting (5.24) from (5.26) and testing the difference by $w = H_\varepsilon(v_k - \theta_k)$, where H_ε is the regularization of the Heaviside function H,

$$H_\varepsilon(v) = \begin{cases} 0 & \text{if } v \leq 0 \\ v/\varepsilon & \text{if } v \in (0,\varepsilon) \\ 1 & \text{if } v \geq \varepsilon \end{cases}, \qquad (5.27)$$

we obtain, since $v_k < v_{k-1}$, that

$$\int_\Omega c(\chi_k) \left(\left(e_1^R(v_k) - e_1^R(v_{k-1})\right) - \left(e_1^R(\theta_k) - e_1^R(\theta_{k-1})\right)\right) H_\varepsilon(v_k - \theta_k)\,\mathrm{d}x \leq 0\,. \qquad (5.28)$$

Assume that $\theta_{k-1} \geq v_{k-1}$ (this is true for $k = 1$). For $\varepsilon \searrow 0$, (5.28) yields $\theta_k \geq v_k$, and by induction we get $\theta_k \geq v_k > v_n$ for all $k = 1, \ldots, n$. By (5.25), we have

$$e_1(v_k) - e_1(v_{k-1}) = -C\tau v_k^2$$

with $C = c_R/c_*$. Under Hypo. 4.1 (ii), the function $G(z) := -\int_z^{v_0} \frac{c_1(s)}{s^2}\,\mathrm{d}s$ is increasing in $(0, v_0]$, $G(0+) = -\infty$, $G(v_0) = 0$. Moreover, by the Mean Value Theorem, there exists $s_k \in [v_k, v_{k-1}]$ such that

$$\frac{G(v_k) - G(v_{k-1})}{e_1(v_k) - e_1(v_{k-1})} = \frac{G'(s_k)}{c_1(s_k)} = \frac{1}{s_k^2} \le \frac{1}{v_k^2},$$

hence $G(v_{k-1}) - G(v_k) \le C\tau$, that is, $G(v_n) \ge -Cn\tau$. We thus see that $\theta_k \ge v_n \ge G^{-1}(-Cn\tau) = G^{-1}(-CT) =: \theta^\flat(T)$. This concludes the proof of the lower bound for θ_k.

5.3. **Estimates.** Now, we perform the estimates we need in order to pass to the limit as $\tau \searrow 0$ in (5.12)–(5.14). The right hand side of (5.12) is bounded from above, by virtue of (5.22), by $C(T,R)(\theta_{k-1} + 1)$, where $C(T,R)$ is, here and in the sequel, any sufficiently large constant depending only on T and R, and independent of k and τ. Testing (5.12) by $w = \theta_k - \theta_{k-1}$, we obtain, using (5.22), that

$$\int_\Omega \left(\frac{1}{\tau} c(\chi_k) \left(e_1^R(\theta_k) - e_1^R(\theta_{k-1}) \right) (\theta_k - \theta_{k-1}) + \kappa(\chi_{k-1}) \nabla \theta_k \nabla(\theta_k - \theta_{k-1}) \right) \mathrm{d}x$$

$$+ \int_\Omega c_R(\theta_k - \theta_{k-1})^2 (\theta_k + \theta_{k-1})\,\mathrm{d}x + \int_{\partial\Omega} h(x)(\theta_k - \theta_{k\Gamma})(\theta_k - \theta_{k-1})\,\mathrm{d}\sigma(x)$$

$$\le C(T,R) \int_\Omega |\theta_k - \theta_{k-1}|(\theta_{k-1} + 1)\,\mathrm{d}x.$$

Using the lower bound for θ_k and (5.22) again, and choosing $\chi_{-1} = \chi_0$, we get

$$\frac{1}{\tau C_1(T)} \int_\Omega |\theta_k - \theta_{k-1}|^2\,\mathrm{d}x + \int_\Omega \left(\kappa(\chi_{k-1})|\nabla\theta_k|^2 - \kappa(\chi_{k-2})|\nabla\theta_{k-1}|^2 \right)\,\mathrm{d}x$$

$$+ \int_{\partial\Omega} h(x) \left((\theta_k - \theta_{k\Gamma})^2 - (\theta_{k-1} - \theta_{(k-1)\Gamma})^2 \right)\,\mathrm{d}\sigma(x)$$

$$\le \int_\Omega \left(\kappa(\chi_{k-1}) - \kappa(\chi_{k-2}) \right)|\nabla\theta_{k-1}|^2\,\mathrm{d}x + \int_{\partial\Omega} h(x)|\theta_{k\Gamma} - \theta_{(k-1)\Gamma}|\,|\theta_k - \theta_{k\Gamma}|\,\mathrm{d}\sigma(x)$$

$$+ \tau C(T,R) \int_\Omega (\theta_{k-1} + 1)^2\,\mathrm{d}x$$

$$\le \tau C(T,R) \Big(\int_\Omega (1 + |\theta_{k-1}| + |\nabla\theta_{k-1}|)^2\,\mathrm{d}x$$

$$+ \int_{\partial\Omega} h(x)(|\overset{\bullet}{\theta}_{k\Gamma}|^2 + (\theta_k - \theta_{k\Gamma})^2)\,\mathrm{d}\sigma(x) \Big),$$

where $C_1(T)$ is a positive constant depending on T but not on τ. The elementary inequality $\theta_k^2 - \theta_{k-1}^2 = (\theta_k - \theta_{k-1})(\theta_k + \theta_{k-1}) \le \frac{1}{2\tau}(\theta_k - \theta_{k-1})^2 + \frac{\tau}{2}(\theta_k + \theta_{k-1})^2$ yields that $\frac{1}{\tau C_1(T)}|\theta_k - \theta_{k-1}|^2 \ge \frac{2}{C_1(T)}(\theta_k^2 - \theta_{k-1}^2) - \frac{\tau}{C_1(T)}(\theta_k + \theta_{k-1})^2$. This enables us to rewrite the above inequality in the form

$$q_k - q_{k-1} \le \tau C(q_k + q_{k-1} + b_{k-1}), \tag{5.29}$$

with

$$q_k = \int_\Omega \left(\frac{2}{C_1(T)}\theta_k^2 + \kappa(\chi_{k-1})|\nabla\theta_k|^2 \right)\,\mathrm{d}x + \int_{\partial\Omega} h(x)(\theta_k - \theta_{k\Gamma})^2\,\mathrm{d}\sigma(x),$$

$$b_k = 1 + \int_{\partial\Omega} h(x)|\overset{\bullet}{\theta}_{k\Gamma}|^2\,\mathrm{d}\sigma(x),$$

and $C := C(T, R)$. Inequality (5.29) is equivalent to

$$q_k \le \frac{1 + \tau C}{1 - \tau C} q_{k-1} + \frac{\tau C}{1 - \tau C} b_{k-1} , \tag{5.30}$$

which yields

$$q_k \; \le \; \left(\frac{1 + \tau C}{1 - \tau C} \right)^k q_0 + \tau \frac{C}{1 - \tau C} \sum_{j=0}^{k-1} b_j \left(\frac{1 + \tau C}{1 - \tau C} \right)^{k-1-j} \; \le \; \mathrm{e}^{3k\tau C} \left(q_0 + \tau \sum_{j=0}^{k-1} b_j \right) \tag{5.31}$$

holding true for $\tau \le 1/(3C)$. We conclude for all $m = 1, \ldots, n$ that

$$\frac{1}{\tau} \sum_{k=1}^{m} \int_\Omega |\theta_k - \theta_{k-1}|^2 \; \mathrm{d}x + \int_\Omega |\nabla \theta_m|^2 \; \mathrm{d}x + \int_{\partial\Omega} h(x)(\theta_m - \theta_{m\Gamma})^2 \; \mathrm{d}\sigma(x) \le C(T, R). \tag{5.32}$$

Then, we introduce the piecewise constant and piecewise linear interpolants, for $t \in [(k-1)\tau, k\tau)$, $k = 1, \ldots, n$, by the formula

$$\underline{\theta}^{(\tau)}(x,t) = \theta_{k-1}(x), \quad \bar{\theta}^{(\tau)}(x,t) = \theta_k(x), \quad \hat{\theta}^{(\tau)}(x,t) = \theta_{k-1}(x) + (t - (k-1)\tau) \; \overset{\bullet}{\theta}_k(x) , \tag{5.33}$$

with a similar notation for U, χ, θ_Γ, and P_0. In particular, we set

$$\hat{e}^{(\tau)}(x,t) = e_1^R(\theta_{k-1}(x)) + \frac{1}{\tau}(t - (k-1)\tau)(e_1^R(\theta_k(x)) - e_1^R(\theta_{k-1}(x))) .$$

The estimate (5.32) gives immediately that

$$\hat{\theta}_t^{(\tau)} \text{ bounded in } L^2(0, T; L^2(\Omega)) ,$$

$$\nabla \bar{\theta}^{(\tau)} \text{ bounded in } L^\infty(0, T; L^2(\Omega)) ,$$

$$\int_0^T \int_\Omega \left(|\underline{\theta}^{(\tau)} - \hat{\theta}^{(\tau)}|^2 + |\bar{\theta}^{(\tau)} - \hat{\theta}^{(\tau)}|^2 \right)(x,t) \; \mathrm{d}x \, \mathrm{d}t \le C(T, R)\tau^2 .$$

The inequality $|\nabla \hat{\theta}^{(\tau)}(x,t)| \le \max\{|\nabla \theta_k(x)|, |\nabla \theta_{k-1}(x)|\} \le |\nabla \theta_k(x)| + |\nabla \theta_{k-1}(x)|$ for $t \in [(k-1)\tau, k\tau)$ entails that also $\nabla \hat{\theta}^{(\tau)}$ are bounded in $L^\infty(0, T; L^2(\Omega))$. By compact embedding of $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ into $C^0([0, T]; L^2(\Omega))$, letting τ tend to 0 and passing to subsequences if necessary, we get the convergences

$$\left. \begin{array}{rcll} \hat{\theta}^{(\tau)} & \to & \theta & \text{strongly in } C^0([0, T]; L^2(\Omega)) , \\ \underline{\theta}^{(\tau)} \to \theta \, , \; \bar{\theta}^{(\tau)} & \to & \theta & \text{strongly in } L^2(0, T; L^2(\Omega)) , \\ \hat{\theta}_t^{(\tau)} & \to & \theta_t & \text{weakly in } L^2(0, T; L^2(\Omega)) , \\ \nabla \bar{\theta}^{(\tau)} & \to & \nabla \theta & \text{weakly* in } L^\infty(0, T; L^2(\Omega)) . \end{array} \right\} \tag{5.34}$$

Now we estimate $\nabla \chi_k$ and ∇U_k as follows. From Eq. (5.13) it follows that

$$(\overset{\bullet}{U}_k(x) - \overset{\bullet}{U}_k(y))(U_k(x) - U_k(y)) \le C(T, R) \Big(|U_k(x) - U_k(y)|^2 + |\chi_k(x) - \chi_k(y)|^2$$

$$+ \; |\chi_{k-1}(x) - \chi_{k-1}(y)|^2 + |\theta_{k-1}(x) - \theta_{k-1}(y)|^2 + |x - y|^2 \Big)$$

and, analogously, from (5.14), we obtain

$$(\overset{\bullet}{\chi}_k(x) - \overset{\bullet}{\chi}_k(y))(\chi_k(x) - \chi_k(y)) \le C(T, R) \Big(|U_k(x) - U_k(y)|^2 + |\chi_k(x) - \chi_k(y)|^2$$

$$+ \; |\chi_{k-1}(x) - \chi_{k-1}(y)|^2 + |\theta_{k-1}(x) - \theta_{k-1}(y)|^2 \Big) .$$

Summing up the two previous inequalities, we get

$$
\begin{aligned}
(U_k(x) - U_k(y))^2 &+ (\chi_k(x) - \chi_k(y))^2 \leq (U_{k-1}(x) - U_{k-1}(y))^2 \\
&+ (\chi_{k-1}(x) - \chi_{k-1}(y))^2 \\
&+ \tau C(T,R)\Big((U_k(x) - U_k(y))^2 + (\chi_k(x) - \chi_k(y))^2 + (U_{k-1}(x) - U_{k-1}(y))^2 \\
&+ (\chi_{k-1}(x) - \chi_{k-1}(y))^2 + (\theta_{k-1}(x) - \theta_{k-1}(y))^2 + (x - y)^2\Big).
\end{aligned}
$$

We are again in the situation of Eq. (5.29), with $q_k = (U_k(x) - U_k(y))^2 + (\chi_k(x) - \chi_k(y))^2$, $b_k = (\theta_{k-1}(x) - \theta_{k-1}(y))^2 + (x - y)^2$. Hence, by (5.31), we obtain

$$
\begin{aligned}
(U_k(x) &- U_k(y))^2 + (\chi_k(x) - \chi_k(y))^2 \\
&\leq C(T,R)\Big((U_0(x) - U_0(y))^2 + (\chi_0(x) - \chi_0(y))^2 \\
&\qquad + \tau \sum_{j=0}^{k-1} (\theta_j(x) - \theta_j(y))^2 + (x - y)^2\Big).
\end{aligned}
\tag{5.35}
$$

We now claim that

$$
\nabla \hat{U}^{(\tau)}, \ \nabla \hat{\chi}^{(\tau)} \text{ are bounded in } L^\infty(0, T; L^2(\Omega)).
\tag{5.36}
$$

In fact, (5.35) is an inequality of the form $|w(x) - w(y)|^2 \leq \frac{1}{n} \sum_{j=0}^{n-1} |v_j(x) - v_j(y)|^2$ for a.e. $x, y \in \Omega$, for functions $w \in L^2(\Omega)$, $v_j \in H^1(\Omega)$. We choose an arbitrary C^1-function ψ with compact support $\bar{\Omega}_0 \subset \Omega$, a unit vector e_i in the direction of the i-th coordinate, and any $\delta > 0$ sufficiently small such that $\bar{\Omega}_0 + se_i \subset \Omega$ for all $s \in [-\delta, \delta]$. An easy computation yields

$$
\begin{aligned}
\frac{1}{\delta} \int_\Omega w(x)(\psi(x) - \psi(x - \delta e_i))\,\mathrm{d}x &= -\frac{1}{\delta} \int_{\Omega_0} \psi(x)(w(x + \delta e_i) - w(x))\,\mathrm{d}x \\
&\leq \frac{1}{\delta} \|\psi\|_{L^2(\Omega)} \left(\int_{\Omega_0} |w(x + \delta e_i) - w(x)|^2\,\mathrm{d}x\right)^{1/2} \\
&\leq \frac{1}{\delta} \|\psi\|_{L^2(\Omega)} \left(\frac{1}{n} \sum_{j=0}^{n-1} \int_{\Omega_0} |v_j(x + \delta e_i) - v_j(x)|^2\,\mathrm{d}x\right)^{1/2} \\
&= \frac{1}{\delta} \|\psi\|_{L^2(\Omega)} \left(\frac{1}{n} \sum_{j=0}^{n-1} \int_{\Omega_0} \left|\int_0^\delta \frac{\partial v_j}{\partial x_i}(x + se_i)\,\mathrm{d}s\right|^2 \mathrm{d}x\right)^{1/2} \\
&\leq \frac{1}{\delta^{1/2}} \|\psi\|_{L^2(\Omega)} \left(\frac{1}{n} \sum_{j=0}^{n-1} \int_0^\delta \int_{\Omega_0} \left|\frac{\partial v_j}{\partial x_i}(x + se_i)\right|^2 \mathrm{d}x\,\mathrm{d}s\right)^{1/2} \\
&\leq \|\psi\|_{L^2(\Omega)} \left(\frac{1}{n} \sum_{j=0}^{n-1} \left\|\frac{\partial v_j}{\partial x_i}\right\|_{L^2(\Omega)}^2\right)^{1/2}.
\end{aligned}
\tag{5.37}
$$

Letting δ tend to 0 in (5.37), we obtain

$$
\int_\Omega w(x) \frac{\partial \psi}{\partial x_i}(x)\,\mathrm{d}x \leq \|\psi\|_{L^2(\Omega)} \left(\frac{1}{n} \sum_{j=0}^{n-1} \left\|\frac{\partial v_j}{\partial x_i}\right\|_{L^2(\Omega)}^2\right)^{1/2}
$$

for all smooth test functions with compact support in Ω, hence

$$\left\| \frac{\partial w}{\partial x_i} \right\|_{L^2(\Omega)} \leq \left(\frac{1}{n} \sum_{j=0}^{n-1} \left\| \frac{\partial v_j}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Using now the previous estimate on $\nabla \bar{\theta}^\tau$ and the assumptions on the initial conditions, we get (5.36) from (5.35).

By virtue of (5.22), $\hat{U}_t^{(\tau)}, \hat{\chi}_t^{(\tau)}$ are bounded in $L^\infty(\Omega_T)$. Furthermore,

$$\int_0^T \int_\Omega |\bar{U}^{(\tau)} - \hat{U}^{(\tau)}|^2(x,t) \; \mathrm{d}x \, \mathrm{d}t \;\; \leq \;\; C(T,R)\tau^2,$$

$$\int_0^T \int_\Omega \left(|\underline{\chi}^{(\tau)} - \hat{\chi}^{(\tau)}|^2 + |\bar{\chi}^{(\tau)} - \hat{\chi}^{(\tau)}|^2 \right)(x,t) \; \mathrm{d}x \, \mathrm{d}t \;\; \leq \;\; C(T,R)\tau^2,$$

so that the convergences (5.34) take place also for U and χ. We now rewrite (5.12)–(5.14) in terms of the functions $\underline{\theta}^{(\tau)}, \bar{\theta}^{(\tau)}, \hat{e}^{(\tau)}, \underline{\chi}^{(\tau)}, \bar{\chi}^{(\tau)}, \hat{\chi}^{(\tau)}, \bar{U}^{(\tau)}, \hat{U}^{(\tau)}, \bar{\theta}_\Gamma^{(\tau)}, \bar{P}_0^{(\tau)}$. The above estimates allow us to pass to the limit as $\tau \searrow 0$ and obtain a solution for the following truncated problem

$$\int_\Omega c(\chi) e_1^R(\theta)_t w(x) \, \mathrm{d}x + \int_\Omega \kappa(\chi) \nabla\theta \cdot \nabla w(x) \, \mathrm{d}x$$

$$= \int_{\partial\Omega} h(x)(\theta_\Gamma(x,t) - \theta)w(x) \, \mathrm{d}\sigma(x)$$

$$- \int_\Omega \left(U_t \mathcal{A}(U,\chi,x,t) + \chi_t \big(\mathcal{C}(U,\chi) + c'(\chi)(e_1^R(\theta) - f_1(1)) \big) \right) w(x) \, \mathrm{d}x, \quad (5.38)$$

$$U_t - Q_R(\theta) = -\mathcal{A}(U,\chi,x,t), \tag{5.39}$$

$$\gamma(\theta)\chi_t + \mathcal{B}_R(\chi,\theta) + \partial I(\chi) \ni -\mathcal{C}(U,\chi), \tag{5.40}$$

where

$$\mathcal{B}_R(\chi,\theta) := c'(\chi)(f_1^R(\theta) - f_1(1)) - 2Q_R(\theta), \tag{5.41}$$

$\mathcal{A}$ and $\mathcal{C}$ are defined in (4.2), (4.4), and (5.38) is to be satisfied for all test functions $w \in W^{1,2}(\Omega)$ and a.e. $t \in (0,T)$, while (5.39)–(5.40) hold a.e. in Ω_T.

The next step consists in proving that θ remains uniformly bounded also from above independently of R, so that the truncation does not become active if R is sufficiently large. Note first that the "truncated" version of the entropy balance (3.6) reads here

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (c(\chi)s_1^R(\theta) + 2\chi + U) \, \mathrm{d}x = \int_{\partial\Omega} \frac{h(x)}{Q_R(\theta)}(\theta_\Gamma(x,t) - \theta) \, \mathrm{d}\sigma(x) \tag{5.42}$$

$$+ \int_\Omega \left(\frac{\kappa(\chi)|\nabla Q_R(\theta)|^2}{Q_R(\theta)^2} + \frac{\gamma(\theta)\chi_t^2}{Q_R(\theta)} + \frac{U_t^2}{Q_R(\theta)} \right) \mathrm{d}x.$$

Arguing as in Section 3, we obtain the following counterpart of the extended energy balance (3.10),

$$\int_\Omega \left(c(\chi)(e_1^R(\theta) - f_1(1)) + \frac{\lambda(\chi)}{2}(U - 1 + \chi)^2 \right)(x, t)\,\mathrm{d}x$$

$$+ \int_\Omega \left(U + 2\chi - gx_3 U \right)(x, t)\,\mathrm{d}x + \frac{K_\Gamma}{2}\left(U_\Omega(t) + P_0(t) + \frac{g\zeta_\Gamma}{K_\Gamma} \right)^2$$

$$+ \bar{\theta}_\Gamma \int_0^t \int_\Omega \left(\frac{\kappa(\chi)|\nabla Q_R(\theta)|^2}{Q_R^2(\theta)} + \frac{\gamma(\theta)\chi_t^2}{Q_R(\theta)} + \frac{U_t^2}{Q_R(\theta)} \right)(x, \xi)\,\mathrm{d}x\,\mathrm{d}\xi$$

$$+ \int_0^t \int_{\partial\Omega} \frac{h(x)}{Q_R(\theta)}(\theta - \theta_\Gamma(x, \xi))(Q_R(\theta) - \bar{\theta}_\Gamma)\,\mathrm{d}\sigma(x)\,\mathrm{d}\xi$$

$$= \quad E^0 + E_\Gamma^0 - \bar{\theta}_\Gamma S^0 + \bar{\theta}_\Gamma \int_\Omega \left(c(\chi)s_1^R(\theta) + 2\chi + U \right)(x, t)\,\mathrm{d}x$$

$$+ \int_0^t K_\Gamma (P_0)_t(\xi) \left(U_\Omega(\xi) + P_0(\xi) + \frac{g\zeta_\Gamma}{K_\Gamma} \right)\,\mathrm{d}\xi, \tag{5.43}$$

which holds for every solution to (5.38)–(5.40) and every $t \in (0, T)$.

5.4. Uniform upper bound for θ. We choose R large enough such that $B(R) > \theta^* \geq \bar{\theta}_\Gamma$. Then $(\theta - \theta_\Gamma(x, \xi))(Q_R(\theta) - \bar{\theta}_\Gamma) \geq (Q_R(\theta) - \theta_\Gamma(x, \xi))(Q_R(\theta) - \bar{\theta}_\Gamma)$. We may therefore argue as at the end of Section 3 and obtain from (5.43) for all $t \in (0, T)$ that

$$\int_\Omega \left(e_1^R(\theta) + U^2 \right)(x, t)\,\mathrm{d}x + \int_0^t \int_{\partial\Omega} \frac{h(x)}{Q_R(\theta)}\left(Q_R(\theta) - \sqrt{\bar{\theta}_\Gamma \theta_\Gamma(x, \xi)} \right)^2\,\mathrm{d}\sigma(x)\,\mathrm{d}\xi$$

$$+ \int_0^t \int_\Omega \left(\frac{\kappa(\chi)|\nabla Q_R(\theta)|^2}{Q_R^2(\theta)} + \frac{\gamma(\theta)\chi_t^2}{Q_R(\theta)} + \frac{U_t^2}{Q_R(\theta)} \right)(x, \xi)\,\mathrm{d}x\,\mathrm{d}\xi$$

$$\leq C\left(1 + \int_0^t \int_{\partial\Omega} h(x)\left(\sqrt{\theta_\Gamma(x, \xi)} - \sqrt{\bar{\theta}_\Gamma} \right)^2\,\mathrm{d}\sigma(x)\,\mathrm{d}\xi + \int_0^t |(P_0)_t(\xi)|\,\mathrm{d}\xi \right). \tag{5.44}$$

In order to perform the Moser iteration scheme on θ as in [11, Prop. 3.6], we need first to estimate U and U_t in terms of θ. Rewriting (5.39) as

$$U_t + \lambda(\chi)U = Q_R(\theta) + G(x, t)$$

where, by virtue of (5.44), $G(x, t)$ is bounded above by a positive constant G_0. Denoting $\hat{\lambda}(x, t) := \int_0^t \lambda(\chi(x, s))\,\mathrm{d}s$, we obtain the formula

$$U(x, t) = \mathrm{e}^{-\hat{\lambda}(x, t)}U_0(x) + \int_0^t \mathrm{e}^{\hat{\lambda}(x, \xi) - \hat{\lambda}(x, t)}\left(Q_R(\theta) + G \right)(x, \xi)\,\mathrm{d}\xi.$$

Using Hypo. 4.1 (iii), we get the estimate

$$|U(x, t)| \leq |U_0(x)| + \int_0^t \mathrm{e}^{\underline{\lambda}(t - \xi)}Q_R(\theta)(x, \xi)\,\mathrm{d}\xi + \frac{G_0}{\underline{\lambda}} \tag{5.45}$$

and

$$|U_t(x, t)| \leq \bar{\lambda}|U(x, t)| + Q_R(\theta)(x, t) + G_0. \tag{5.46}$$

Now we are ready in order to start the Moser iteration scheme. Choose in (5.38) $w(x) = u^p$, $u = \psi_R(\theta) := (Q_R(\theta) - R)^+$, with any $p > 1$ and with R larger than the constants in Hypothesis 4.1. Then

$$\int_\Omega c(\chi)(e_1^R(\theta))_t u^p \; \mathrm{d}x + \frac{4p}{(p+1)^2} \int_\Omega \kappa(\chi) |\nabla u^{\frac{p+1}{2}}|^2 \, \mathrm{d}x + \int_{\partial\Omega} h(x) u^{p+1} \, \mathrm{d}s(x)$$

$$\leq - \int_\Omega \left(U_t \mathcal{A}(U, \chi, x, t) - \chi_t \big(\mathcal{C}(U, \chi) - c'(\chi)(e_1^R(\theta) - f_1(1)) \big) \right) u^p \, \mathrm{d}x \,. \qquad (5.47)$$

Put $E_p^R(\theta) = \int_0^\theta c_1^R(r) \psi_R^p(r) \, \mathrm{d}r$. Then, we can rewrite (5.47) as

$$\int_\Omega \big(c(\chi) E_p^R(\theta) \big)_t \, \mathrm{d}x + \frac{4p}{(p+1)^2} \int_\Omega \kappa(\chi) |\nabla u^{\frac{p+1}{2}}|^2 \, \mathrm{d}x + \int_{\partial\Omega} h(x) u^{p+1} \, \mathrm{d}s(x)$$

$$\leq - \int_\Omega U_t \mathcal{A}(U, \chi, x, t) u^p \, \mathrm{d}x$$

$$- \int_\Omega \chi_t \left(\mathcal{C}(U, \chi) u^p + c'(\chi) \big((e_1^R(\theta) - f_1(1)) u^p - E_p^R(\theta) \big) \right) \, \mathrm{d}x \,. \qquad (5.48)$$

We now prove that the last integral in (5.48) is non-positive if R is sufficiently large. First of all let us note that, if $\chi_t = 0$ then it vanishes. Hence, let us consider the case $\chi_t \neq 0$. Then, from (5.40) it follows that $\chi_t(\gamma(\theta)\chi_t + \mathcal{B}_R(\chi, \theta) + \mathcal{C}(U, \chi)) = 0$, hence

$$\chi_t = -\frac{1}{\gamma(\theta)} \left(\mathcal{B}_R(\chi, \theta) + \mathcal{C}(U, \chi) \right) \,.$$

The last integral in (5.48) is of the form $- \int_\Omega \frac{1}{\gamma(\theta)} I_1 \times I_2 \, \mathrm{d}x$, where

$$
\begin{aligned}
I_1 &:= \chi_t = -\mathcal{C}(U, \chi) + 2Q_R(\theta) + c'(\chi)(f_1(1) - f_1^R(\theta)), \\
I_2 &:= \mathcal{C}(U, \chi) u^p + c'(\chi) \big((e_1^R(\theta) - f_1(1)) u^p - E_p^R(\theta) \big) \\
&= \big(\mathcal{C}(U, \chi) - c'(\chi) f_1(1) \big) u^p + c'(\chi) \big(e_1^R(\theta) u^p - E_p^R(\theta) \big) \,.
\end{aligned}
$$

We can now estimate from below the last term as follows

$$e_1^R(\theta) u^p - E_p^R(\theta) = \int_0^\theta p e_1^R(r) u^{p-1} \, \mathrm{d}r \geq e_1(R) u^p \,.$$

We have $I_2 = 0$ if $\theta \leq R$, while for $\theta > R$ we have by Hypothesis 4.1 (i)

$$
\begin{aligned}
I_1 &\geq -|\mathcal{C}(U, \chi)| + 2R + \underline{c} |f_1^R(R) - f_1(1)|, \\
I_2 &\geq u^p \left(-|\mathcal{C}(U, \chi)| + \underline{c} f_1(1) + \underline{c} e_1^R(R) \right) \,.
\end{aligned}
$$

By virtue of (5.22), we have $|\mathcal{C}(U, \chi)| \leq C(B^2(R) + 1)$. Referring to (5.11), we conclude that there exists $R_0 > 1$ larger than all constants in Hypothesis 4.1 such that for $R \geq R_0$ we have in (5.48)

$$- \int_\Omega \chi_t \left(\mathcal{C}(U, \chi) u^p + c'(\chi) \big((e_1^R(\theta) - f_1(1)) u^p - E_p^R(\theta) \big) \right) \, \mathrm{d}x \leq 0 \,.$$

Let us fix now $R > R_0$ and continue the Moser estimate, rewriting (5.48) as follows

$$\int_\Omega \big(c(\chi) E_p^R(\theta) \big)_t \, \mathrm{d}x + \frac{4p}{(p+1)^2} \int_\Omega \kappa(\chi) |\nabla u^{\frac{p+1}{2}}|^2 \, \mathrm{d}x + \int_{\partial\Omega} h(x) u^{p+1} \, \mathrm{d}s(x)$$

$$\leq - \int_\Omega U_t \mathcal{A}(U, \chi, x, t) u^p \, \mathrm{d}x \,. \qquad (5.49)$$

We have $c(\chi)E_p^R(u) \geq \frac{c_* c^*}{p+1} u^{p+1}$, $\kappa(\chi) \geq \kappa_*$. Integrating (5.49) in time, we obtain, using Hypo. 4.1 (i),(ii),(iv), as well as the estimates (5.45)–(5.46) and the fact that $u(x,0) \equiv 0$, that

$$\frac{c_* c^*}{p+1} \int_\Omega u^{p+1}(x,t) \, \mathrm{d}x + \frac{4p\kappa_*}{(p+1)^2} \int_0^t \int_\Omega \left| \nabla u^{\frac{p+1}{2}} \right|^2 (x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi$$

$$\leq \int_0^t \int_\Omega u^p(x,\xi) \left(1 + Q_R(\theta(x,\xi)) + \int_0^\xi e^{-\underline{\lambda}(\xi - \eta)} Q_R(\theta(x,\eta)) \, \mathrm{d}\eta \right) r(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \, .$$

The function $r(x,t) = C\mathcal{A}(U,\chi,x,t)$, where C is a suitable constant, has norm in $L^\infty(0,T;L^2(\Omega))$ bounded independently of R by virtue of (5.44). Note that $Q_R(\theta) \leq u + R$. Hence, the function $v := u/R$ satisfies for all $p > 1$ the inequality

$$\frac{c_* c^*}{p+1} \int_\Omega v^{p+1}(x,t) \, \mathrm{d}x + \frac{4p\kappa_*}{(p+1)^2} \int_0^t \int_\Omega \left| \nabla v^{\frac{p+1}{2}} \right|^2 (x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi$$

$$\leq \int_0^t \int_\Omega v^p(x,\xi) \left(1 + v(x,\xi) + \int_0^\xi e^{-\underline{\lambda}(\xi - \eta)} v(x,\eta) \, \mathrm{d}\eta \right) r(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi.$$

The argument of [10, Prop. 4.5] yields $\|v\|_{L^\infty(\Omega_T)} \leq \bar{C}$ with a constant $\bar{C}$ independent of R and T. Consequently,

$$\|Q_R(\theta)\|_{L^\infty(\Omega_T)} \leq (1 + \bar{C})R.$$

Choosing R sufficiently large such that $B(R) > (1+\bar{C})R$, we can remove the truncation from (5.38)–(5.40), concluding in this way the proof of existence of a bounded solution to (4.5)–(4.7). If moreover (3.12) holds, then $r \in L^\infty(0,\infty;L^2(\Omega))$, and the upper bound holds globally in Ω_∞. Indeed, the lower bound for θ in Subsection 5.2 is independent of the time step τ and is preserved when $\tau \searrow 0$.

6. **Uniqueness and continuous data dependence.** In this Section, we prove uniqueness and continuous data dependence of solutions under the more restrictive assumption that $\kappa(r) = \bar{\kappa} \in \mathbb{R}^+$ for all $r \in \mathbb{R}^+$.

In what follows, we denote by $R_0, R_1, R_2, \ldots$ suitable constants that possibly depend on T, but not on the solutions. We first rewrite Eq. (4.5) in the form

$$\int_\Omega (c(\chi)(e_1(\theta) - f_1(1)))_t w(x) \, \mathrm{d}x + \int_\Omega \bar{\kappa} \nabla \theta \cdot \nabla w(x) \, \mathrm{d}x$$

$$+ \int_{\partial\Omega} h(x)(\theta - \theta_\Gamma) w(x) \, \mathrm{d}\sigma(x) = - \int_\Omega \left(U_t \mathcal{A}(U,\chi,x,t) + \chi_t \mathcal{C}(U,\chi) \right) w(x) \, \mathrm{d}x \, .$$

$$(6.1)$$

We fix two sets of data and two corresponding solutions, and label them with indices 1 and 2. We denote $\hat{\theta} = \theta_1 - \theta_2$, $\hat{\chi} = \chi_1 - \chi_2$, $\hat{\chi}_0 = \chi_{01} - \chi_{02}$, $\hat{\theta}_0 = \theta_{01} - \theta_{02}$, $\hat{\theta}_\Gamma = \theta_{\Gamma 1} - \theta_{\Gamma 2}$, $\hat{U} = U_1 - U_2$, $\hat{U}_0 = U_{01} - U_{02}$, $\hat{\Theta}(x,t) = \int_0^t \hat{\theta}(x,\tau) \, \mathrm{d}\tau$, $\hat{\Theta}_\Gamma(x,t) = \int_0^t \hat{\theta}_\Gamma(x,\tau) \, \mathrm{d}\tau$, $\hat{P}_0 = P_{01} - P_{02}$. Within the range $\theta^\flat(T) \leq \theta \leq \theta^\sharp(T)$ and $\chi \in [0,1]$, $|\chi_t| \leq C$ of admissible values for the solutions, and, thanks to Hypo. 4.1, all nonlinearities in (4.5)–(4.7) are Lipschitz continuous. We integrate the difference of the two equations (6.1), written for (θ_1, U_1, χ_1) and (θ_2, U_2, χ_2), from 0 to t, and

test by $w = \theta_1 - \theta_2$. This yields

$$
\int_\Omega |\hat{\theta}(x,t)|^2 \, \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \left(R_0 \int_\Omega |\nabla \hat{\Theta}(x,t)|^2 \, \mathrm{d}x + R_1 \int_{\partial\Omega} h(x)(\hat{\Theta} - \hat{\Theta}_\Gamma)^2(x,t) \, \mathrm{d}\sigma(x) \right)
$$

$$
\leq R_2 \Big(\|\hat{\theta}_0\|^2_{L^2(\Omega)} + \int_0^t |\hat{P}_0(\xi)|^2 \, \mathrm{d}\xi + \int_{\partial\Omega} h(x)|\hat{\Theta} - \hat{\Theta}_\Gamma|\,|\hat{\theta}_\Gamma|(x,t) \, \mathrm{d}\sigma(x)
$$

$$
+ \int_\Omega \Big(\int_0^t \big(|\hat{\chi}_t(x,\xi)| + |\hat{U}_t(x,\xi)| + |\hat{\chi}(x,\xi)| + |\hat{U}(x,\xi)| \big) \, \mathrm{d}\xi \Big)^2 \, \mathrm{d}x \Big). \tag{6.2}
$$

Repeating the argument of [10, Proposition 4.3] or [4, Proposition 3.4] about the L^1-Lipschitz continuity of solution operators to gradient flows, we obtain for the solutions to (4.6)–(4.7) for a.e. $(x,t) \in \Omega_T$ the estimate

$$
\int_0^t \big(|\hat{\chi}_t(x,\tau)| + |\hat{U}_t(x,\tau)| \big)(x,\xi) \, \mathrm{d}\xi + |\hat{\chi}(x,t)| + |\hat{U}(x,t)| \tag{6.3}
$$

$$
\leq R_3 \left(\int_0^t |\hat{P}_0(\xi)| \, \mathrm{d}\xi + |\hat{\chi}_0(x)| + |\hat{U}_0(x)| + \int_0^t \left(|\hat{\theta}(x,\xi)| + \int_\Omega |\hat{\theta}(y,\xi)| \, \mathrm{d}y \right) \mathrm{d}\xi \right).
$$

Integrating (6.2) from 0 to t and using (6.3) together with Gronwall's argument, we obtain for each $t \in [0,T]$ the estimate

$$
\int_0^t \int_\Omega |\hat{\theta}(x,\xi)|^2 \, \mathrm{d}x \, \mathrm{d}\xi + \int_\Omega |\hat{\chi}(x,t)|^2 \, \mathrm{d}x + \int_\Omega |\hat{U}(x,t)|^2 \, \mathrm{d}x
$$

$$
\leq R_4 \Big(\|\hat{\theta}_0\|^2_{L^2(\Omega)} + \|\hat{\chi}_0\|^2_{L^2(\Omega)} + \|\hat{U}_0\|^2_{L^2(\Omega)} + \int_0^t |\hat{P}_0(\xi)|^2 \, \mathrm{d}\xi
$$

$$
+ \int_0^t \int_{\partial\Omega} h(x)\hat{\theta}_\Gamma^2(x,\xi) \, \mathrm{d}\sigma(x) \, \mathrm{d}\xi \Big). \tag{6.4}
$$

This concludes the proof of uniqueness of solutions and of Theorem 4.2.

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