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THE PREISACH HYSTERESIS MODEL: ERROR BOUNDS FOR NUMERICAL IDENTIFICATION AND INVERSION

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Abstract. A structure analysis of the Preisach model in a variational setting is carried out by means of an auxiliary hyperbolic equation with memory variable playing the role of time, and amplitude of cycles as spatial variable. Using this representation, we propose an algorithm and derive error estimates for the identification of the Preisach density function and for an approximate inversion of the Preisach operator.

Introduction. This text offers some mathematical background for numerical treatment of the Preisach hysteresis model. It was motivated by stimulating discussions with Ciro Visone and Daniele Davino at the University del Sannio in Benevento about the inversion method described in [3], and the author is grateful for useful suggestions and practical comments.

We work here with the Preisach operator in a variational setting following [8], restricting ourselves to time discrete inputs and outputs similarly as in [1]. The extension to arbitrary regulated, and especially continuous functions of time, is straightforward in terms of the Kurzweil integral variational formulation as in [11]. In this framework, the Nemytskii operator and the Prandtl-Ishlinskii operator turn out to be special cases of the Preisach model.

An explicit inversion formula for the Prandtl-Ishlinskii operator was derived in [7]. For a general Preisach operator, no inversion formula is known. The first paper dealing with the question of invertibility of Preisach operators is [2]. The proof of existence and continuity of the Preisach inverse there deals with evolving curves in the Preisach plane and is based on geometrical intuition. We present here a new purely analytical and short proof of the Lipschitz continuity of the inverse Preisach operator with respect to the sup-norm, propose algorithms for identification of the Preisach density and real time numerical inversion, and derive error estimates for the schemes. This is different from [4], where the Preisach density is identified by means of a least squares technique. Here, in Section 4, assuming output measurement error δ, the density is approximated with error of the order $\delta^{1/3}$. The inversion error is proportional to the discretization error and remains bounded independently of the number of cycles.

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Both the identification and inversion problems can then be reduced to properties of solutions to an auxiliary wave equation with Cauchy data, where the cycle amplitude can be interpreted as spatial variable, and the memory parameter plays the role of time. As a by-product, we obtain an analytical formula for superpositions of the Preisach operator with the Nemytskii operator and with the Prandtl-Ishlinskii operator, extending thus the classification of hysteresis operators from [9, 10] to more general, in particular non-symmetric, cases.

The paper is organized as follows. In Section 1 we introduce the Preisach operator, establish its basic properties, and state the main results. Section 2 is devoted to a structure analysis of the Preisach model and superposition formulas. The proof of the Lipschitz continuity of the Preisach operator and its inverse is carried out in Section 3. The identification algorithm is investigated in Section 4, and error estimates for our numerical Preisach inversion algorithm are obtained in Section 5.

1. Time discrete Preisach operator.

We consider evolution processes on possibly infinite discrete time sequences. To this aim, we define the space

$$\ell^\infty = \{\mathbf{u} = \{u_j\}_{j=0}^\infty : \sup_{j\in\mathbb{N}\cup\{0\}} |u_j| < \infty\}\,. \tag{1.1}$$

endowed with seminorms $|\mathbf{u}|_k = \sup_{0\le j\le k} |u_j|$ for $k \in \mathbb{N}\cup\{0\}$. It is a Banach space with norm $|\mathbf{u}|_\infty = \sup_{j\in\mathbb{N}\cup\{0\}} |u_j|$.

For a given parameter $r \ge 0$, an input $\mathbf{u} = \{u_j\}_{j=0}^\infty \in \ell^\infty$, and an initial condition $\lambda_{-1}(r) \in \mathbb{R}$, the sequence

$$\lambda_j(r) = \max\{u_j - r, \min\{\lambda_{j-1}(r), u_j + r\}\} \tag{1.2}$$

defines the *discrete play operator* with threshold r. Obviously,

$$\lambda_j(0) = u_j \tag{1.3}$$

for all j. The full one-parameter system of play operators for all $r \ge 0$ gives a complete characterization of all hysteresis operators with return point memory, see [1, Theorem 2.7.7]. It is convenient to introduce the set

$$\Lambda = \{\lambda \in W^{1,\infty}_{\rm loc}(0,\infty) : |\lambda'(r)| \le 1 \text{ a.e.}\} \tag{1.4}$$

of admissible memory configurations (the memory state space), where the prime denotes derivative with respect to the memory variable r. It is easy to see that if $\lambda_{j-1} \in \Lambda$, then λ_j given by (1.2) also belongs to Λ. Given $\lambda_{-1} \in \Lambda$ and $\mathbf{u} \in \ell^\infty$, we use for the play operator the notation

$$\boldsymbol{\lambda}(r) = \{\lambda_j(r)\}_{j=0}^\infty = \mathfrak{p}_r[\mathbf{u}, \lambda_{-1}]\,. \tag{1.5}$$

The play operator is Lipschitz continuous in the following sense.

Lemma 1.1. *Let $\mathbf{u}^{(i)} \in \ell^\infty$ and $\lambda_{-1}^{(i)}$ be given, and let $\boldsymbol{\lambda}^{(i)}(r) = \mathfrak{p}_r[\mathbf{u}^{(i)}, \lambda_{-1}^{(i)}]$, $i = 1, 2$. Then for all $j \in \mathbb{N}\cup\{0\}$ and $r \ge 0$ we have*

$$|\lambda_j^{(1)}(r) - \lambda_j^{(2)}(r)| \le \max\{|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|_j, |\lambda_{-1}^{(1)}(r) - \lambda_{-1}^{(2)}(r)|\}\,, \tag{1.6}$$

where the symbol $|\cdot|_j$ denotes the seminorm defined at the beginning of this section.

Proof. We first check that (1.2) can be characterized by the variational inequality

$$|u_j - \lambda_j(r)| \le r\,, \tag{1.7}$$

$$(\lambda_j(r) - \lambda_{j-1}(r))(u_j - \lambda_j(r) - y) \ge 0 \quad \forall |y| \le r\,. \tag{1.8}$$

Indeed, let (1.2) be fulfilled. Then $u_j - r \le \lambda_j(r) \le u_j + r$, hence (1.7) holds. Assuming (1.7), condition (1.2) is equivalent to the statement

If $\lambda_j(r) > \lambda_{j-1}(r)$ then $\lambda_j(r) = u_j - r$, and if $\lambda_j(r) < \lambda_{j-1}(r)$ then $\lambda_j(r) = u_j + r$.

This proves the equivalence of (1.2) with (1.7) + (1.8). We thus have

$$\begin{aligned}
(\lambda_j^{(1)}(r) - \lambda_{j-1}^{(1)}(r))((u_j^{(1)} - u_j^{(2)}) - (\lambda_j^{(1)}(r) - \lambda_j^{(2)}(r))) &\ge 0\,, \\
(\lambda_j^{(2)}(r) - \lambda_{j-1}^{(2)}(r))((u_j^{(2)} - u_j^{(1)}) - (\lambda_j^{(2)}(r) - \lambda_j^{(1)}(r))) &\ge 0\,.
\end{aligned}$$

Summing up the above inequalities yields

$$((\lambda_j^{(1)}(r) - \lambda_j^{(2)}(r)) - (\lambda_{j-1}^{(1)}(r) - \lambda_{j-1}^{(2)}(r)))((\lambda_j^{(1)}(r) - \lambda_j^{(2)}(r)) - (u_j^{(1)} - u_j^{(2)})) \le 0\,,$$

which implies in turn

$$|\lambda_j^{(1)}(r) - \lambda_j^{(2)}(r)| \le \max\{|\lambda_{j-1}^{(1)}(r) - \lambda_{j-1}^{(2)}(r)|, |u_j^{(1)} - u_j^{(2)}|\}\,,$$

and (1.6) follows by induction. □

The variational formulation of the Preisach operator as a nonlinear combination of plays goes back to [8]. Here, in the discrete case, we define the Preisach output sequence $\mathbf{w} = \{w_j\}_{j=0}^{\infty}$ by the integral formula

$$w_j = f(u_j) + \int_0^{\infty} g(r, \lambda_j(r))\,\mathrm{d}r \quad \text{for} \quad j = 0, 1, 2, \dots\,, \tag{1.9}$$

where f, g are functions satisfying Hypothesis 1.2 below, and λ_j is defined by (1.2). Formula (1.9) enables us to introduce the Preisach operator P defined in (a subset of) $\ell^\infty \times \Lambda$ with values in ℓ^∞, and rewrite (1.9) in the form

$$\mathbf{w} = \{w_j\}_{j=0}^{\infty} = P[\mathbf{u}, \lambda_{-1}]. \tag{1.10}$$

An important special case – the so-called Prandtl-Ishlinskii operator – corresponds to the choice

$$f(v) = \Gamma'(0)v\,, \quad g(r, v) = \Gamma''(r)v\,, \tag{1.11}$$

where $\Gamma : [0, \infty) \to [0, \infty)$ is a given increasing function with locally Lipschitz continuous derivative such that $\Gamma(0) = 0$.

Hypothesis 1.2. We consider functions $f : \mathbb{R} \to \mathbb{R}$ and $\mu : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ with the following properties.

(i) f is locally Lipschitz continuous, $f(0) = 0$, $0 < b < f'(u) < F(|u|)$ a.e., where $F : [0, \infty) \to (0, \infty)$ is a nondecreasing function;
(ii) $\mu \in L^\infty((0, \infty) \times \mathbb{R})$, $|\mu(r, v)| \le M$ a.e., $g(r, v) = \int_0^v \mu(r, \xi)\,\mathrm{d}\xi$;
(iii) $\exists \mu_0 \in L^1(0, \infty)$: $\mu_0 \ge 0$ a.e., $\int_0^\infty \mu_0(r)\,\mathrm{d}r = b_0 < b$, $\mu(r, v) \ge -\mu_0(r)$ a.e.

We see that the integral in (1.9) is not necessarily well defined for all choices of the initial condition $\lambda_{-1} \in \Lambda$. We therefore consider a special class of initial conditions

$$\Lambda_K = \{\lambda \in \Lambda : \lambda(r) = 0 \text{ for } r \ge K\} \tag{1.12}$$

for $K > 0$. If $\lambda_{-1} \in \Lambda_K$ and $\mathbf{u} = \{u_j\}_{j=0}^{\infty}$ is an arbitrary input sequence, we easily prove by induction that λ_j defined by (1.2) has the property

$$\lambda_j \in \Lambda_{K_j}\,, \quad K_j = \max\{K, |\mathbf{u}|_j\}\,. \tag{1.13}$$

Hence, formula (1.9) is meaningful whenever $\lambda_{-1} \in \Lambda_K$ for some $K > 0$.

For the reader who is more familiar with the original "non-ideal relay" definition of the Preisach operator as in [12], let us recall that $\mu(r, v)$ is the relative density distribution of relays with thresholds $v - r$ and $v + r$, and the curve $v = \lambda_j(r)$ describes the interface between the $+1$ and -1 regions in the Preisach half-plane with coordinates $r > 0$, $v \in \mathbb{R}$ (that is, rotated by 225° with respect to the Preisach (α, β)-half-plane (cf. [6, 13]) with $\alpha = v - r$, $\beta = v + r$). The equivalence of the two concepts is proved in [8].

Analytical properties of the Preisach operator and its inverse are investigated in detail in [2]. We state here the result on local Lipschitz continuity of P and global Lipschitz continuity of its inverse in new form and give a simple analytical proof, which will enable us to derive error bounds for numerical inversion.

Theorem 1.3. *Let Hypothesis 1.2 hold, and let $\lambda_{-1}^{(1)}, \lambda_{-1}^{(2)} \in \Lambda_K$ be given for some $K > 0$.*

(i) *Let $\mathbf{u}^{(i)} = \{u_j^{(i)}\}_{j=0}^{\infty} \in \ell^\infty$, $i = 1, 2$, be given sequences, and let $\mathbf{w}^{(i)} = \{w_j^{(i)}\}_{j=0}^{\infty}$ be the output sequences defined by (1.9). Then for all $j \in \mathbb{N} \cup \{0\}$ we have*

$$|w_j^{(1)} - w_j^{(2)}| \le \max\left\{(F(K_j) + MK_j)|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|_j, M\int_0^\infty |\lambda_{-1}^{(1)}(r) - \lambda_{-1}^{(2)}(r)|\,\mathrm{d}r\right\}, \tag{1.14}$$

where $K_j = \max\{K, |\mathbf{u}^{(i)}|_j, i = 1, 2\}$

(ii) *Let $\mathbf{w}^{(i)} = \{w_j^{(i)}\}_{j=0}^{\infty} \in \ell^\infty$ be given sequences, and let K be a constant such that $|\mathbf{w}^{(i)}|_\infty \le (b - b_0)K$, $i = 1, 2$. Then there exist uniquely determined sequences $\mathbf{u}^{(i)} = \{u_j^{(i)}\}_{j=0}^{\infty}$ such that (1.9) holds, $|\mathbf{u}^{(i)}|_\infty \le K$ for $i = 1, 2$, and for all $j \in \mathbb{N} \cup \{0\}$ we have*

$$|u_j^{(1)} - u_j^{(2)}| \le \frac{2}{b - b_0}|\mathbf{w}^{(1)} - \mathbf{w}^{(2)}|_j + \max\left\{1, \frac{K^*}{b - b_0}\right\} \max_{r \ge 0} |\lambda_{-1}^{(1)}(r) - \lambda_{-1}^{(2)}(r)|, \tag{1.15}$$

where $K^* = 2(F(K) + MK)$.

We postpone the proof of Theorem 1.3 to Section 3 and investigate first some structure properties of the Preisach operator.

2. **Auxiliary results.**

We associate with f and g from Hypothesis 1.2 a Cauchy problem for the wave equation

$$\left.\begin{aligned} S_{rr} - S_{vv} &= g(r, v)\,, \\ S(0, v) &= 0\,, \\ S_r(0, v) &= f(v)\,. \end{aligned}\right\} \tag{2.1}$$

Here and in the sequel, the indices $_r$ and $_v$ denote partial derivatives with respect to r and v, respectively.

We see that the Preisach memory variable r thus plays the role of time in (2.1). We call S the *generating function of the Preisach operator P*. It is given by the integral formula

$$S(r, v) = \frac{1}{2}\int_{v-r}^{v+r} f(\xi)\,\mathrm{d}\xi + \frac{1}{2}\int_0^r \int_{v-r+\varrho}^{v+r-\varrho} g(\varrho, \xi)\,\mathrm{d}\xi\,\mathrm{d}\varrho\,. \tag{2.2}$$

In PDE terminology, the integration domain

$$\mathcal{T}(r, v) = \{(\varrho, \xi) \in \mathbb{R}^2 : 0 < \varrho < r, |v - \xi| < r - \varrho\} \tag{2.3}$$

in (2.2) is called the *characteristic triangle with vertex (r, v)*.

For convenience, we write down explicit formulas for partial derivatives of S, which we need in the sequel.

$$\left.\begin{array}{rcl} S_r(r,v) & = & \frac{1}{2}(f(v+r)+f(v-r)) \\ & & +\frac{1}{2}\int_0^r (g(\varrho, v+r-\varrho)+g(\varrho, v-r+\varrho))\,\mathrm{d}\varrho\,, \\ S_v(r,v) & = & \frac{1}{2}(f(v+r)-f(v-r)) \\ & & +\frac{1}{2}\int_0^r (g(\varrho, v+r-\varrho)-g(\varrho, v-r+\varrho))\,\mathrm{d}\varrho\,, \\ S_{rv}(r,v) & = & \frac{1}{2}(f'(v+r)+f'(v-r)) \\ & & +\frac{1}{2}\int_0^r (\mu(\varrho, v+r-\varrho)+\mu(\varrho, v-r+\varrho))\,\mathrm{d}\varrho\,, \\ S_{vv}(r,v) & = & \frac{1}{2}(f'(v+r)-f'(v-r)) \\ & & +\frac{1}{2}\int_0^r (\mu(\varrho, v+r-\varrho)-\mu(\varrho, v-r+\varrho))\,\mathrm{d}\varrho\,, \\ S_{rr}(r,v) & = & \frac{1}{2}(f'(v+r)-f'(v-r)) \\ & & +\frac{1}{2}\int_0^r (\mu(\varrho, v+r-\varrho)-\mu(\varrho, v-r+\varrho))\,\mathrm{d}\varrho + g(r,v)\,. \end{array}\right\} \tag{2.4}$$

Note that S_v is precisely what the engineers call *Everett function*, that is, the integral of the Preisach density over the characteristic triangle $\mathcal{T}(r,v)$ including the corresponding part of the coordinate line $r=0$, see, e.g., [5]. The interface lines between the $+1$ and -1 regions in the Preisach model stated in terms of non-ideal relays as, for example, in [2], follow the characteristic directions of the hyperbolic equation (2.1).

In the particular case of the Prandtl-Ishlinskii operator (1.11), the function S has the form

$$S(r,v) = \Gamma(r)v\,, \tag{2.5}$$

so that the Everett function $\Gamma = S_v$ is independent of v.

For S as in (2.2), set

$$\left.\begin{array}{rcl} S_+(r,v) & = & f'(v+r) + \int_0^r \mu(\varrho, v+r-\varrho)\,\mathrm{d}\varrho\,, \\ S_-(r,v) & = & f'(v-r) + \int_0^r \mu(\varrho, v-r+\varrho)\,\mathrm{d}\varrho\,. \end{array}\right\} \tag{2.6}$$

By Hypothesis 1.2, we have $S_+ \ge b - b_0$, $S_- \ge b - b_0$, and $S_{rv} = \frac{1}{2}(S_+ + S_-)$, $S_{vv} = \frac{1}{2}(S_+ - S_-)$.

Let now $\lambda \in \Lambda$ be arbitrary. We define the function $\varphi_\lambda : [0,\infty) \to [0,\infty)$ by the formula

$$\varphi_\lambda(r) = S_v(r, \lambda(r)) \quad \text{for} \quad r \ge 0\,. \tag{2.7}$$

Then $\varphi_\lambda(0) = 0$, and for a. e. $r > 0$ we have

$$\varphi_\lambda'(r) = S_{rv}(r,\lambda(r)) + S_{vv}(r,\lambda(r))\lambda'(r) = \frac{1}{2}(1+\lambda'(r))S_+ + \frac{1}{2}(1-\lambda'(r))S_- \ge b - b_0. \tag{2.8}$$

Hence, the inverse $\varphi_\lambda^{-1} : [0,\infty) \to [0,\infty)$ is increasing and Lipschitz continuous with Lipschitz constant $1/(b-b_0)$.

The following result plays a crucial role in our analysis.

Lemma 2.1. *Let Hypothesis 1.2 hold, and let $\lambda \in \Lambda_K$ be given for some $K > 0$. For $s \ge 0$ set*

$$\lambda_*(s) = \left(S_r(r,\lambda(r)) + \int_r^\infty g(\varrho, \lambda(\varrho))\,\mathrm{d}\varrho \right)\Bigg|_{r=\varphi_\lambda^{-1}(s)}\,. \tag{2.9}$$

Then $\lambda_* \in \Lambda$.

Proof of Lemma 2.1. For all $r \geq 0$ we have

$$\lambda_*(\varphi_\lambda(r)) = S_r(r, \lambda(r)) + \int_r^\infty g(\varrho, \lambda(\varrho))\,\mathrm{d}\varrho\,.$$

The chain rule yields

$$\begin{aligned} \frac{\mathrm{d}\lambda_*}{\mathrm{d}s}(\varphi_\lambda(r))\varphi_\lambda'(r) &= S_{rv}(r,\lambda(r))\lambda'(r) + S_{rr}(r,\lambda(r)) - g(r,\lambda(r)) \\ &= S_{rv}(r,\lambda(r))\lambda'(r) + S_{vv}(r,\lambda(r))\,, \end{aligned}$$

hence

$$\begin{aligned} \frac{\mathrm{d}\lambda_*}{\mathrm{d}s}(\varphi_\lambda(r)) &= \frac{S_{rv}(r,\lambda(r))\lambda'(r) + S_{vv}(r,\lambda(r))}{S_{vv}(r,\lambda(r))\lambda'(r) + S_{rv}(r,\lambda(r))} \\ &= \frac{(1+\lambda'(r))S_+ - (1-\lambda'(r))S_-}{(1+\lambda'(r))S_+ + (1-\lambda'(r))S_-}\,. \end{aligned}$$

We see that $\mathrm{d}\lambda_*/\,\mathrm{d}s \in [-1,1]$ almost everywhere, which we wanted to prove. Note also the implications

$$\lambda'(r) = 1 \Rightarrow \frac{\mathrm{d}\lambda_*}{\mathrm{d}s}(\varphi_\lambda(r)) = 1, \quad \lambda'(r) = -1 \Rightarrow \frac{\mathrm{d}\lambda_*}{\mathrm{d}s}(\varphi_\lambda(r)) = -1\,. \tag{2.10}$$

□

Lemma 2.2. *Let $\lambda_{-1} \in \Lambda_K$ be given for some $K > 0$, let $\mathbf{u} = \{u_j\}_{j=0}^\infty \in \ell^\infty$ be a given input sequence, and let $\mathbf{w} = \{w_j\}_{j=0}^\infty$ be given by the Preisach formula (1.9). For $s \geq 0$ set*

$$\lambda_j^*(s) = \left(S_r(r,\lambda_j(r)) + \int_r^\infty g(\varrho,\lambda_j(\varrho))\,\mathrm{d}\varrho\right)\Bigg|_{r=\varphi_{\lambda_j}^{-1}(s)} \quad \text{for } j \in \mathbb{N}\cup\{-1,0\}, \tag{2.11}$$

$$\hat\lambda_{-1}(s) = \lambda_{-1}^*(s)\,, \tag{2.12}$$

$$\hat\lambda_j(s) = \max\{w_j - s, \min\{\hat\lambda_{j-1}(s), w_j + s\}\} \quad \text{for} \quad j \in \mathbb{N}\cup\{0\}\,. \tag{2.13}$$

Then for all $j \in \mathbb{N}\cup\{0\}$ and $s \geq 0$ we have $\hat\lambda_j(s) = \lambda_j^*(s)$.

Proof of Lemma 2.2. By definition, the assertion holds for $j = -1$. We proceed by induction, and assume that it holds for some $j-1$. We consider first the case $u_j \geq u_{j-1}$ and set

$$r_j = \min\{r \geq 0 : u_j \leq \lambda_{j-1}(r) + r\}\,.$$

Then

$$\lambda_j(r) = \begin{cases} u_j - r & \text{for} \quad r \in [0, r_j)\,, \\ \lambda_{j-1}(r) & \text{for} \quad r \geq r_j\,. \end{cases} \tag{2.14}$$

In agreement with (1.3) and (1.9), set $u_{-1} = \lambda_{-1}(0)$, $w_{-1} = \hat\lambda_{-1}(0) = \lambda_{-1}^*(0)$. We have

$$\begin{aligned} w_j - w_{j-1} &= f(u_j) - f(u_{j-1}) + \int_0^\infty \int_{\lambda_{j-1}(r)}^{\lambda_j(r)} \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \qquad (2.15) \\ &\geq b(u_j - u_{j-1}) - \int_0^{r_j}\int_{u_{j-1}-r}^{u_j - r} \mu_0(r)\,\mathrm{d}v\,\mathrm{d}r \\ &\geq (b - b_0)(u_j - u_{j-1}) \geq 0\,. \end{aligned}$$

We have used the fact that in $[0, r_j)$, the function $r \mapsto \lambda_{j-1}(r) + r$ is nondecreasing, hence $\lambda_{j-1}(r) \geq u_{j-1} - r$. It follows from (2.13) that putting

$$\hat{s}_j = \min\{s \geq 0 : w_j \leq \hat{\lambda}_{j-1}(s) + s\}\,,$$

we have

$$\hat{\lambda}_j(s) = \begin{cases} w_j - s & \text{for } \; s \in [0, \hat{s}_j)\,, \\ \hat{\lambda}_{j-1}(s) & \text{for } \; s \geq \hat{s}_j\,. \end{cases} \tag{2.16}$$

The next step consists in putting

$$s_j^* = \varphi_{\lambda_{j-1}}(r_j) = S_v(r_j, \lambda_{j-1}(r_j))\,.$$

For $s \geq s_j^*$ we have $r := \varphi^{-1}_{\lambda_{j-1}}(s) \geq r_j$, hence $\lambda_j(r) = \lambda_{j-1}(r)$, and

$$s = S_v(r, \lambda_{j-1}(r)) = S_v(r, \lambda_j(r)) = \varphi_{\lambda_j}(r)\,.$$

Consequently,

$$\lambda_j^*(s) = \left(S_r(r, \lambda_{j-1}(r)) + \int_r^\infty g(\varrho, \lambda_{j-1}(\varrho))\,\mathrm{d}\varrho\right)\Bigg|_{r=\varphi^{-1}_{\lambda_{j-1}}(s)} = \lambda^*_{j-1}(s) = \hat{\lambda}_{j-1}(s)$$

by induction hypothesis. On the other hand,

$$\begin{aligned} w_j - \lambda_j^*(s_j^*) \;&=\; f(u_j) + \int_0^{r_j} g(\varrho, \lambda_j(\varrho))\,\mathrm{d}\varrho - S_r(r_j, \lambda_j(r_j)) \\ &= \frac{1}{2}(f(u_j) - f(u_j - 2r_j)) + \frac{1}{2}\int_0^{r_j}(g(\varrho, u_j - \varrho) - g(\varrho, u_j - 2r_j + \varrho))\,\mathrm{d}\varrho \\ &= S_v(r_j, \lambda_j(r_j)) \;=\; s_j^*\,. \end{aligned}$$

Note that $\lambda_j^*(0) = \hat{\lambda}_j(0) = w_j$. The function $s \mapsto \lambda_j^*(s) + s - w_j$ is nondecreasing in $[0, s_j^*]$ and vanishes at the endpoints of the interval, hence it is constant, that is, $\lambda_j^*(s) = w_j - s$ for $s \in [0, s_j^*]$.

Set $s_j^+ = \max\{\hat{s}_j, s_j^*\}$, $s_j^- = \min\{\hat{s}_j, s_j^*\}$. The functions $\hat{\lambda}_j$ and λ_j^* coincide on $[0, s_j^-] \cup [s_j^+, \infty]$, and for $s \in [s_j^-, s_j^+]$ we have

$$\hat{\lambda}_j(s) - \lambda_j^*(s) = \begin{cases} \hat{\lambda}_{j-1}(s) + s - w_j & \text{if } \; \hat{s}_j \leq s_j^*, \\ -\hat{\lambda}_{j-1}(s) - s + w_j & \text{if } \; \hat{s}_j > s_j^*. \end{cases}$$

In both cases, this is a monotone function which vanishes at the endpoints of the interval $[s_j^-, s_j^+]$, hence $\hat{\lambda}_j = \lambda_j^*$. The argument is similar if $u_j \leq u_{j-1}$. This completes the proof of Lemma 2.2. □

Note that (2.13) defines the play operator applied to $\mathbf{w}$, or, in other words, the play operator superposed to the Preisach operator applied to $\mathbf{u}$. We now derive further superposition formulas, which generalize analogous results in [9], where the argument is based on the investigation of memory sequences, and therefore is valid only for symmetric nonlinearities. Here, no symmetry is assumed.

Proposition 2.3. *Let P be the Preisach operator (1.9)–(1.10) satisfying Hypothesis 1.2, let $h : \mathbb{R} \to \mathbb{R}$ be an increasing locally Lipschitz continuous function such that $h(0) = 0$, and let Q be the Prandtl-Ishlinskii operator (1.11). Let S be the generating function of P according to (2.1).*

(i) *Let* $\int_0^\infty g(\frac{1}{2}(h(r)-h(-r)),\frac{1}{2}(h(r)+h(-r)))\,\mathrm{d}r<\infty$. *Then the superposed operator* $P^h=P\circ h$ *is a Preisach operator with generating function*

$$S^h(r,v)=\int_0^v S_q(p(r,\xi),q(r,\xi))\,\mathrm{d}\xi\,, \tag{2.17}$$

where

$$\begin{aligned} p(r,v) &= \frac{1}{2}(h(v+r)-h(v-r))\,,\\ q(r,v) &= \frac{1}{2}(h(v+r)+h(v-r))\,. \end{aligned}$$

(ii) *Let there exist* $R>0$ *such that* $S_r(r,0)=0$ *for* $r>R$. *Then* $P^Q=Q\circ P$ *is a Preisach operator with generating function*

$$S^Q(r,v)=\int_0^v \Gamma(S_\xi(r,\xi))\,\mathrm{d}\xi\,. \tag{2.18}$$

(iii) *Let* $\lim_{r\to\infty}\Gamma(r)=\infty$. *Then the superposition* $h\circ Q$ *is a Preisach operator if and only if at least one of the functions* h, Γ *is linear.*

In particular, it follows immediately from (2.18) and (1.11) that the composition $Q_1\circ Q_2$ of two Prandtl-Ishlinskii operators Q_1,Q_2 with Everett functions Γ_1,Γ_2, respectively, is again a Prandtl-Ishlinskii operator with Everett function $\Gamma_1\circ\Gamma_2$, so that the inverse of a Prandtl-Ishlinskii operator is explicitly obtained by inverting the Everett function Γ. This result goes back to [7]. In general, no explicit formula for the inverse Preisach operator P^{-1} is known. This is possible only in special cases like $Q^{-1}\circ h^{-1}=(h\circ Q)^{-1}$ as in (iii), see [14]. This also shows that the inverse of a Preisach operator cannot be expected to be Preisach in general.

Proof of Proposition 2.3. (i) Let us consider a function $\lambda\in\Lambda_K$ for some $K>0$. The Nemytskii operator h is a special case of the Preisach operator with $f=h$ and $g=0$. Hence, in view of (2.4)–(2.9) and Lemma 2.2, it suffices to construct two functions $\tilde f$ and $\tilde g$ such that the following implication holds:

$$\begin{aligned} &s=p(r,\lambda(r)),\ \tilde\lambda(s)=q(r,\lambda(r))\\ &\qquad\Longrightarrow\quad f(\tilde\lambda(0))+\int_0^\infty g(s,\tilde\lambda(s))\,\mathrm{d}s=\tilde f(\lambda(0))+\int_0^\infty \tilde g(r,\lambda(r))\,\mathrm{d}r\,. \end{aligned} \tag{2.19}$$

We have by substitution

$$\int_0^\infty g(s,\tilde\lambda(s))\,\mathrm{d}s=\int_0^\infty (S_{pp}-S_{qq})(p,q)(p_r+\lambda'(r)p_v)(r,\lambda(r))\,\mathrm{d}r\,. \tag{2.20}$$

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}r}\big(S_p(p(r,\lambda(r)),q(r,\lambda(r)))\big)=(p_r+\lambda'(r)p_v)S_{pp}+(q_r+\lambda'(r)q_v)S_{pq}\,, \tag{2.21}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}\big(S^h_r(r,\lambda(r))\big)=S^h_{rr}+\lambda'(r)(S_{pq}p_r+S_{qq}q_r)\,. \tag{2.22}$$

Furthermore, $S^h_{vv}(r,v)=S_{pq}p_v+S_{qq}q_v$ and $p_r=q_v$, $p_v=q_r$. Hence,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}r}\big(S^h_r(r,\lambda(r))-S_p(p(r,\lambda(r)),q(r,\lambda(r)))\big)\\ &\qquad=S^h_{rr}-S^h_{vv}-(p_r+\lambda'(r)p_v)(S_{pp}-S_{qq})(p,q)\,. \end{aligned} \tag{2.23}$$

Integrating (2.23) over r from 0 to ∞ and using (2.20), we obtain (2.19) with $\tilde{f}(v) = f(h(v)) = S^h_r(0, v)$ and $\tilde{g}(r, v) = S^h_{rr} - S^h_{vv}$, which we wanted to prove.

(ii) We proceed in a similar way, with λ_* as in (2.9), with the intention to find $\hat{f}$, $\hat{g}$ such that

$$\Gamma'(0)\lambda_*(0) + \int_0^\infty \Gamma''(s)\lambda_*(s)\,\mathrm{d}s = \hat{f}(\lambda(0)) + \int_0^\infty \hat{g}(r, \lambda(r))\,\mathrm{d}r\,. \tag{2.24}$$

We have

$$\begin{aligned}\Gamma'(0)\lambda_*(0) + \int_0^\infty \Gamma''(s)\lambda_*(s)\,\mathrm{d}s &= -\int_0^\infty \Gamma'(s)\lambda'_*(s)\,\mathrm{d}s \\ &= -\int_0^\infty \Gamma'(S_v(r, \lambda(r)))(S_{vv} + \lambda'(r)S_{rv})\,\mathrm{d}r\,.\end{aligned} \tag{2.25}$$

Using the identities $S^Q_v = \Gamma'(S_v)$ and

$$\frac{\mathrm{d}}{\mathrm{d}r}\big(S^Q_r(r, \lambda(r))\big) = S^Q_{rr} + \lambda'(r)\Gamma'(S_v(r, \lambda(r)))S_{rv}\,, \tag{2.26}$$

we obtain

$$\Gamma'(0)\lambda_*(0) + \int_0^\infty \Gamma''(s)\lambda_*(s)\,\mathrm{d}s = S^Q_r(0, \lambda(0)) + \int_0^\infty (S^Q_{rr} - S^Q_{vv})(r, \lambda(r))\,\mathrm{d}r\,, \tag{2.27}$$

and (2.24) follows with $\hat{f}(v) = S^Q_r(0, v)$, $\hat{g} = S^Q_{rr} - S^Q_{vv}$ as in (2.1).

(iii) Assume that there exist functions f and g such that for every $\lambda \in \Lambda_K$ we have

$$h\left(\Gamma'(0)\lambda(0) + \int_0^\infty \Gamma''(r)\lambda(r)\,\mathrm{d}r\right) = f(\lambda(0)) + \int_0^\infty g(r, \lambda(r))\,\mathrm{d}r\,. \tag{2.28}$$

For given numbers $0 < a < b < c$ such that $u \ge 2b - c$, we consider the functions

$$\lambda(r) = \begin{cases} -u + r & \text{for } r \in [0, b)\,, \\ -u + 2b - r & \text{for } r \in [b, c)\,, \\ -u + 2b - 2c + r & \text{for } r \in [c, 2c - 2b + u)\,, \\ 0 & \text{for } r \ge 2c - 2b + u\,, \end{cases} \tag{2.29}$$

$$\bar{\lambda}(r) = \begin{cases} -u + 2a - r & \text{for } r \in [0, a)\,, \\ \lambda(r) & \text{for } r \ge a\,. \end{cases} \tag{2.30}$$

We evaluate the left hand side of (2.28) for λ and $\bar{\lambda}$ using the integration-by-parts formula

$$\Gamma'(0)\lambda(0) + \int_0^\infty \Gamma''(r)\lambda(r)\,\mathrm{d}r = -\int_0^\infty \Gamma'(r)\lambda'(r)\,\mathrm{d}r\,.$$

Subtracting the identities (2.28) written for $\bar{\lambda}$ and λ, we thus obtain

$$\begin{aligned}&h(2\Gamma(a) - 2\Gamma(b) + 2\Gamma(c) - \Gamma(2c - 2b + u)) - h(-2\Gamma(b) + 2\Gamma(c) - \Gamma(2c - 2b + u)) \\ &= f(-u + 2a) - f(-u) + \int_0^a (g(r, -u + 2a - r) - g(r, -u + r))\,\mathrm{d}r\,.\end{aligned} \tag{2.31}$$

The right hand side of (2.31) is independent of b and c. We differentiate (2.31) with respect to b and with respect to c, add the two results and obtain for all a, b, c, u as above that

$$\begin{aligned}&\big(h'(2\Gamma(a) - 2\Gamma(b) + 2\Gamma(c) - \Gamma(2c - 2b + u)) \\ &\quad - h'(-2\Gamma(b) + 2\Gamma(c) - \Gamma(2c - 2b + u))\big)\big(\Gamma'(c) - \Gamma'(b)\big) = 0\,.\end{aligned} \tag{2.32}$$

Let now $x \in \mathbb{R}$ be arbitrarily given, and let us assume that Γ' is non-constant in $[0, \infty)$. Then there exist $c_0 > b_0 > 0$ such that $\Gamma'(b_0) \neq \Gamma'(c_0)$. Set $z = (x/\Gamma(b_0))^+$. We find $b \in [b_0, c_0]$ and $c \geq \Gamma^{-1}((z+2)\Gamma(b))$ such that $\Gamma'(b) \neq \Gamma'(c)$. We now claim that $u \geq 2b - c$ can be chosen in such a way that

$$-2\Gamma(b) + 2\Gamma(c) - \Gamma(2c - 2b + u) = x\,. \tag{2.33}$$

Indeed, we have

$$-2\Gamma(b)+2\Gamma(c)-\Gamma(2c-2b+u) \geq z\Gamma(b)+\Gamma(c)-\Gamma(2c-2b+u) \geq x+\Gamma(c)-\Gamma(2c-2b+u)\,,$$

and (2.33) follows. By virtue of (2.32), for all $x \in \mathbb{R}$ and all $\delta \in (0, 2\Gamma(b_0))$ we have $h'(x+\delta) - h'(x) = 0$, hence h is linear. □

3. **Proof of Theorem 1.3.** Inequality (1.14) is an immediate consequence of Lemma 1.1. Indeed, we have

$$\begin{aligned} w_j^{(1)} - w_j^{(2)} &= f(u_j^{(1)}) - f(u_j^{(2)}) + \int_0^{K_j} \int_{\lambda_j^{(2)}(r)}^{\lambda_j^{(1)}(r)} \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \\ &\leq F(K_j)|u_j^{(1)} - u_j^{(2)}| + M \int_0^{K_j} |\lambda_j^{(2)}(r) - \lambda_j^{(1)}(r)|\,\mathrm{d}r\,, \end{aligned}$$

and (1.14) follows easily.

Part (ii) is more involved. Let $\mathbf{w} = \{w_j\}_{j=0}^\infty$ and $\lambda_{-1} \in \Lambda_K$ be given. As in Lemma 2.2, we define $\lambda_{-1}^*(s)$ by (2.11) and put $w_{-1} = \lambda_{-1}^*(0)$, $u_{-1} = \lambda_{-1}(0)$. The first line of (2.15) can now be viewed as an equation for unknown u_j. More precisely, for $u \in \mathbb{R}$ we define the function

$$W(u) = f(u) - f(u_{j-1}) + \int_0^{r_u} \int_{\lambda_{j-1}(r)}^{\lambda_u(r)} \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r\,, \tag{3.1}$$

where

$$r_u = \min\{r \geq 0 : |u - \lambda_{j-1}(r)| \leq r\}\,.$$

We have $r_u = 0$ if $u = u_{j-1}$, and

$$\lambda_u(r) = \begin{cases} \lambda_{j-1}(r) & \text{for } \ r \geq r_u\,, \\ u - r & \text{for } \ r \in [0, r_u) \quad \text{if } \ u > u_{j-1}\,, \\ u + r & \text{for } \ r \in [0, r_u) \quad \text{if } \ u < u_{j-1}\,. \end{cases} \tag{3.2}$$

The identity $W(u_{j-1}) = 0$ is obvious. For $u > u_{j-1}$, we use Fubini's Theorem and rewrite (3.1) in the form

$$\begin{aligned} W(u) &= f(u) - f(u_{j-1}) + \int_0^{r_u} \int_{\lambda_{j-1}(r)}^{u-r} \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \\ &= f(u) - f(u_{j-1}) + \int_{u_{j-1}}^{u} \int_0^{r_v} \mu(r, v-r)\,\mathrm{d}r\,\mathrm{d}v\,. \end{aligned}$$

Similarly, for $u < u_{j-1}$ we obtain

$$W(u) = f(u) - f(u_{j-1}) + \int_{u_{j-1}}^{u} \int_0^{r_v} \mu(r, v+r)\,\mathrm{d}r\,\mathrm{d}v\,.$$

Hence, W is absolutely continuous and $W'(u) \geq b - b_0$ a.e. This allows us to conclude that the equation $W(u) = w_j - w_{j-1}$ has a unique solution $u = u_j$. It remains to check that $|u_j| \le K$ for all j. It is certainly true for $j = -1$, Assume that it holds for some $j-1$, and assume for example that $w_j > w_{j-1}$, $u_j > K$. Then $r_{u_j} = u_j$, $\lambda_{u_j}(r) = \max\{u_j - r, 0\}$, and we directly obtain from (1.9) that

$$\begin{aligned} w_j = f(u_j) + \int_0^\infty g(r, \lambda_{u_j}(r))\,\mathrm{d}r &= f(u_j) + \int_0^{u_j}\int_0^{u_j - r} \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \\ &\ge (b - b_0)u_j > (b-b_0)K, \end{aligned} \tag{3.3}$$

which is a contradiction. The existence part is thus complete.

To prove inequality (1.15), we define as in (2.11)

$$\lambda_j^{*(i)}(s) = \left(S_r(r, \lambda_j^{(i)}(r)) + \int_r^\infty g(\varrho, \lambda_j^{(i)}(\varrho))\,\mathrm{d}\varrho \right)\Big|_{r = \varphi^{-1}_{\lambda_j^{(i)}}(s)} \tag{3.4}$$

for $s \ge 0$, $j \in \mathbb{N} \cup \{0\}$, and $i = 1, 2$. We define

$$r_j = \min\{r \ge 0 : \lambda_j^{(1)}(r) = \lambda_j^{(2)}(r)\}, \quad s_j = \varphi_{\lambda_j^{(1)}}(r_j) = \varphi_{\lambda_j^{(2)}}(r_j)\,. \tag{3.5}$$

Assume that for example

$$r_j > 0\,, \lambda_j^{(1)}(r) > \lambda_j^{(2)}(r) \quad \text{for} \quad r \in [0, r_j)\,. \tag{3.6}$$

then

$$w_j^{(i)} - \lambda_j^{*(i)}(s_j) = f(u_j^{(i)}) - S_r(r_j, \lambda_j^{(i)}(r_j)) + \int_0^{r_j} g(\varrho, \lambda_j^{(i)}(\varrho))\,\mathrm{d}\varrho\,, \tag{3.7}$$

hence

$$\begin{aligned} &(w_j^{(1)} - w_j^{(2)}) - (\lambda_j^{*(1)}(s_j) - \lambda_j^{*(2)}(s_j)) \\ &\quad = (f(u_j^{(1)}) - f(u_j^{(2)})) + \int_0^{r_j} (g(\varrho, \lambda_j^{(1)}(\varrho)) - g(\varrho, \lambda_j^{(2)}(\varrho)))\,\mathrm{d}\varrho \\ &\quad = (f(u_j^{(1)}) - f(u_j^{(2)})) + \int_0^{r_j} \int_{\lambda_j^{(2)}(\varrho)}^{\lambda_j^{(1)}(\varrho)} \mu(\varrho, v)\,\mathrm{d}v\,\mathrm{d}\varrho \\ &\quad \ge b(u_j^{(1)} - u_j^{(2)}) - \int_0^{r_j} \mu_0(\varrho)(\lambda_j^{(1)}(\varrho) - \lambda_j^{(2)}(\varrho))\,\mathrm{d}\varrho \\ &\quad \ge b(u_j^{(1)} - u_j^{(2)}) - b_0 \max\{|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|_j, \max_{r\ge 0} |\lambda_{-1}^{(1)}(r) - \lambda_{-1}^{(2)}(r)|\}\,. \end{aligned}$$

This yields

$$\begin{aligned} b|u_j^{(1)} - u_j^{(2)}| \quad \le \quad & |w_j^{(1)} - w_j^{(2)}| + \max\{|\mathbf{w}^{(1)} - \mathbf{w}^{(2)}|_j, \max_{s\ge 0} |\lambda_{-1}^{*(1)}(s) - \lambda_{-1}^{*(2)}(s)|\} \\ & + b_0 \max\{|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|_j, \max_{r\ge 0} |\lambda_{-1}^{(1)}(r) - \lambda_{-1}^{(2)}(r)|\}\,. \end{aligned} \tag{3.8}$$

Without assumption (3.6), inequality (3.8) holds as well. Indeed, we have $u_j^{(1)} = u_j^{(2)}$ if $r_j = 0$, and if $u_j^{(1)} < u_j^{(2)}$, then we simply interchange $u_j^{(1)}$ and $u_j^{(2)}$.

We first prove that

$$\max_{s\ge 0} |\lambda_{-1}^{*(1)}(s) - \lambda_{-1}^{*(2)}(s)| \le K^* \max_{r\ge 0} |\lambda_{-1}^{(1)}(r) - \lambda_{-1}^{(2)}(r)|\,, \quad K^* = 2(F(K) + KM)\,. \tag{3.9}$$

Indeed, let $s > 0$ be arbitrary. Set $r_i = \varphi^{-1}_{\lambda^{(i)}_{-1}}(s)$, that is,

$$s = S_v(r_1, \lambda^{(1)}_{-1}(r_1)) = S_v(r_2, \lambda^{(2)}_{-1}(r_2))\,,$$

and assume that $r_1 \ge r_2$. If $r_1 \ge K$, then

$$s = S_v(r_1, \lambda^{(1)}_{-1}(r_1)) = S_v(r_1, 0) = S_v(r_1, \lambda^{(2)}_{-1}(r_1)) = S_v(r_2, \lambda^{(2)}_{-1}(r_2))\,,$$

hence $r_1 = r_2$ and $\lambda^{*(1)}_{-1}(s) = \lambda^{*(2)}_{-1}(s)$. It suffices thus to consider that case $r_2 \le r_1 < K$. Set $D = \max_{r\ge 0} |\lambda^{(1)}_{-1}(r) - \lambda^{(2)}_{-1}(r)|$. We have

$$\begin{aligned}
&\lambda^{*(1)}_{-1}(s) - \lambda^{*(2)}_{-1}(s) \qquad (3.10)\\
&= S_r(r_1, \lambda^{(1)}_{-1}(r_1)) - S_r(r_1, \lambda^{(2)}_{-1}(r_1)) + \int_{r_1}^{\infty} g(r, \lambda^{(1)}_{-1}(r)) - g(r, \lambda^{(2)}_{-1}(r))\,\mathrm{d}r\\
&+ S_r(r_1, \lambda^{(2)}_{-1}(r_1)) - S_r(r_2, \lambda^{(2)}_{-1}(r_2)) - \int_{r_2}^{r_1} g(r, \lambda^{(2)}_{-1}(r))\,\mathrm{d}r\,.
\end{aligned}$$

By hypothesis, the inequality $r_1 \pm \lambda^{(i)}_{-1}(r_1) \le K$ holds for $i = 1, 2$, hence

$$\begin{aligned}
&\left| S_r(r_1, \lambda^{(1)}_{-1}(r_1)) - S_r(r_1, \lambda^{(2)}_{-1}(r_1)) + \int_{r_1}^{\infty} g(r, \lambda^{(1)}_{-1}(r)) - g(r, \lambda^{(2)}_{-1}(r))\,\mathrm{d}r \right|\\
&\qquad \le D(F(K) + MK).
\end{aligned}$$

Furthermore, set $s_1 = S_v(r_1, \lambda^{(2)}_{-1}(r_1)) \ge s$. Then

$$S_r(r_1, \lambda^{(2)}_{-1}(r_1)) - S_r(r_2, \lambda^{(2)}_{-1}(r_2)) - \int_{r_2}^{r_1} g(r, \lambda^{(2)}_{-1}(r))\,\mathrm{d}r = \lambda^{*(2)}_{-1}(s_1) - \lambda^{*(2)}_{-1}(s)\,,$$

where

$$|\lambda^{*(2)}_{-1}(s_1) - \lambda^{*(2)}_{-1}(s)| \le s_1 - s = |S_v(r_1, \lambda^{(2)}_{-1}(r_1)) - S_v(r_1, \lambda^{(1)}_{-1}(r_1))| \le D(F(K)+MK),$$

and (3.9) follows from (3.10).

Taking in (3.8) the maximum over j, we obtain, using (3.9), that

$$\begin{aligned}
b|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|_j \;\le\;& |\mathbf{w}^{(1)} - \mathbf{w}^{(2)}|_j + \max\{|\mathbf{w}^{(1)} - \mathbf{w}^{(2)}|_j, K^* \max_{r\ge 0} |\lambda^{(1)}_{-1}(r) - \lambda^{(2)}_{-1}(r)|\}\\
&+ b_0 \max\{|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|_j, \max_{r\ge 0} |\lambda^{(1)}_{-1}(r) - \lambda^{(2)}_{-1}(r)|\}\,. \qquad (3.11)
\end{aligned}$$

This is an inequality of the form

$$bx \le y + \max\{y, K^* a\} + b_0 \max\{x, a\} \qquad (3.12)$$

with $x = |\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|_j$, $y = |\mathbf{w}^{(1)} - \mathbf{w}^{(2)}|_j$, $a = \max_{r\ge 0} |\lambda^{(1)}_{-1}(r) - \lambda^{(2)}_{-1}(r)|$, which yields

$$x \le \frac{2}{b - b_0} y + \max\left\{1, \frac{K^*}{b - b_0} a\right\}\,. \qquad (3.13)$$

Estimate (1.15) now follows from (3.13).

4. **Identification.** Here, the constitutive functions f, g in (1.9) are assumed to be *unknown*, and we propose an algorithm to determine them with a controlled error. To this aim, we use again formula (2.9) that we rewrite in the form

$$\lambda_*(s) = \left(S_r(r, \lambda(r)) + \int_r^\infty g(\varrho, \lambda(\varrho))\,\mathrm{d}\varrho\right), \ s = S_r(r, \lambda(r))\,. \tag{4.1}$$

For simplicity, we assume the asymptotically symmetric case

$$\exists K > 0: \ r + |v| > K \implies f(v) = -f(-v)\,, \ g(r,v) = -g(r,-v)\,. \tag{4.2}$$

This is not a real restriction: We see from (1.13) that if $|\mathbf{u}|_\infty < K$ and $\lambda_{-1} \in \Lambda_K$, the values of f and g outside $\mathcal{T}(K, 0)$ (cf. (2.3)) never come into play in the formula (1.9).

We first derive a few special properties of the play system.

Lemma 4.1. *Let $r_0 \ge r_1 > r_2 > \cdots > r_n = 0$ be a sequence, and let $\lambda \in \Lambda_K$ be such that*

$$\lambda(r_{j-1}) - \lambda(r_j) = \pm(-1)^j (r_{j-1} - r_j) \quad \textit{for} \quad j = 1, \ldots, n\,. \tag{4.3}$$

Put $\lambda_{-1}(r) = \lambda(\max\{r, r_0\})$ for $r \ge 0$, and

$$u_j = \lambda(r_j) \pm (-1)^j r_j \quad \textit{for} \quad j = 0, 1, \ldots, n\,, \tag{4.4}$$

with an arbitrary (e.g. constant) extension for $j > n$. Let $\boldsymbol{\lambda}(r) = \{\lambda_j(r)\}_{j=0}^\infty = \mathfrak{p}_r[\mathbf{u}, \lambda_{-1}]$. Then $\lambda(r) = \lambda_n(r)$ for all $r \ge 0$.

Proof. The sequence $\{\lambda_j(r)\}_{j=0}^\infty$ is defined by the formula (1.2). Assume for definiteness that

$$\left\{ \begin{array}{rcll} \lambda(r_{j-1}) - \lambda(r_j) & = & (-1)^j (r_{j-1} - r_j) & \text{for} \quad j = 1, \ldots, n\,, \\ u_j & = & \lambda(r_j) + (-1)^j r_j & \text{for} \quad j = 0, 1, \ldots, n\,, \end{array} \right. \tag{4.5}$$

the other case is analogous. We prove by induction that for all $j = 0, 1, \ldots, n$ we have

$$\lambda_j(r) = \left\{ \begin{array}{ll} u_j - (-1)^j r & \text{for} \quad r \in [0, r_j)\,, \\ \lambda(r) & \text{for} \quad r \ge r_j\,. \end{array} \right. \tag{4.6}$$

For $j = 0$ and $r < r_0$ we have

$$\lambda_0(r) = \max\{u_0 - r, \min\{u_0 + r, u_0 - r_0\}\} = u_0 - r\,,$$

while for $r \ge r_0$ we use the fact that $\lambda \in \Lambda$ to conclude that

$$\lambda_0(r) = \max\{\lambda(r_0) + r_0 - r, \min\{\lambda(r_0) + r_0 + r, \lambda(r)\}\} = \lambda(r)\,,$$

which is precisely (4.6). Let us assume now that (4.6) holds for $j - 1$, that is,

$$\lambda_{j-1}(r) = \left\{ \begin{array}{ll} u_{j-1} + (-1)^j r & \text{for} \quad r \in [0, r_{j-1})\,, \\ \lambda(r) & \text{for} \quad r \ge r_{j-1}\,. \end{array} \right. \tag{4.7}$$

We have $\lambda'_{j-1}(r) = \lambda'(r) = (-1)^j$ for $r \in (r_j, r_{j-1})$, hence $\lambda_{j-1}(r) = \lambda(r)$ for $r \ge r_j$. From the formula

$$\lambda_j(r) = \max\{\lambda(r_j) + (-1)^j r_j - r, \min\{\lambda(r_j) + (-1)^j r_j + r, \lambda_{j-1}(r)\}\}$$

we immediately obtain that $\lambda_j(r) = \lambda(r)$ for $r \ge r_j$. On the other hand, we have $\lambda_j(0) = u_j = \lambda(r_j) + (-1)^j r_j$, $\lambda_j(r_j) = \lambda(r_j)$, which entails $\lambda'_j(r) = (-1)^j$ in $(0, r_j)$, and (4.6) follows. The induction step is thus complete and for j we obtain the assertion. □

Lemma 4.2. *Let* (4.2) *hold, and let* $\lambda \in \Lambda_K$ *be as in Lemma 4.1. Let* λ_* *be given by* (4.1)*, and put*

$$s_j = S_v(r_j, \lambda(r_j)) \quad \textit{for} \quad j = 0, 1, \dots, n\,, \tag{4.8}$$

where S *is the function* (2.2)*. Then* $\lambda_* \in \Lambda_{K_*}$ *with* $K_* = S_v(K, 0)$*, and*

$$\lambda_*(s_{j-1}) - \lambda_*(s_j) = \pm(-1)^j (s_{j-1} - s_j) \quad \textit{for} \quad j = 1, \dots, n\,. \tag{4.9}$$

Proof. We have $\lambda'(r) = \pm(-1)^j$ in (r_j, r_{j-1}), hence $\lambda_*'(s) = \pm(-1)^j$ in (s_j, s_{j-1}) by (2.10), and the assertion follows. □

We now fix a constant $R \in (0, K)$, with the goal to identify the functions f, g in the characteristic triangle $\mathcal{T}(R, 0)$. We choose a discretization parameter $m \in \mathbb{N}$, and for $\alpha, \beta = -m, \dots, m$, we define the lines

$$\begin{aligned} \mathcal{L}_\alpha &= \left\{(r, v) \in \mathbb{R}^2 : v - r = \alpha \tfrac{R}{m}\right\}\,, \\ \mathcal{L}^\beta &= \left\{(r, v) \in \mathbb{R}^2 : v + r = \beta \tfrac{R}{m}\right\}\,, \end{aligned}$$

and their intersection points

$$z_\alpha^\beta = \mathcal{L}_\alpha \cap \mathcal{L}^\beta = (r_\alpha^\beta, v_\alpha^\beta)\,,\ r_\alpha^\beta = (\beta - \alpha)\frac{R}{2m}\,,\ v_\alpha^\beta = (\beta + \alpha)\frac{R}{2m}\,.$$

The set

$$D_R^{(m)} = \{z_\alpha^\beta : -m \le \alpha \le \beta \le m\}$$

forms a grid of $\mathcal{T}(R, 0)$, with square cells

$$\mathcal{Q}_\alpha^\beta = \left\{(r, v) \in \mathbb{R} : \alpha\frac{R}{m} < v - r < (\alpha + 1)\frac{R}{m}\,,\ (\beta - 1)\frac{R}{m} < v + r < \beta\frac{R}{m}\right\} \tag{4.10}$$

for $\beta \ge \alpha + 2$, and triangular cells $\mathcal{T}(z_{\beta-1}^\beta)$ for $b = -m + 1, \dots, m$, see Figure 1.

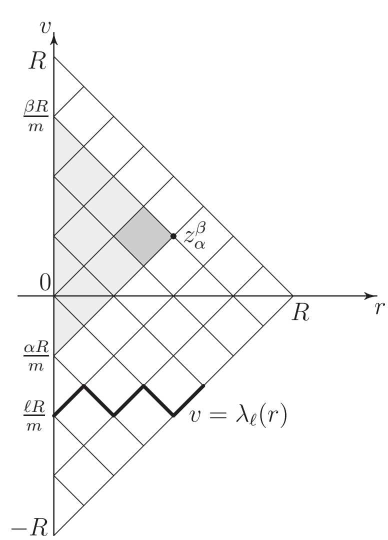

Figure 1. A square cell $\mathcal{Q}_\alpha^\beta$ is represented in dark gray, a characteristic triangle $\mathcal{T}(z_\alpha^\beta)$ in light gray, the thick line represents the memory path $v = \lambda^{(\gamma)}(r)$.

We now define a sequence of functions $\lambda^{(\gamma)}$ from Λ_R, $\gamma = -m, \dots, m-1$, such that

$$\lambda^{(\gamma)}(0) = v_\gamma^\gamma = \gamma\frac{R}{m}\,,\ \lambda^{(\gamma)\prime} = (-1)^{k-1} \ \text{ for } \ r \in \Big((k-1)\frac{R}{2m}, k\frac{R}{2m}\Big)\,,\ k = 1, \dots, p_\gamma\,, \tag{4.11}$$

where p_γ is defined by the formula

$$\left\{ \begin{array}{rcl} p_\gamma \frac{R}{2m} + \left|\lambda^{(\gamma)}\left(p_\gamma \frac{R}{2m}\right)\right| & = & R\,, \\ (p_\gamma + 1)\frac{R}{2m} + \left|\lambda^{(\gamma)}\left((p_\gamma+1) \frac{R}{2m}\right)\right| & > & R\,. \end{array} \right. \tag{4.12}$$

We then extend $\lambda^{(\gamma)}$ for $r \geq p_\gamma R/(2m)$ in an arbitrary way, for example $\lambda^{(\gamma)} = (R-r)^+$ if $\gamma \geq 0$, $\lambda^{(\gamma)} = -(R-r)^+$ if $\gamma < 0$, see Figure 1. A straightforward computation yields

$$p_\gamma = \left\{ \begin{array}{ll} 2(m-\gamma) & \text{if } \ \gamma \geq 0\,, \\ 2(m+\gamma)+1 & \text{if } \ \gamma < 0\,. \end{array} \right. \tag{4.13}$$

Each function $\lambda^{(\gamma)}$ satisfies the hypotheses of Lemma 4.1, with the choice $r_0 = r_1 = p_\gamma$, $r_j = r_1 - (j-1)R/(2m)$.

Furthermore, for each grid point $z_\alpha^\beta \in D_R^{(m)}$ but $z_m^m = (0, R)$ there exist uniquely determined $\gamma \in \{-m, \dots, m-1\}$ and $p \in \{0, \dots, p_\gamma\}$ such that

$$z_\alpha^\beta = \Big(p\frac{R}{2m}, \lambda^{(\gamma)}\Big(p\frac{R}{2m}\Big)\Big)\,. \tag{4.14}$$

Indeed, it suffices to put

$$p = \beta - \alpha\,,\ \gamma = \alpha + \left[\frac{p}{2}\right]\,, \tag{4.15}$$

where $[\cdot]$ denotes the integer part of a real number.

The identification algorithm is defined as follows: For each $\lambda = \lambda^{(\gamma)}$ we define $\mathbf{u}$ and $\boldsymbol{\lambda}$ as in Lemma 4.1, and set $\mathbf{w}^{(\gamma)} = P[\mathbf{u}, \lambda_{-1}^{(\gamma)}]$. By Lemmas 2.2 and 4.2, the sequence $\boldsymbol{\lambda}_*(s) = \mathfrak{p}_s[\mathbf{w}, \lambda_{-1}^*]$ is given by (2.11), with λ_j as in (1.2). From Lemmas 4.1, 4.2 it follows that $w_j^{(\gamma)}$ can be represented in terms of λ_* defined in (4.1). We have in particular

$$\begin{array}{rcl} w_j^{(\gamma)} & = & \lambda_*(s_j) \pm (-1)^j s_j\,, \\ w_{j-1}^{(\gamma)} & = & \lambda_*(s_{j-1}) \pm (-1)^{j-1} s_{j-1}\,, \end{array}$$

hence

$$w_j^{(\gamma)} - w_{j-1}^{(\gamma)} = \lambda_*(s_j) - \lambda_*(s_{j-1}) \pm (-1)^j (s_j + s_{j-1})\,.$$

Together with (4.9), this yields

$$w_j^{(\gamma)} - w_{j-1}^{(\gamma)} = \pm 2(-1)^j s_j \ \text{ for } \ j = 1, \dots, n\,. \tag{4.16}$$

We conclude that

$$s_j = S_v(r_j, \lambda(r_j)) = \frac{1}{2}|w_j^{(\gamma)} - w_{j-1}^{(\gamma)}|\,. \tag{4.17}$$

If the values $w_j^{(\gamma)}$ are obtained from measurements, then the values of $S_v(z_\alpha^\beta)$ are available for all $-m \leq \alpha \leq \beta \leq m$. Indeed, this follows from (4.14) if $\alpha < \beta$, while $S_v(z_\beta^\beta) = 0$ for all $\beta \in \{-m, \dots, m\}$. With μ as in Hypothesis 1.2, we now set

$$A_\alpha^\beta = \int_{\mathcal{Q}_\alpha^\beta} \mu(\varrho, \xi)\,\mathrm{d}\varrho\,\mathrm{d}\xi\,. \tag{4.18}$$

From (2.4) it follows

$$S_v(z_\alpha^\beta) = \frac{1}{2}\Big(f\Big(\beta\frac{R}{m}\Big) - f\Big(\alpha\frac{R}{m}\Big)\Big) + \frac{1}{2}\int_{\mathcal{T}(z_\alpha^\beta)} \mu(r,v)\,\mathrm{d}r\,\mathrm{d}v\,. \tag{4.19}$$

Hence,

$$A_\alpha^\beta = 2\left(S_v(z_\alpha^\beta) + S_v(z_{\alpha+1}^{\beta-1}) - S_v(z_{\alpha+1}^\beta) - S_v(z_\alpha^{\beta-1})\right) \tag{4.20}$$

for all $-m \le \alpha < \beta \le m$, $\beta \ge \alpha + 2$.

Assume now that the measured values $\hat{w}_j^{(\gamma)}$ differ from the the accurate outputs $w_j^{(\gamma)}$ by an error δ, that is,

$$\left|\hat{w}_j^{(\gamma)} - w_j^{(\gamma)}\right| \le \delta \tag{4.21}$$

for all γ and all j. The approximate values $\hat{S}_\alpha^\beta$ of $S_v(z_\alpha^\beta)$ and $\hat{A}_\alpha^\beta$ of A_α^β then, by virtue of (4.17) and (4.20), satisfy for all $-m \le \alpha < \beta \le m$ the estimates

$$\left|\hat{S}_\alpha^\beta - S_v(z_\alpha^\beta)\right| \le \delta\,, \tag{4.22}$$

and

$$\left|\hat{A}_\alpha^\beta - A_\alpha^\beta\right| \le 8\delta \quad \text{if} \quad \beta \ge \alpha + 2\,. \tag{4.23}$$

For $\alpha = \beta - 1$ we have

$$S_v(z_{\beta-1}^\beta) = \frac{1}{2}\Big(f\Big(\beta\frac{R}{m}\Big) - f\Big((\beta-1)\frac{R}{m}\Big)\Big) + \frac{1}{2}\int_{\mathcal{T}(z_{\beta-1}^\beta)} \mu(r,v)\,\mathrm{d}r\,\mathrm{d}v\,. \tag{4.24}$$

Equations (4.18) and (4.24) suggest to define the approximate values of $\hat{f}$ of f and $\hat{\mu}$ of μ by the formula

$$\hat{\mu}(r,v) \quad = \quad \begin{cases} 2m^2\hat{A}_\alpha^\beta & \text{for} \quad (r,v) \in \mathcal{Q}_\alpha^\beta\,, \qquad \beta \ge \alpha + 2\,, \\ 2m^2\hat{A}_{\beta-2}^\beta & \text{for} \quad (r,v) \in \mathcal{T}(z_{\beta-1}^\beta)\,, \quad \beta \ge -m+2\,, \\ 2m^2\hat{A}_{-m}^{-m+2} & \text{for} \quad (r,v) \in \mathcal{T}(z_{-m}^{-m+1})\,. \end{cases} \tag{4.25}$$

$$\hat{f}'(v) \quad = \quad \frac{2m}{R}\hat{S}_{\beta-1}^\beta \quad \text{for} \quad v \in \left((\beta-1)\frac{R}{m}, \beta\frac{R}{m}\right)\,,\ \beta = -m+1,\dots,m \tag{4.26}$$

We now state and prove the following identification result.

Proposition 4.3. *Let Hypothesis 1.2 hold, and let both f' and μ be Lipschitz continuous on $\mathcal{T}(R,0)$ with a Lipschitz constant L, that is,*

$$\begin{cases} |f'(v) - f'(\tilde{v})| \ \le \ L|v - \tilde{v}| & \forall v \in [-R,R]\,, \\ |\mu(r,v) - \mu(\tilde{r},\tilde{v})| \ \le \ L\max\{|r-\tilde{r}|, |v-\tilde{v}|\} & \forall (r,v) \in \mathcal{T}(R,0)\,. \end{cases} \tag{4.27}$$

Let the output data $w_j^{(\gamma)}$ be given with error δ as in (4.21). Then we have

$$\begin{cases} |\hat{f}'(v) - f'(v)| \ \le \ 2\delta\frac{m}{R} + (M + 4LR^2)\frac{1}{4Rm} & \textit{for a.e. } v \in [-R,R]\,, \\ |\hat{\mu}(r,v) - \mu(r,v)| \ \le \ \frac{2LR}{m} + 16\delta m^2 & \textit{for a.e. } (r,v) \in \mathcal{T}(R,0)\,. \end{cases} \tag{4.28}$$

Remark 4.4. We see that the error may become large when we decrease the grid size. The optimal value for m in terms of the measurement error δ is of the order $\delta^{-1/3}$, and the error is then of the order $\delta^{1/3}$ as mentioned in the Introduction.

Proof of Proposition 4.3. For $\beta \geq \alpha + 2$ and $(r, v) \in \mathcal{Q}_\alpha^\beta$ we have by (4.23) that

$$\left|\hat{\mu}(r, v) - 2m^2 \int_{\mathcal{Q}_\alpha^\beta} \mu(\varrho, \xi)\, \mathrm{d}\varrho\, \mathrm{d}\xi\right| < 16\delta m^2 . \tag{4.29}$$

On the other hand, by the Lipschitz continuity of μ,

$$\left|\mu(r, v) - 2m^2 \int_{\mathcal{Q}_\alpha^\beta} \mu(\varrho, \xi)\, \mathrm{d}\varrho\, \mathrm{d}\xi\right| < \frac{LR}{m} . \tag{4.30}$$

Hence,

$$|\hat{\mu}(r, v) - \mu(r, v)| \leq \frac{LR}{m} + 16m^2\delta \quad \forall (r, v) \in \mathcal{Q}_\alpha^\beta \ \text{ if } \ \beta \geq \alpha + 2 . \tag{4.31}$$

For $(r, v) \in \mathcal{T}(z_{\beta-1}^\beta)$, we proceed in a similar way. Inequality (4.29) still holds with $\alpha = \beta - 2$ if $\beta \geq -m + 2$, and with $\alpha = -m$, $\beta = -m + 2$ otherwise. Inequality (4.30) is satisfied with right hand side $(2LR)/m$ instead of $(LR)/m$, which proves the second estimate in (4.28).

For $v \in ((\beta - 1)\frac{R}{m}, \beta\frac{R}{m})$ we have by (4.24) and (4.26) that

$$\left|\hat{f}'(v) - \frac{m}{R}\Big(f\Big(\beta\frac{R}{m}\Big) - f\Big((\beta - 1)\frac{R}{m}\Big)\Big)\right| < \delta\frac{2m}{R} + \frac{M}{4mR} . \tag{4.32}$$

The Lipschitz continuity of f' yields

$$\left|f'(v) - \frac{m}{R}\Big(f\Big(\beta\frac{R}{m}\Big) - f\Big((\beta - 1)\frac{R}{m}\Big)\Big)\right| < L\frac{R}{m} , \tag{4.33}$$

and the first estimate in (4.28) follows. ☐

5. **Numerical inversion.**

We proceed in principle as at the beginning of the proof of Theorem 1.3. Given a sequence $\mathbf{w} = \{w_j\}_{j=0}^\infty \in \ell^\infty$ and an initial configuration $\lambda_{-1} \in \Lambda_K$, we define λ_{-1}^* by (2.11), set $u_{-1} = \lambda_{-1}(0)$, $w_{-1} = \lambda_{-1}^*(0)$, and determine u_j consecutively for $j = 0, 1, 2, \dots$ as solutions $u = u_j$ of the equation $W(u) = w_j - w_{j-1}$. The problem here is that the Preisach measure is known only approximately, and that the equation $W(u) = w_j - w_{j-1}$ can also be solved only approximately, with u_j from an a priori given discrete set of admissible values.

We denote by $\mathbb{Z}$ the set of all integers, assume that Hypothesis 1.2 holds, and make the following discretization assumptions. The identification algorithm in Section 4 offers an example of admissible approximations, cf. Proposition 4.3 and Remark 4.4.

Hypothesis 5.1. A discretization parameter $h > 0$, and functions $f_h : \mathbb{R} \to \mathbb{R}$, $\mu_h : (0, \infty) \times \mathbb{R} \to \mathbb{R}$, f_h locally Lipschitz, $\mu_h \in L^\infty((0, \infty) \times \mathbb{R})$ are given such that

- (i) $|f_h(u) - f(u)| \leq h \ \forall u \in \mathbb{R}$, $f_h'(u) \geq b$ a.e.;
- (ii) $\int_{-R}^R \int_0^R |\mu_h(r, v) - \mu(r, v)|\, \mathrm{d}r\, \mathrm{d}v < MR^2 h$ for all $R > 0$, $\mu_h(r, v) \geq -\beta(r)$ a.e.;
- (iii) We define Λ_K^h as the set of all $\lambda \in \Lambda_K$ such that $\lambda'(r) \in \{-1, 1\}$ is constant in $(kh, (k+1)h)$ and $\lambda(kh) \in h\mathbb{Z}$ for all $k \in \mathbb{N} \cup \{0\}$, $k \leq K/h$, and assume that $\lambda_{-1}^h \in \Lambda_K^h$ is given such that $\max_{r \geq 0} |\lambda_{-1}^h(r) - \lambda_{-1}(r)| < h$;
- (iv) The approximate solutions u_j^h belong to the set $2h\mathbb{Z}$.

We now define the approximate Preisach operator $\mathbf{w} = P_h[\mathbf{u}, \lambda_{-1}^h]$ by the formula

$$w_j = f_h(u_j) + \int_0^\infty g_h(r, \lambda_j^h(r))\, \mathrm{d}r \quad \text{ for } \ j = 0, 1, 2, \dots , \tag{5.1}$$

analogous to (1.9), with g_h related to μ_h as in Hypothesis 1.2, and $\boldsymbol{\lambda}^h(r) = \{\lambda_j^h(r)\}_{j=0}^\infty = \mathfrak{p}_r[\mathbf{u}, \lambda_{-1}^h]$ for any sequence $\mathbf{u} = \{u_j\}_{j=0}^\infty \in \ell^\infty$.

Solving the equation $\mathbf{w} = P_h[\mathbf{u}, \lambda_{-1}^h]$ for given $\mathbf{w}$ involves numerical integration over a 2D domain limited by the curves $v = \lambda_{j-1}^h(r)$ and $v = \lambda_j^h(r)$, cf. the first line of (2.15). The following Lemma shows that the integration domain has a convenient form.

Lemma 5.2. *Let λ_{-1}^h and $\mathbf{u}^h = \{u_j^h\}_{j=0}^\infty$ be as in Hypothesis 5.1, let $|u_j^h| \le K$ for all $j \in \mathbb{N} \cup \{0\}$, and let $\boldsymbol{\lambda}^h(r) = \mathfrak{p}_r[\mathbf{u}^h, \lambda_{-1}^h]$. Then $\lambda_j^h \in \Lambda_K^h$ for all $j \in \mathbb{N} \cup \{0\}$.*

Proof. We proceed by induction. Let the assertion hold for $j-1$, and assume for example that $u_j^h > u_{j-1}^h = \lambda_{j-1}^h(0)$. Then

$$\lambda_j^h(r) = \begin{cases} u_j^h - r & \text{for } r < r_j\,, \\ \lambda_{j-1}^h(r) & \text{for } r \ge r_j\,, \end{cases} \tag{5.2}$$

where $r_j = \min\{r > 0 : r + \lambda_{j-1}^h(r) = u_j^h\}$. The function $\nu(r) := r + \lambda_{j-1}(r)$ is nondecreasing, $\nu'(r) \in \{0, 2\}$ is constant in $(kh, (k+1)h)$ for $k \le K/h$. We have $\nu(0) \in 2h\mathbb{Z}$, hence $\nu(kh) \in 2h\mathbb{Z}$ for all $k \in \mathbb{N} \cup \{0\}$, and $\nu(r) \notin 2h\mathbb{Z}$ for all $r \notin h\mathbb{Z}$. We conclude that $r_j \in h\mathbb{Z}$, which completes the proof. □

The approximate inversion algorithm is defined as follows. We set

$$\lambda_{-1}^{*h}(s) = \left(S_r^h(r, \lambda_{-1}^h(r)) + \int_r^\infty g_h(\varrho, \lambda_{-1}^h(\varrho))\, \mathrm{d}\varrho \right)\Bigg|_{r = \varphi^{-1}_{\lambda_{-1}^h}(s)}\,, \tag{5.3}$$

and $u_{-1}^h = \lambda_{-1}^h(0)$, $w_{-1}^h = \lambda_{-1}^{*h}(0)$. The function S^h is related to f_h and μ_h as in (2.1). We have by Hypothesis 5.1 that $|w_{-1}^h - w_{-1}| \le (F(K) + MK)h$.

We assume that the discretization parameter h is sufficiently small and choose $K_h > K$ such that $(F(K_h) + MK_h)h < (b - b_0)(K_h - K)$. We now continue by induction, and assume that we have constructed u_i^h, w_i^h for $i = -1, 0, 1, \ldots, j-1$ in such a way that

$$\left.\begin{aligned} & w_i^h = f_h(u_i^h) + \textstyle\int_0^\infty g_h(\varrho, \lambda_i^h(\varrho))\, \mathrm{d}\varrho\,, \\ & u_i^h \in 2h\mathbb{Z}\,, \\ & |w_i^h - w_i| \le (F(K_h) + MK_h)h\,, \\ & |u_i^h| \le K_h \end{aligned}\right\} \tag{5.4}$$

for all $i = -1, 0, 1, \ldots, j-1$. Let u_j^h run over the set $\{2hk : k = -K_h/h \ldots, K_h/h\}$. The distance between two consecutive values of u_j^h is $2h$, so that the distance between two consecutive values of w_j^h is at most $2(F(K_h) + MK_h)h$. Hence, w_j^h can be chosen in such a way that (5.4) holds for $i = j$. We now state the numerical approximation result. It shows that the above algorithm does not accumulate the error in time and is independent of the number of time steps.

Theorem 5.3. *Let Hypotheses 1.2, 5.1 hold, and let $K > 0$, $\mathbf{w} \in \ell^\infty$ such that $|\mathbf{w}|_\infty < (b - b_0)K$ and $\lambda_{-1} \in \Lambda_K$ be given. Let $\mathbf{u} \in \ell^\infty$ be the solution of the equation $\mathbf{w} = P[\mathbf{u}, \lambda_{-1}]$, and let $\mathbf{u}^h, \mathbf{w}^h$ be constructed by the algorithm (5.4). Then there exists a constant $C^* > 0$ depending only on the data such that $|\mathbf{u} - \mathbf{u}^h|_\infty \le Ch$.*

Proof of Theorem 5.3. We denote by P_h the operator which with $\mathbf{u}^h$ and λ_{-1}^h associates $\mathbf{w}^h$. Then $\mathbf{w}^h = P_h[\mathbf{u}^h, \lambda_{-1}^h]$, and we set $\hat{\mathbf{w}}^h = P[\mathbf{u}^h, \lambda_{-1}^h]$. By Hypothesis

5.1 (i), (ii) we have $|\mathbf{w}^h - \hat{\mathbf{w}}^h|_\infty \le (1 + MK_h^2)h$. Furthermore, by (1.15), there exist positive constants C_1, C_2, C_3 such that

$$|\mathbf{u} - \mathbf{u}^h|_\infty \le C_1(h + |\mathbf{w} - \hat{\mathbf{w}}^h|_\infty) \le C_2(h + |\mathbf{w} - \mathbf{w}^h|_\infty) \le C_3 h\,,$$

and the proof is complete. □

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